1. Suppose that $X$ and $Y$ are normed linear spaces and that $T : X \longrightarrow Y$ is a linear operator. Show that if $T$ is continuous at a single point, then it is continuous and hence bounded. [*10 points*]

2. Let $E$ be a measurable set, $1 \leq p < \infty$, $q$ the conjugate of $p$, and $\mathcal{S}$ a dense subset of $L^q(E)$. Show that if $g \in L^p(E)$ and $\int_E f \cdot g = 0$ for all $f \in \mathcal{S}$, then $g = 0$ almost everywhere. [*10 points*]

3. Let $X$ be a normed linear space.

   (i) Show that if $\{x_n\}$ is a sequence in $X$ that converges to $x$ weakly, then $x$ is unique. [*2 points*]

   (ii) Show that the strong convergence implies the weak convergence, but the converse is not true in general. [*4 points*]

   (iii) Suppose that $X$ is a Hilbert space and $x_n \to x$ weakly in $X$. Show that if $\|x_n\| \to \|x\|$, then $x_n \to x$ strongly in $X$. [*4 points*]

4. Suppose that $\mathcal{K} : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ is continuous, where $-\infty < a < b < \infty$ and $H = L^2(a, b)$ is a Hilbert space. A linear operator $T : H \longrightarrow H$ is defined by

$$Tu(x) = \int_a^b \mathcal{K}(x, y)u(y)dy, \qquad \text{for all } u \in H.$$

   (i) Show that the operator $T$ is bounded. [*3 points*]

   (ii) Show that $T^*$ defined by $T^*u = \int_a^b \mathcal{K}(y, x)u(y)dy$ is an adjoint operator of $T$ and hence $T$ is self-adjoint if $\mathcal{K}(x, y) = \mathcal{K}(y, x)$ for all $x, y \in [a, b]$. [*3 points*]

   (iii) Show that $Tu \in C[a, b]$, for $u \in H$. [*4 points*]