1. [10 points] Let f be a non negative Lebesgue measurable function on E with $m(E) < \infty$. Denote $E_k = f^{-1}\left([k, k+1)\right), k \in \mathbb{N}$. Prove that

$$\int_E f < +\infty$$

if and only if

$$\sum_{k=1}^{\infty} k \cdot m(E_k) < +\infty.$$

2. [10 points] Do the following.

   (a) Give a definition of a Lebesgue measurable function on $\mathbb{R}$.

   (b) Assume that the function f defined on $(0,1)$ is differentiable at each point on $(0,1)$. Prove that its derivative f' is Lebesgue measurable on $(0,1)$.

   (c) Let f be *any* function defined on $\mathbb{R}$. Define a new function g by

$$g(x) \equiv \chi_{\mathbb{Q}}(x) \cdot f(x), \quad x \in [0,1],$$

   where $\mathbb{Q}$ is the set of rational numbers and $\chi_{\mathbb{Q}}$ is the characteristic function of $\mathbb{Q}$. Prove that g is Lebesgue measurable on $\mathbb{R}$.

3. [10 points] Let m and n be natural numbers $m, n \geq 1$ and $m < n$. Assume that $E_1, E_2, \cdots, E_n$ are n Lebesgue measurable subsets of $[0,1]$ with the property that each point $x \in [0,1]$ is in at least m of above subsets. Prove that there exists at least one E_j such that $m(E_j) \geq \frac{m}{n}$.

4. [10 points] Do the following.

   (a) State the definition of *bounded variation.*

   (b) Let f be the function defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1 - x, & \text{if } x \in (0, 1); \\ 7, & \text{if } x = 1. \end{cases}$$

   Prove that f is of bounded variation. Find the total variation of f.

   (c) Write the function f into the form of difference of two increasing functions.