1. [10 points] Let E be a subset of $\mathbb{R}$ with finite outer measure $m^*(E)$ and $a < b < c$.

   (a) Prove that
   $$m^*(E \cap [a, c]) = m^*(E \cap [a, b]) + m^*(E \cap [b, c]).$$

   (b) Define the function f_E by
   $$f_E(x) \equiv m^*\big(E \cap (-\infty, x]\big), \quad \forall x \in \mathbb{R}.$$

   Prove that f_E is *increasing and continuous.*

2. [10 points] Let $f(x)$ and $g(x)$ be Lebesgue measurable functions on $[0, 1]$.

   (a) Prove by definition that $|f(x)|$ is Lebesgue measurable on $[0, 1]$.

   (b) Prove by definition that $h(x) \equiv \min(f(x), g(x))$ is Lebesgue measurable on $[0, 1]$.

3. [10 points] Let $f_n(x) \equiv \dfrac{\sin x}{n^{-2} + \cos x}$, $x \in \left[0, \dfrac{\pi}{2}\right]$, $n \in \mathbb{N}$, and let $\nu(x)$ be a bounded non-negative measurable function on $\mathbb{R}$. Do the following.

   (a) Prove that the limit $\lim_{n\to\infty} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} f_n(x)\mathrm{d}x$ exists.

   (b) Find the limit value in (a).

   (c) Prove that, for $\alpha \in (0, \pi/2)$,

   $$\lim_{n\to\infty} \int_0^\alpha f_n(x)\nu(x)\mathrm{d}x = \lim_{n\to\infty} \int_0^\alpha \tan(x)\nu(x)\mathrm{d}x,$$

   and this limit is finite.

4. [10 points] Do the following.

   (a) State the definition of an *absolutely continuous function on a finite interval* $[a, b]$.

   (b) Let $P(x)$ be a polynomial function on $[0, 1]$. Prove that $P(x)$ is absolutely continuous on $[0, 1]$.