Asymptotic Approximation of the Random Walks with Heavy tails.

by

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A dissertation submitted to the faculty
The University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Doctorate of Philosophy in
Applied Mathematics.

Charlotte

2013

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Abstract

AGBOR ANDU. The Asymptotic Approximation of the Random Walk with heavy tails. (Under the direction of DR. BORIS VAINBERG)

The main result of this dissertation concerns the asymptotics, uniform in t and x, of the probability distribution of a random walk with heavy tails. The random walk is a Markov process and thus can be characterized in terms of their generators. We impose certain conditions on the Fourier transform of the kernel of the generator, which still allow us to consider rather general class of processes on Z^d. The process we consider can be viewed as a generalization of the simple symmetric walk (in continuous time) for which both the central limit theorem and large deviation results are well-known.

For problems with heavy tails, the analogue of the central limit theorem is the convergence of the properly normalized process to the stable laws. In terms of probability densities, these limit theorems give the asymptotics of $p(t, x, 0)$ when x is of order $t^{1/\alpha}$.

For the class of random walk under consideration, we obtain the asymptotics of $p(t, x, 0)$ uniformly in t and x for all $t > 1$, $x \in R^d$, covering, in particular, the regime of the central limit theorem and large deviations.

ACKNOWLEDGEMENT

My immense gratitude to my advisor, Dr. Boris Vainberg for his guidance, supervision, motivation and insightfulness in the subject matter. His broad knowledge in applied mathematics helped me tremendously towards the realization and completion of this dissertation.

I am also grateful to Dr. Stanislav Molchanov for his support with useful material for the project.

Last but not the least, I extend my acknowledgement to my Wife, Mrs. Roseline J. Agbor and my son Ken R. Agbor for their moral support and financial sacrifices without which this would have been a piped dream.

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Chapter 1. Literature Review and Introduction

1.1 Literature Review

Research in the area of large deviations for random walks with heavy-tailed jumps began in the second half of the twentieth century. At first the main efforts was, of course, concentrated on studying the deviations of the sum S_n of r.v's. Here one should first of all mention the papers by C. Heyde [6, 7], S.V. Nagaev [8, 9], A.V. Nagaev [10, 11], D.H. Fuk and S.V. Nagaev [12], L.V. Rozovski [13, 14] and others. These established the basic principle by which asymptotics of $P(S_n > x)$ are formed: the main contributions to the probability of interest comes from trajectories that contain large jumps.

Later on, papers began to appear in which this principle was used to find the distribution of the maximum $\bar{S}_n$ of partial sums and also to solve more general boundary problems for random walks (I.F. Pinelis [15], V.V. Godovanchuk [16], A.A. Borokov [17, 18].Somewhat aside from this were papers devoted to the probabilities of large deviations of maximum of a random walk with negative drift. The general first results were obtain by A.A. Bokorov in [18], while more complete versions (for subexponential summands) were established by N. Veraverbeke [19]and D.A. Korshunov [20].

A.A. Bokorov [21, 22, 23] began a systematic study of large deviations for random walks with regularly distributed jumps. Then the papers [24, 25, 26, 27, 28] and some others appeared, in which the derived results were extended to semi-exponential and regular exponentially decaying distributions, to multivariate random walks,to the case of non-identically distributed summands and so on. As a result, a whole range interesting problems arose, unified by the general approach to their solution and a system of interconnected rather advanced results were, as a rule, quite close to unimprovable. As these problems and results were, moreover, of considerable interest for applications, the idea of writing a thesis on all this became quite natural.

This thesis concerns the asymptotic behaviour of the probabilities of rare events related to large deviations of the trajectories of random walk whose jump distribution decays not very fast at infinity and possess some form of 'regular behaviour'.

Random Walks form a classical object of probability theory, the study of which is of tremendous theoretical interest. They constitute a mathematical model of great importance for applications in mathematical statistics, risk theory, queueing theory and so on.

Large deviations and rare events are of great interest in all these applied areas, since computing the asymptotic of large deviation probabilities enables one to find for example, small error probabilities in mathematical statistics, small ruin probabilities in risk theory, small buffer overflow probabilities in queueing theory, and so on.

Slowly decaying and, in particular, regular distributions present, when one is studying large deviation probabilities, an alternative to distributions decaying exponentially fast at infinity (for which cramer's condition hold). The first classical results in large deviation theory were obtained for the case of distribution decaying exponentially fast. However, this condition of exponential decay fails in many applied problems.

For regular distribution, the large deviation probabilities are mostly formed by contributions from the distribution tails(on account of the large jumps in the random walk trajectory).As a result, analytical methods prove to be efficient, and everything is determined by the behaviour of the laplace transform of the jump distributions.

1.2 Introduction

The main result of this Thesis concerns the asymptotics, uniform in t and x, of the probability density of a random walk with heavy tails. The random walks are Markov Processes (section 2.2) and thus can be characterized in terms of their generators (section 3.1). We impose certain conditions on the Fourier transform of the kernel of the generator, which still allow us to consider a rather general class of process on Z^d. The processes we consider can be viewed as a generalization of the simple symmetric walk (in continuous time) for which both the central limit theorem and large deviation results are well-known.

For problems with heavy tails, the analogue of the central limit theorem is the convergence of the properly normalized process to the stable laws (section 2.2). In terms of probability densities, these limit theorems give the asymptotics of $p(t, x, 0)$ when x is of order $t^{1/\alpha}$ [29].

For the class of random walks under consideration, we obtain the asymptotics of $p(t, x, 0)$ uniformly in t and x for all $t > 1$, $x \in R^d$, covering, in particular, the regime of the limit theorem and large deviations.

