Real Analysis II, Problem 1 **Name:**

1. Let X be a compact space and let A be a closed subset of X. Prove directly from the definition that A is compact.

2. Let H be a Hilbert space with inner product (x, y) for $x, y \in H$ and let $\{x_n\} \subset H$ such that $\{x_n\}$ converges weakly to $x \in H$. Prove that $\|x_n - x\| \to 0$ if and only if $\lim_{n\to\infty} \|x_n\| = \|x\|$.

3. Let H be a Hilbert space with inner product (x, y) for $x, y \in H$ and let $\{x_n\} \subset H$ such that $\{x_n\}$ converges weakly to $x \in H$. Prove that $\{\|x_n\|\}$ is bounded, i.e., there is a constant M independent of n such that $\|x_n\| \leq M$ for all indices n.

4. Let $(X, \mathcal{B}, \mu)$ be a complete measure space and let $f \in L^p(\mu)$, $1 < p < \infty$. Let $\{T_n\}$ be a convergent sequence of bounded operators on $L^p(\mu)$, i.e., there exists a $T \in \mathcal{L}(L^p(\mu))$ such that $\|T_n - T\| \to 0$. Let $g \in L^q(\mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$, and suppose there exists $\{g_n\} \subset L^q(\mu)$ such that $\|g_n - g\|_q \to 0$. Prove that

$$\int_X (T_n f) g_n d\mu \to \int_X (Tf) g d\mu.$$