UNCC ID:

1. Suppose that E is a Legesgue measurable subset of $\mathbb{R}$ with finite measure, and let $1 \leq p_1 < p_2 < \infty$.
   (a) Use Hölder's Inequality to show that $L^{p_2}(E) \subseteq L^{p_1}(E)$.

   (b) Show that if $(f_n)_{n\geq 1} \to f$ in $L^{p_2}(E)$, then $(f_n)_{n\geq 1} \to f$ in $L^{p_1}(E)$.

2. Suppose that X is a Banach space and $P : X \to X$ is a bounded linear map such that $P \circ P = P$ (where $\circ$ denotes composition). Let $Y = P(X)$. Prove that $P : X \to Y$ is an open mapping.

3. Suppose that H is a Hilbert space and V is a nonempty closed subset of H such that if $u, v \in V$ then $\lambda u + (1-\lambda)v \in V$ for all $\lambda \in [0, 1]$. Prove that there exists a unique $u \in V$ such that $\|u\| \leq \|v\|$ for all $v \in V$. You may use the Parallelogram Identity:

$$\|a - b\|^2 + \|a + b\|^2 = 2\|a\|^2 + 2\|b\|^2.$$

4. Let H be a Hilbert space with inner procuct $< \cdot, \cdot >$, and let $\mathcal{L}(H)$ be the set of bounded linear operators from H to itself.
   (a) Let $T \in \mathcal{L}(H)$, and suppose that $(T_n)_{n\geq 1} \subset \mathcal{L}(H)$ is a sequence of compact operators such that $\|T_n - T\|$ converges to zero. Prove that T is compact.

   (b) Suppose that $(\varphi_k)_{k\geq 1}$ is an orthonormal basis of H, and let $(\lambda_k)_{k\geq 1}$ be a sequence of real numbers that converges to zero. Define $T : H \to H$ by

$$T(h) = \sum_k \lambda_k < h, \varphi_k > \varphi_k.$$

   Prove that T is compact.