In the case of the simple random walk on the lattice Z^d, the transition probability $p(t, x, y)$ satisfies the standard heat equation

$$\begin{cases} \frac{dp(t,x,y)}{dt} & = \kappa \triangle_x p(t, x, y) = \kappa \triangle_y p(t, x, y) \\ p(0, x, y) & = \delta_y(x), \end{cases}$$

where κ is the diffusion coefficient. The generator $\kappa\triangle$ of a simple symmetric walk is a particular case of the generator

$$\mathcal{L}f(x) = k\sum_{z\neq 0}[f(x+z)-f(x)]q(z)$$

$$q(z) = q(-z) > 0,$$

$$\sum_{z\neq 0} q(z) = 1$$

of the process with heavy tails. Here $q(z)$ is the probability of the jump from one state x_t to another state $x_t + z$ in time dt, which is described by the relation

$$x_{t+dt} = \begin{cases} x_t & \text{with prob. } 1 - kdt, \\ x_t + z & \text{with prob. } kq(z)dt. \end{cases}$$

Indeed, $\mathcal{L} = \kappa\Delta$ if

$$q(z) = \begin{cases} \frac{1}{2d} & \text{if } |z| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We'll, however, consider q that may be positive everywhere. The precise conditions on q will be provided below.

The transition probability $p(t, x, y)$ of a random walk with heavy tails is determined by solving the initial value problem

$$\frac{dp}{dt} = \mathcal{L}_x p, \quad p(0, x, y) = \delta_y(x).$$

We apply the Fourier transform to obtain

$$p(t, x, y) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{i(\varphi,(y-x))-t\phi(\varphi)}\, \mathrm{d}\varphi, \quad d \geq 1,$$

where

$$\phi(\varphi) = k\sum_{z\in Z^d}(1 - e^{i(\varphi,z)})q(z), \quad d \geq 1.$$

We use these results to determine the asymptotics of the transition probability, $p(t,x)$, in both the 1-dimensional case (Sect. 4.2) and the n-dimensional case (sect. 4.3). In determining the asymptotics of p, we assumed that q decays slow (heavy-tail), that is $q(z) \sim \frac{q_0}{|z|^{1+\alpha}}, \quad z \to \infty, \quad 0 < \alpha < 2.$

In fact, the asymptotics of p in the 1-dimensional case is studied, we assume that

$$q(z) = \frac{q_0}{|z|^{1+\alpha}} + \frac{q_1}{|z|^{2+\alpha}} + \frac{q_2}{|z|^{3+\alpha}} + O(\frac{1}{|z|^{4+\alpha}}).$$

Then the following relation is proved for the function $\phi(\varphi)$:

$$\phi(\varphi) \sim c_0|\varphi|^\alpha + O(|\varphi|^\gamma),\ \varphi \to 0,$$

where $\gamma = \min\{2, \alpha+1\}, \quad c_0$ and a_0 are constants. This asymptotics is used to justify the following main result concerning the 1-D case. Without loss of generality, one may assume that $c_0 = 1$. Then the following relations are valid when $d = 1$.

$$p(t,x) = \frac{1}{t^{\frac{1}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right)(1 + o(1)), \quad x^2 + t^2 \to \infty,$$

where

$$F(\sigma) = \int_{-\infty}^{\infty} e^{i\sigma\varphi - |\varphi|^\alpha}\, \mathrm{d}\varphi,$$

and ω is a neighborhood of the set $\{\sigma_i\}_{i=1}^n$ of points σ_i such that

$$F(\sigma) = 0,\ |\sigma_i| < \infty.$$

The asymptotics of $p(t,x)$ in the multidimensional case is similar to the one above. We use notation $L(\varphi)$ for the function $\phi(\varphi)$ in the multidimensional case (to distinguish the cases). We assume that L has an asymptotic behavior at zero similar to one that was established in the one dimensional case. Namely, we assume that

$$L(\varphi) = |\varphi|^\alpha h(\dot{\varphi}) + \sum_{i=1}^{M-1} |\varphi|^{\alpha+i} h_i(\dot{\varphi}) + O(|\varphi|^{\alpha+M}), \quad \varphi \to 0,$$

for some large enough M, $\dot{\varphi} = \frac{\varphi}{|\varphi|}$, $h = h(\dot{\varphi})$ and h_i are smooth functions on the sphere. The asymptotics of $p(t,x)$ is given by

$$p(t,x) = \begin{cases} \frac{1}{t^{\frac{d}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right)(1+o(1)), \quad |x|^2+t^2 \to \infty, \quad \text{if } \frac{x}{t^{\frac{d}{\alpha}}} \notin B(\epsilon), \\ \\ \frac{1}{t^{\frac{d}{\alpha}}} \left[F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) + o(1)\right], \quad |x|^2+t^2 \to \infty, \quad \text{if } \frac{x}{t^{\frac{d}{\alpha}}} \in B(\epsilon), \end{cases}$$

where we denoted by $B(\epsilon)$ an ϵ-neighborhood of the set of points in $\mathbb{R}^d$ where $F(z) = 0$ and

$$F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) = \int_{R^d} e^{i\left(\varphi, \frac{x}{t^{\frac{1}{\alpha}}}\right) - |\varphi|^\alpha h(\dot{\varphi})} \, \mathrm{d}\varphi.$$

The asymptotics of $F(y) = \int_{R^d} e^{i(\varphi,y) - |\varphi|^\alpha h(\dot{\varphi})} \, \mathrm{d}\varphi$, where $\dot{\varphi} = \frac{\varphi}{|\varphi|}$, is shown to be

$$F(y) = |y|^{-d-\alpha} f(\dot{y}) + o(|y|^{-d-\alpha}), \quad |y| \to \infty,$$

where $\dot{y} = \frac{y}{|y|}$ and $f(\dot{y})$ is defined by $h(\dot{\varphi})$ as follows:

$$\int_{R^d} h(\dot{\varphi}) |\varphi|^\alpha e^{iy\varphi} d\varphi = -|y|^{-d-\alpha} f(\dot{y}).$$

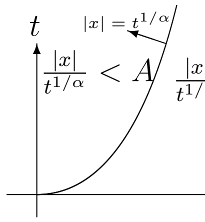
The integral here is understood in the sense of the Fourier transform in the space of distributions. Thus the formula for p can be rewritten in the form

$$\text{a)}\quad p(x,t) = \frac{1}{t^{\frac{d}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) + O\left(\frac{1}{t^{\frac{d}{\alpha}+1}}\right), \qquad \frac{|x|}{t^{\frac{1}{\alpha}}} \leq A,$$

where $\quad A \quad$ is arbitrary,

$$\text{b)}\quad p(x,t) = \frac{t}{|x|^{d+\alpha}} f(\dot{x}) + o\left(\frac{t}{|x|^{d+\alpha}}\right), \qquad \text{if} \quad \frac{|x|}{t^{\frac{1}{\alpha}}} \to \infty,$$

where the regions in the domain of $p(x,t)$ is described by the figure below



Chapter 2. Central Limit Theorem and Regularly Varying functions.

