1. [10 points] Let E be a measurable set and $g \in L^2(E)$. Define a linear functional on $L^2(E)$ by

$$T(f) = \int g \cdot f, \text{ for all } f \in L^2(E).$$

Show that T is a bounded linear functional on $L^2(E)$ and $\|T\|_* = \|g\|_2$. Here $\|\cdot\|_*$ is the operator norm $\|T\|_* = \sup\{T(f) : \|f\| \leq 1\}$.

2. [10 points] Let X and Y be normed spaces. The norm on $X \times Y$ is defined by

$$\|(x, y)\| = \|x\| + \|y\|.$$

If $T_1 : X \longrightarrow Y$ is a closed linear operator and $T_2 \in \mathcal{L}(X, Y)$ (which is the space of bounded linear operators), show that $T_1 + T_2$ is a closed linear operator.

3. [10 points] Let X and Y be normed linear spaces and $T : X \longrightarrow Y$ a liner operator. Show that T is a compact operator if and only if for every bounded sequence $\{x_n\}$ in X, the image sequence $\{T(x_n)\}$ in Y has a convergent subsequence.

4. [10 points] Let X and Y be normed spaces and $T : X \longrightarrow Y$ is a linear operator. Show that T is continuous if and only if T is bounded (that is, there is a constant M such that for all $x \in X$, $\|T(x)\| \leq M\|x\|$.)