1. Let m^* denote the Lebesgue outer measure, and suppose that $E \subset \mathbb{R}$.

   (a) State the definition of $m^*(E)$.

   (b) Now suppose that E is compact. Prove that

$$m^*(E) = \inf\left\{\sum_{n=1}^{N} \ell(I_n) : E \subset \bigcup_{n=1}^{N} I_n\right\},$$

   where the infimum is over all finite collections of bounded open intervals and $\ell(I)$ denotes the length of the interval I.

2. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers tending to zero, and suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions on $\mathbb{R}$. Let

$$E_n = \{x : |f_n(x)| > a_n\},$$

and suppose that $\sum_n m(E_n) < \infty$.

(a) Prove that $\{f_n\}_{n=1}^{\infty}$ converges in measure to zero.

(b) Prove that $\{f_n\}_{n=1}^{\infty}$ converges pointwise almost everywhere to zero.

3. Prove that the following limit exists and find the limit:

$$\lim_{n} \int_{0}^{2} \frac{e^{n(x-1)}}{1+e^{n(x-1)}}.$$

4. Let $f : [0, 1] \to \mathbb{R}$ be Lebesgue integrable, and let $F : [0, 1] \to \mathbb{R}$ be defined by

$$F(x) = \int_0^x f.$$

(a) Prove directly from the definition that F has bounded variation.

(b) Now suppose that $f \geq 0$. Find the total variation of F.