2.1 Central Limit Theorem

Theorem 2.1.1 (Central Limit Theorem). Suppose $X_1, X_2, ...,$ are mutually independent and identically distributed random variables with mean m and finite variance σ^2. Let $S_n = \sum_{k=1}^{n} X_k$. Then we have

$$\lim_{n\to\infty} P\left(a < \frac{S_n - nm}{\sqrt{n}\sigma} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$

uniformly for all $-\infty \leq a < b \leq \infty$.

Proof. We can without loss of geniality show assume that $m = 0$ and $\sigma^2 = 1$ and that if X and Y be independent r.v., then $\mu_{X+Y} = \sqrt{2\pi}\mu_X * \mu_Y$. It follows that

$$\mu S_n = (2\pi)^{\frac{n-1}{2}} \underbrace{\mu * \mu * ... * \mu}_{\text{n times}}$$

where μ denotes the common distribution of the X_ns. Let $Z_n = \frac{s_n}{\sqrt{n}}$. Then,

$$\widehat{\mu Z_n}(t) = \widehat{\mu S_n}\left(\frac{t}{\sqrt{n}}\right) = \frac{1}{\sqrt{2\pi}}\left[\sqrt{2\pi}\hat{\mu}\left(\frac{t}{\sqrt{n}}\right)\right]^n.$$

We get

$$\widehat{\mu Z_n}(t) = \frac{1}{\sqrt{2\pi}}\left[1 - \frac{t^2}{2n} + \alpha\left(\frac{t}{\sqrt{n}}\right)\right]^n.$$

consequently,

$$\lim_{n\to\infty} \widehat{\mu Z_n}(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} = \frac{1}{\sqrt{2\pi}}\hat{g}(t),$$

where g is the Gaussian function.

By Lévy's theorem, the sequence $\{Z_n\}_{n=1}^n$ converges in distribution to a r.v. having distribution $\nu(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2}dx$. We can conclude from here that

$$\lim_{n\to\infty} \int_R f d\mu_{Z_n} = \int_R f d\nu, \qquad f \in C_b(\mathcal{R}).$$

Let $0 < \epsilon < \frac{(b-a)}{2}$. Choose a continuous function f_1 such that $0 \leq f_1 \leq 1$, $f_1(x) = 0$ for $x \notin (a,b)$ and $f_1(x) = 1$ for $x \in [a+\epsilon, b-\epsilon]$. And choose a continuous function f_2 such that $0 \leq f_2 \leq 1$, $f_2(x) = 1$ for $x \in [a,b]$, and $f_2(x) = 0$ for $x \notin (a-\epsilon, b+\epsilon)$. Then

$$\int_R f_1(x) d\mu_{Z_n}(x) \leq \int_{(a,b]} d\mu_{Z_n}(x) \leq \int_R f_2(x) d\mu_{Z_n}(x).$$

Thus,

$$\frac{1}{\sqrt{2\pi}} \int_{a+\epsilon}^{b-\epsilon} e^{-x^2/2}dx \leq \int_R f_1(x) d\nu(x) \leq \liminf_{n\to\infty} \mu_{Z_n}((a,b])$$

and

$$\limsup_{n\to\infty} \mu_{Z_n}((a,b]) \leq \int_R f_2(x) d\nu(x) \leq \frac{1}{\sqrt{2\pi}} \int_{a-\epsilon}^{b+\epsilon} e^{-x^2/2}dx.$$

Because ϵ can be made arbitrarily small, it follows that

$$\frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}dx = \lim_{n\to\infty} \mu_{Z_n}((a,b]) = \lim_{n\to\infty} P\left(a < \frac{S_n}{\sqrt{n}} \leq b\right),$$

as required. □

We can obtain as a special case of central limit theorem, the following result known as DeMoivre-Laplace theorem:

$$\lim_{n\to\infty} P\left(a < \frac{n(E) - np}{\sqrt{np(1-p)}} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}dx$$

uniformly for all $-\infty \leq a < b \leq \infty$. $X_1, X_2, ...$ has a binomial distribution and are iid and have common mean p and variance $p(1-p)$. Note that for $n(E) =$

$X_1 + X_2 + ... + X_n$, we obtain the DeMoivre-Laplace theorem from the central limit theorem.

2.2 Stable Laws

Definition 2.2.1 (slowly varying functions). A positive (Lebesgue) measurable function $L(t)$ is said to be a slowly varying function(svf) as $t \to \infty$ if, for any fixed $v > 0$,

$$\frac{L(vt)}{L(t)} \to 1 \quad \text{as} \quad t \to \infty. \tag{2.1}$$

Definition 2.2.2 (regularly varying functions). A function $V(t)$ is said to be a regularly varying function (of index $-\alpha \in \mathbb{R}$) function(rvf) as $t \to \infty$ if it can be represented as

$$V(t) = t^{-\alpha} L(t), \tag{2.2}$$

where $L(t)$ is an svf as $t \to \infty$.

The definition of an s.v.f(r.v.f) as $t \downarrow 0$ is quite similar. In what follows, the term s.v.f(r.v.f) will always refer, unless otherwise stipulated, a function which is slowly(regularly) varying at infinity.

One can easily see that, similarly to (2.1), the convergence

$$\frac{V(vt)}{V(t)} \to v^{-\alpha} \quad as \quad t \to \infty. \tag{2.3}$$

for any fixed $v > 0$ is a characteristic property of regularly varying functions. Thus, and s.v.f of index 0.

