1. [10 points] Give an example of a sequence of Lebesgue measurable functions $\{f_n, n \in \mathbb{N}\}$ that simultaneously satisfies all three of the following conditions:

   (i) $0 \le f_n(x) \le \dfrac{1}{x}, \quad \forall n \in \mathbb{N};$

   (ii) $\lim_{n\to\infty} f_n(x) = 0, \quad \forall x \in [0,1];$

   (iii) $\lim_{n\to\infty} \int_{[0,1]} f_n(x)\mathrm{d}x = \infty.$

2. [10 points] Do the following.

   (a) Let $E \subset \mathbb{R}$ be a nonempty Lebesgue measurable set and f be a function defined on E. Give a definition that the function f is Lebesgue measurable on E.

   (b) Let f be a function defined on $[0, 1]$. Assume that f is Lebesgue measurable on (α, β) for all $\alpha, \beta \in \mathbb{R}$, $0 < \alpha < \beta < 1$. Prove by definition that f is Lebesgue measurable on $[0, 1]$.

3. [10 points] Let f be a Lebesgue integrable function on $[0,1]$ and $f(x) > 0, \forall x \in [0,1]$.

   (a) Prove that

$$[0,1] = \bigcup_{n\in\mathbb{N}} \left\{ x \in [0,1], f(x) \geq \frac{1}{n} \right\}.$$

   (b) Let $\mathcal{F}$ be the family of all Lebesgue measurable subsets S of $[0,1]$ with property that $m(S) \geq 0.5$. Using the above results in (a), prove that

$$\inf_{S\in\mathcal{F}} \int_S f \mathrm{d}m > 0.$$

4. [10 points] Do the following.

   (a) State the definition of *bounded variation.*

   (b) Let f be the function defined by

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & x \in (0, \frac{2}{\pi}]; \\ 0 & x = 0. \end{cases}$$

   Prove that this function is NOT of bounded variation.