Law of Large Numbers

Let $\xi, \xi_1, \xi_2, ...$ be independent identically distributed (i.i.d) random variables. Put $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} \xi_i, \quad n = 1, 2, ...$$

The following assertions constitute the fundamental classical limit theorems for random walks,$S_n; n \geq 1$.

The strong law of large numbers states that, if there exists a finite expectations $\mathbf{E}\xi$, then as $n \to \infty$,

$$\frac{S_n}{n} \to \mathbf{E}\xi \quad \text{almost surely (a.s)} \tag{2.4}$$

One could call the value $n\mathbf{E}\xi$ the first-order approximation to the sum S_n.

The central limit theorem states that if $\mathbf{E}\xi^2 < \infty$ then. as $n \to \infty$,

$$\zeta_n = \frac{S_n - n\mathbf{E}\xi}{\sqrt{nd}} \Rightarrow \zeta \in \Phi, \tag{2.5}$$

where $d = \text{Var}\xi = \mathbf{E}\xi^2 - (\mathbf{E}\xi)^2$ is the variance of the r.v. ξ, the symbol $\Rightarrow$ denotes weak convergence of the r.v. in distribution and the notation $\zeta \in \Phi$ says that the r.v. ζ has the distribution Φ which is statndard normal, parameters (0,1). $n\mathbf{E}\xi + \zeta\sqrt{nd}$ can be considered the second-order approximation of S_n.

Since the relation Φ is continuous, the relation (2.5) is equivalent to the following one: for any $v \in \mathbb{R}$ we have

$$\mathbf{P}(\zeta_n \geq v) \to \mathbf{P}(\zeta \geq v) \quad \text{as} \quad n \to \infty,$$

and, moreover, this convergence is uniform in v. In other words, for deviations of the form $x = n\mathbf{E}\xi + v\sqrt{nd}$,

$$\mathbf{P}(S_n \geq x) \backsim \mathbf{P}\left(\zeta \geq \frac{x - n\mathbf{E}\xi}{\sqrt{nd}}\right) = 1 - \Phi(v) \quad \text{as} \quad n \to \infty \tag{2.6}$$

uniformly in $v \in [v_1, v_2]$ where $-\infty < v_1 \leq v_2 < \infty$ are fixed numbers and Φ is the statndard normal distribution function.

Convergence to stable laws.

If the expectation of the r.v. ξ is infinite or does not exist, then the first-order approximation for the sum S_n can only be found when the sum of the right and left tails of the distribution of ξ, that is, the function

$$F(t) = \mathbf{P}(\zeta \geq t) + \mathbf{P}(\zeta < -t), \quad t > 0,$$

is regularly varying as $t \to \infty$; it can be represented as

$$F(t) = t^{-\alpha} L(t), \tag{2.7}$$

where $\alpha \in (0, 1]$ and $L(t)$ is a slowly varying function(s.v.f) as $t \to \infty$. The same can be said about the second-approximation for S_n in the case when $\mathbf{E}|\xi| < \infty$ but $\mathbf{E}\xi^2 = \infty$. In this case, we have $\alpha \in [1, 2]$ in (2.7).

For these two cases, we have the following assertion. For simplicity, assume that $\alpha < 2$, $\alpha \neq 1$; we also assume that $\mathrm{E}\xi = 0$ when expectation is finite. We exclude $\alpha = 1$ to avoid the necessity of non-trivial centring of sums S_n when $\mathrm{E}\xi = \pm\infty$ or expectations does exist.

Let $F_+(t) = \mathbf{P}(\xi \geq t)$, let (2.7) hold and let there exist the limit

$$\lim_{t \to \infty} \frac{F_+(t)}{F(t)} = \rho_+ \in [0, 1]$$

Denote by

$$F^{-1}(x) = \inf\{t > 0 : F(t) \leq x\}, \quad x > 0,$$

the (generalized) inverse function for F, and put

$$b(n) = F^{-1}\left(\frac{1}{n}\right) = n^{\frac{1}{\alpha}} L_1(n),$$

where L_1 is also and s.v.f. Then, as $n \to \infty$,

$$\frac{S_n}{b(n)} \Rightarrow \zeta^{(\alpha,\rho)} \in \mathbf{F}_{\alpha,\rho}, \tag{2.8}$$

where $\mathbf{F}_{\alpha,\rho}$ is the standard stable law with parameters α and $\rho = 2\rho_+ - 1$.

We now state some useful general properties of s.v.f(r.v.f). The proof of these properties and related theorems can be found on [29].

Theorem 2.2.1 (Uniform convergence theorem). If $L(t)$ is an s.v.f as $t \to \infty$, then the convergence of (2.1) holds uniformly in v on any interval $[v_1, v_2]$ with $0 < v_1 < v_2 < \infty$.

It follows from the assertion of the theorem that the uniform convergence (2.1) on an interval $[\frac{1}{M}, M]$ will also take place in the case, when as $t \to \infty$, the quantity $M = M(t)$ increases to infinity slowly enough.

Theorem 2.2.2 (Integral Representation). A positive function $L(t)$ is an s.v.f as $t \to \infty$ iff for some $t_0 > 0$ one has

$$L(t) = c(t) exp\left(\int_{t_0}^{t} \frac{\varepsilon(u)}{u} \mathrm{d}u\right), \quad t \geq t_0, \tag{2.9}$$

where $c(t)$ and $\varepsilon(t)$ are measurable functions, with $c(t) \to c \in \mathbb{R}^+$ and $\varepsilon(t) \to 0$ as $t \to \infty$.

2.3 Asymptotic properties

Theorem 2.3.1. i) If L_1 and L_2 are s.v.f's then $L_1 + L_2$, $L_1 L_2$, L_1^b and $L(t) = L_1(at + b)$, where $a \geq 0$ and $b \in \mathbb{R}$ are also s.v.f's.

ii) If L is an s.v.f then for any $\delta > 0$ there exists a $t_\delta > 0$ such that

$$t^{-\delta} \leq L(t) \leq t^{\delta} \quad \text{for all} \quad t \geq t_\delta, \tag{2.10}$$

In other words, $L(t) = t^{o(1)}$ as $t \to \infty$

iii) If L is an s.v.f then for any $\delta > 0$ and $v_0 > 1$ there exists a $t_\delta > 0$ such that for all $v \geq v_0$ and $t \geq t_\delta$,

$$v^{-\delta} \leq \frac{L(vt)}{L(t)} \leq v^{\delta}, \tag{2.11}$$

iv) (Karamata's theorem) If $\alpha > 1$ then, for the r.v.f V in (2.3), one has

$$V^I(t) = \int_t^\infty V(u)\mathrm{d}u \sim \frac{tV(t)}{\alpha - 1} \quad \text{as} \quad t \to \infty. \tag{2.12}$$

If $\alpha < 1$ then

$$V_I(t) = \int_0^t V(u)\mathrm{d}u \sim \frac{tV(t)}{1 - \alpha} \quad \text{as} \quad t \to \infty. \tag{2.13}$$

If $\alpha = 1$ then one has the equalities

$$V_I(t) = tV(t)L_1(t). \tag{2.14}$$

and

$$V^I(t) = tV(t)L_2(t) \quad \text{if} \quad \int_0^\infty V(u)\mathrm{d}u < \infty, \tag{2.15}$$

where the $L_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2$ are s.v.f's.

v) For an r.v.f V of index $-\alpha < 0$ put

$$\sigma(t) = V^{-1}(1/t) = \inf\{u : V(u) < 1/t\}$$

then $\sigma(t)$ is an r.v.f of index $1/\alpha$:

$$\sigma(t) = t^{1/\alpha} L_1(t),$$

where L_1 is an s.v.f. If the function L has property

$$L(tL^{1/\alpha}(t)) \sim L(t),$$

as $t \to \infty$ then

$$L_1(t) \sim L^{1/\alpha}\left(t^{1/\alpha}\right).$$

Similar assertion hold for functions that are slowly or regularly varying as $t \downarrow 0$.

Observe that Theorem 1.1 and the inequality (2.11) we also obtain the following property of s.v.f's: for any $\delta > 0$ there exists a $t_\delta > 0$ such that for all t and v satisfying the inequalities $t \geq t_\delta$, $vt \geq t_\delta$ one has

$$(1-\delta)\min\{v^\delta, v^{-\delta}\} \leq \frac{L(vt)}{L(t)} \leq (1+\delta)\max\{v^\delta, v^{-\delta}\}, \tag{2.16}$$

2.4 The convergence of distribution of sums of random variables with regularly varying tails to stable laws.

As is known, in case $\mathbf{E}\xi^2 < \infty$ one has the central limit theorem, which states that the distribution of the norrmalized sums $S_n = \sum_{i=1}^n \xi_i$ of independent r.v's $\xi_i = \xi$converge to the normal law as $n \to \infty$.

If $\mathbf{E}\xi^2 = \infty$ then the situation noticeably changes. In this case, the convergence of the distribution of appropriately normalized sums S_n to a limiting law will only take place for r.v's with regularly varying distribution tails.

From the proof of central limit theorem by the method of characteristic functions (ch.f.), it is seen that the nature of the limiting distribution for S_n is defined the behaviour of the ch.f.

$$f(\lambda) = \mathbf{E}e^{i\lambda\xi}, \qquad \lambda \in \mathbb{R}$$

of ξ in the vicinity of zero. If $\mathbf{E}\xi^2 = 0$ and $\mathbf{E}\xi^2 = d < \infty$ then,as $n \to \infty$,

$$f\left(\frac{\mu}{\sqrt{n}}\right) = 1 + \frac{f'(0)\mu}{\sqrt{n}} + \frac{f''(0)\mu^2}{2n} + o\left(\frac{1}{n}\right) = 1 - \frac{d\mu^2}{2n} + o\left(\frac{1}{n}\right), \tag{2.17}$$

It is the relation that defines the asymptotic behaviour of the ch.f. $f^n\left(\frac{\mu}{\sqrt{n}}\right)$ of $\frac{S_n}{\sqrt{n}}$, which leads to the limiting normal law. In case $\mathbf{E}\xi^2=\infty$ (so that $f''(0)$ does not exist) we will use the same method, but, in order to obtain the 'right' asymptotic of $f\left(\frac{\mu}{b(n)}\right)$ under a suitable scaling $b(n)$, we will have to impose regular variation conditions on the 'two-sided' tails

$$F(t)=\mathbf{F}((-\infty,-t))+\mathbf{F}([t,\infty))=\mathbf{P}(\xi\notin[-t,t)),\quad t>0.$$

As before, the functions

$$F_+(t)=\mathbf{F}([t,\infty))=\mathbf{P}(\xi\geq t),\quad F_-(t)=\mathbf{F}((-\infty,-t))=\mathbf{P}(\xi<-t)$$

will be referred to as the right and the left tails of the distribution of ξ,respectively.

Assume that the following condition holds for some $\alpha\in(0,2]$ and $\rho\in[-1,1]$:

$[\mathbf{R}_{\alpha,\rho}]$ The two-sided tail $F(t)=F_-(t)+F_+(t)$ is an r.v.f. at infinity, i.e. it has representation of the form

$$F(t)=t^{-\alpha}L_F(t),\quad \alpha\in(0,2], \tag{2.18}$$

where $L_F(t)$ is an s.v.f; in addition there exists the limit

$$\lim_{t\to\infty}\frac{F_+(t)}{F(t)}=\rho_+=\frac{1}{2}(\rho+1)\in[0,1]. \tag{2.19}$$

If $\rho_+>0$ then clearly the right tail $F_+(t)$ admits a representation of the form

$$F_+(t)=V(t)=t^{-\alpha}L(t),\quad \alpha\in(0,2],\quad L(t)\sim\rho_+L_F(t)$$

If $\rho_+=0$ then the right tail $F_+(t)=o(F(t))$ need not be regularly.

It follows from (2.19) that there also exists the limit

$$\lim_{t\to\infty} \frac{F_+(t)}{F(t)} = \rho_- = 1 - \rho_+.$$

If $\rho_>0$ the similarly, the left tail $F_-(t)$ admits a representation of the form

$$F_-(t) = W(t) = t^{-\alpha} L_W(t), \quad \alpha \in (0,2], \quad L_W(t) \sim \rho_- L_F(t)$$

If $\rho_- = 0$ then the left tail $F_-(t) = o(F(t))$ is not assumed to be regularly varying.

The parameters $\rho_\pm$ are connected to the parameters ρ from conditions $[\mathbf{R}_{\alpha,\rho}]$ by the relations

$$\rho = \rho_+ - \rho_- = 2\rho_+ - 1.$$

Evidently, for $\rho < 2$ one has $\mathbf{E}\xi^2 = \infty$, so that the representation (2.17) ceases to hold,and the central limit theorem is inapplicable. In what follows in situation where $\mathbf{E}\xi$ exists and is finite we will always assume, without loss of generality that,

$$\mathbf{E}\xi = 0.$$

Since $F(t)$ in non-increasing, the (generalized) inverse function $F^{-1}(u)$, understood as

$$F^{-1}(u) = \inf\{t > 0 : F(t) < u\},$$

always exists. If $F(t)$ is strictly monotone and continuous then $b = F^{-1}(u)$ is the unique solution of the equation

$$F(b) = u, \qquad u \in (0,1).$$

Put

$$\zeta_n = \frac{S_n}{b(n)},$$

where the scaling factor $b(n)$ is defined in the case $\alpha < 2$ by

$$b(n) = F^{-1}\left(\frac{1}{n}\right) \tag{2.20}$$

It is obvious that in the case $\rho_+ > 0$ the scaling factor $b(n)$ is connected to the function $\sigma(n) = V^{-1}(1/n)$.

For $\alpha = 2$ we put

$$b(n) = Y^{-1}(1/n), \tag{2.21}$$

where

$$\begin{aligned} Y(t) &= 2t^{-2}\int_0^t yF(y)\mathrm{dy} \\ &= 2t^{-2}\left(\int_0^t yV(y)\mathrm{dy} + \int_0^t yW(y)\mathrm{dy}\right) \\ &\sim t^{-2}\mathbf{E}[\xi^2; -t \le \xi \le] = t^{-2}L_Y(t) \end{aligned} \tag{2.22}$$

and L_Y is and s.v.f(See Theorem 2.3.1 iv)). From Theorem 2.3.1 v) it follows also that if (2.18) holds then

$$b(n) = n^{1/\alpha}L_b(n), \quad \alpha \le 2,$$

where L_b is an s.v.f.

Theorem 2.4.1. Let condition $[\mathbf{R}_{\alpha,\rho}]$ be satisfied. Then the following assertions hold true.

i) For $\alpha \in (0, 2)$, $\alpha \neq 1$, and the scaling factor (2.20), we have

$$\zeta_n \Rightarrow \zeta^{(\alpha,\rho)} \quad \text{as} \quad n \to \infty$$

where the distribution $[\mathbf{F}_{\alpha,\rho}]$ of all r.v. $\zeta^{(\alpha,\rho)}$ depends only on the parameters α

and ρ and has a ch.f. $f^{(\alpha,\rho)}(\lambda)$ given by

$$f^{(\alpha,\rho)}(\lambda) = \mathbf{E}e^{i\lambda\zeta^{(\alpha,\rho)}} = \exp\{|\lambda|^{\alpha}B(\alpha,\rho,\phi)\}$$

where $\phi = \mathrm{sign}\lambda$,

$$B(\alpha,\rho,\phi) = \Gamma(1-\alpha)\left(i\rho\phi\sin\frac{\alpha\pi}{2} - \cos\frac{\alpha\pi}{2}\right)$$

and for $\alpha \in (1,2)$ we put $\Gamma(1-\alpha) = \frac{\Gamma(2-\alpha)}{1-\alpha}$.

ii) When $\alpha = 1$, for the sequence ζ_n with scaling factor (2.20) to converge to a limiting law the former, generally speeking, needs to be centered. More precisely, we have

$$\zeta_n - A_n \Rightarrow \zeta^{(1,\rho)} \quad \text{as} \quad n \to \infty,$$

where

$$A_n = \frac{n}{b(n)}[V_I(b(n)) - W_I(b(n))] - \rho C,$$

$C \approx 0.5772$ is the Euler constant and

$$f^{(1,\rho)}(\lambda) = \mathrm{E}e^{i\lambda\zeta^{(\alpha,\rho)}} = \exp\left(-\frac{\pi|\lambda|}{2} - i\rho\lambda\ln|\lambda|\right).$$

If $n[V_I(b(n)) - W_I(b(n))] = o(b(n))$, then $\rho = 0$ and one can put $A_n = 0$.

If $\mathbf{E}\xi = 0$, then

$$A_n = \frac{n}{b(n)}[V^I(b(n)) - W^I(b(n))] - \rho C.$$

If $\mathbf{E}\xi = 0$, $\rho \neq 0$, then $\rho A_n \to -\infty$ as $n \to \infty$.

iii) For $\alpha = 2$ and scaling factor (2.21),

$$\zeta_n \Rightarrow \zeta^{(2,\rho)} = \zeta \quad \text{as} \quad n \to \infty, \qquad f^{(2,\rho)}(\lambda) = \mathrm{E}e^{i\lambda\zeta} = e^{-\lambda^2/2},$$

so that ζ has the standard normal distribution that is independent of ρ.

Remark 1

We can easily verify that in extreme cases $\rho = \pm 1$ the ch.f's $B(\alpha, \rho, \phi)$, $f^{(1,\rho)}(\lambda)$ (defined above) of stable distributions with $\alpha < 2$ admit the following simpler representations:

$$f^{(1,\rho)}(\lambda) = \exp\{-\Gamma(1-\alpha)(-i\lambda)^{\alpha}\}, \quad \alpha \in (0,2), \quad \alpha \neq 1,$$

$$f^{(1,1)}(\lambda) = \exp\{(-i\lambda)\ln(-i\lambda)\}; \quad f^{(\alpha,-1)}(\lambda) = f^{(\alpha,1)}(-\lambda), \quad \alpha \leq 2.$$

Remark 2

From representation of A_n (above) for the centring sequence $\{A_n\}$ in the $\alpha = 1$ it follows that if there exists $\mathbf{E}\xi = 0$ then the boundedness of the sequence implies that $\rho = 0$. The converse assertion, that in case $\mathbf{E}\xi = 0$ the relation $\rho = 0$ implies the boundedness of $\{A_n\}$, is false.

Indeed, let ξ be an r.v. with $\mathbf{E}\xi = 0$ such that for $t \geq t_0 > 0$ one has

$$V(t) = \frac{1}{2t\ln^2 t}, \quad W(t) = V(t)\left[1 + \frac{1}{L_2(t)}\right], \quad L_2(t) = \ln\ln t.$$

Then $\rho = 0, \quad F(t) \sim t^{-1}\ln^{-2} t, \quad b(n) \sim n\ln^{-2} n \quad$ and

$$V^I(t) = \frac{1}{2\ln t}, \quad W^I(t) = V^I(t) + \frac{1+o(1)}{L_2(t)\ln(t)},$$

so that

$$W^I(t) - V^I(t) \sim \frac{1}{L_2(t)\ln t}.$$

Therefore

$$A_n = \frac{(1+o(t))\ln^2 n}{L_2(b(n))\ln b(n)} - \rho C \sim \frac{\ln n}{\ln\ln n} \to \infty \quad \text{as} \quad n \to \infty.$$

Remark 3

If $\alpha < 2$ then from the properties of s.v.f (Theorem (2.3.1) iv) we have that, as $n \to \infty$,

$$\int_0^t yF(y)dy = \int_0^t y^{1-\alpha}L_F(y)dy \sim \frac{1}{2-\alpha}t^{2-\alpha}L_F(t) = \frac{1}{2-\alpha}t^2F(t).$$

Hence for $\alpha < 2$ one has $Y(t) \sim 2(2-\alpha)^{-1}F(x)$,

$$Y^{-1}\left(\frac{1}{n}\right) \sim F^{-1}\left(\frac{2-\alpha}{2n}\right) \sim \left(\frac{2}{2-\alpha}\right)^{1/\alpha} F^{-1}\left(\frac{1}{n}\right)$$

However, when $\alpha = 2$ and $d = \mathbf{E}\xi < \infty$, we have

$$Y(t) \sim t^{-2}d, \quad b(n) = Y^{-1}\left(\frac{1}{n}\right) \sim \sqrt{nd}.$$

Thus, scaling (2.21) is 'transitional' between the scaling of (2.20) (up to the constant factor $2/(2-\alpha)^{1/\alpha}$) and the standard scaling $\sqrt{nd}$ in the central limit theorem in the case $\mathbf{E}\xi^2 < \infty$. This also means that the scaling (2.21) is 'universal' and can be used for all $\alpha \leq 2$. However, for $\alpha < 2$ the scaling (2.20) is simpler and easier to deal with, and this why it will be used the present exposition.

The proof of Theorem (2.4.1) essentially uses the form of the scaling sequence $b(n)$ and thereby helps to establish direct connection between the zones of 'normal' distribution and large deviations. This proof can be found in [29].

Recall that $\mathbf{F}_{\alpha,\rho}$ denotes the distribution of $\zeta^{(\alpha,\rho)}$. The parameter α assumes values from the half-interval $(0, 2]$ and the parameter $\rho = \rho_+ - \rho_-$ can assume any value from the closed interval $[-1, 1]$.

It follows from Theorem (2.4.1)that each $\mathbf{F}_{\alpha,\rho}, \quad 0 < \alpha \leq 2, \quad -1 \leq \rho \leq 1$ is limiting for distributions of suitably normalized sums of i.i.d. r.v's. The law of large numbers implies the the degenerate distribution $\mathbf{I}_a$ concentrated at some point a is also a limiting one. The totality of all these distributions will be denoted by $\mathfrak{G}_o$. Further, it is not hard to see that $\mathbf{F} \in \mathfrak{G}_o$ then the distribution obtained from $\mathbf{F}$ by scale and shift transformation, that is a distribution $\mathbf{F}_{\{a,b\}}$ given, for some fixed $b > 0$ and a, by the relation

$$\mathbf{F}_{\{a,b\}}(B) = \mathbf{F}\left(\frac{B-a}{b}\right), \quad \text{where} \quad \left(\frac{B-a}{b}\right) = \{u \in \mathbb{R} : ub + a \in B\},$$

is also limiting(for the distribution of $(S_n - a_n)/b_n$ as $n \to \infty$, with suitable $\{a_n\}$ and $\{b_n\}$).

Let $\xi, \xi_1, \xi_2, ...$ be independent identically distributed (i.i.d) random variables. Put $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} \xi_i, \quad n = 1, 2, ...$$

The following assertions constitute the fundamental classical limit theorems for random walks,$S_n; n \geq 1$.

Chapter 3. Infinitesimal Matrix

3.1 Markov Processes with a finite state Space

3.1.1 Markov Chains

Let Ω be the space of sequences $(\omega_0, ..., \omega_n)$, where $\omega_k \in X = \{x^1, ..., x^r\}$, $0 \leq k \leq n$. Without loss of generality, we may identify X with the set of the first r integers, $X = \{1, ..., r\}$.

Let P be a probability measure on Ω. Sometimes we shall denote by ω_k the random variable which assigns the value of the k^{th} element to the sequence $\omega = (\omega_0, ..., \omega_n)$. It is usually clear from the context whether ω_k stands for such a random variable or simply the k^{th} element of a particular sequence. We shall denote the probability of the sequence $(\omega_0, ..., \omega_n)$ by $p(\omega_0, ..., \omega_n)$. Thus

$$p(i_0, ..., i_n) = P(\omega_0 = i_0, ..., \omega_n = i_n).$$

Assume that we are given a probability distribution $\mu = (\mu_1, ..., \mu_r)$ on X and a stochastic matrices $P(1), ..., P(n)$ with $P(k) = (p_{ij}(k))$.

Definition 3.1.1. The Markov chain with the state space X generated by the initial distribution μ on X and the stochastic matrices $P(1), ..., P(n)$ is the probability measure P on Ω such that

$$P(\omega_0 = i_0, ..., \omega_n = i_n) = \mu_{i_0}.p_{i_0 i_1}(1)...p_{i_{n-1} i_n}(n) \tag{3.1}$$

for each $i_0, ..., i_n \in X$.

The elements of X are called the states of the Markov chain. Let us check that (3.1) defines a probability measure on Ω. The inequality $P(\omega_0 = i_0, ..., \omega_n = i_n) \geq$

0 is clear. It remains to show that

$$\sum_{i_0=1}^{r} ... \sum_{i_n=1}^{r} P(\omega_0 = i_0, ..., \omega_n = i_n) = 1.$$

We have

$$\sum_{i_0=1}^{r} ... \sum_{i_n=1}^{r} P(\omega_0 = i_0, ..., \omega_n = i_n)$$
$$= \sum_{i_0=1}^{r} ... \sum_{i_n=1}^{r} \mu_{i_0}.p_{i_0 i_1}(1)...p_{i_{n-1} i_n}(n).$$

We now perform the summation over all values of i_n. Note that i_n is only present in the last factor in each term of the sum, and the sum $\sum_{i_n=1}^{r} p_{i_{n-1}i_n}(n)$ is equal to one, since the matrix $P(n)$ is stochastic. We then fix $i_0, ..., i_{n-2}$, and sum over all the values of i_{n-1}, and so on. In the end we obtain $\sum_{i_0=1}^{r} \mu_{i_0}$, which is equal to one, since μ is a probability distribution.

In the same way one can prove the following statement:

$$P(\omega_0 = i_0, ..., \omega_n = i_k) = \mu_{i_0}.p_{i_0 i_1}(1)...p_{i_{k-1} i_k}(k)$$

for any $1 \leq i_0, ..., i_k \leq r, k \leq n$. This equality shows that the induced probability distribution on the space of sequences of the form $(\omega_0, ..., \omega_k)$ is also a Markov chain generated by the initial distributionμ and the stochastic matrices $P(1), ..., P(k)$.

The matrices $P(k)$ are called the transition probability matrices, and the matrix entry $p_{ij}(k)$ is called the transition probability from the state i to the j at time k. The use of the of these terms is justified by the following calculation.

Assuming that $P(\omega_0 = i_0, ..., \omega_{k-2} = i_{k-2}, \omega_{k-1} = i) > 0$. We consider the conditional probability $P(\omega_k = j | \omega_0 = i_0, ..., \omega_{k-2} = i_{k-2}, \omega_{k-1} = i)$. By definition of the measure P,

$$
\begin{aligned}
P(\omega_k = j|\omega_0 = i_0, ...,\\
&\omega_{k-2} = i_{k-2}, \omega_{k-1} = i)\\
&= \frac{P(\omega_0 = i_0, ..., \omega_{k-2} = i_{k-2}, \omega_{k-1} = i, \omega_k = j)}{P(\omega_0 = i_0, ..., \omega_{k-2} = i_{k-2}, \omega_{k-1} = i)}\\
&= \frac{\mu_{i_0} p_{i_0 i_1}(1)...p_{i_{k-2} i}(k-1).p_{ij}(k)}{\mu_{i_0} p_{i_0 i_1}(1)...p_{i_{k-2} i}(k-1)}\\
&= p_{ij}(k).
\end{aligned}
$$

The right-hand side here does not depend on $i_0, ..., i_{k-2}$. This property is sometimes used as a definition of a chain, It is also easy to see that $P(\omega_k = j|\omega_{k-1} = i) = p_{ij}(k)$.

Definition 3.1.2. A Markov chain is said to be homogeneous if $P(k) = P$ for a matrix P which does not depend on, k, $1 \leq k \leq n$.

The notion of a homogeneous Markov chain can be understood as a generalization of the notion of a sequence of independent identical trials. Indeed, it all the rows of the stochastic matrix $P = (p_{ij})$ are equally to $(p_1, ..., p_r)$, where $(p_1, ..., p_r)$ is a probability distribution on X, then the Markov with such a matrix P and the initial distribution $(p_1, ..., p_r)$ is a sequence of independent identical trials.

In what follows we consider only homogeneous Markov Chains. Such chains can be represented with the help of graphs. The vertices of the graph are the elements of X. The vertices i and j are connected by an oriented edge if $p_{ij} > 0$. A sequence of states $(i_0, i_1, ..., i_n)$ which has a positive probability can be represented as a path of length n on the graph starting at the point i_0, then going to the point i_1, and so on. Therefore, homogeneous Markov chain can be represented as a probability distribution on the space of paths of length n on the graph.

Let us consider the conditional probabilities $P(\omega_{s+l} = j|\omega_l = i)$. It is assumed here that $P(\omega_l = i) > 0$. We claim that

$$P(\omega_{s+l} = j|\omega_l = i) = p_{ij}^{(s)},$$

where $p_{ij}^{(s)}$ are elements of the matrix P^s. Indeed,

$$P(\omega_{s+l} = j|\omega_l = i) = \frac{P(\omega_{s+l} = j, \omega_l = i)}{P(\omega_l = i)}$$

$$= \frac{\sum_{i_0=1}^{r} \dots \sum_{i_{l-1}=1}^{r} \sum_{i_{l+1}=1}^{r} \dots \sum_{i_{s+l-1}=1}^{r} P(\omega_0 = i_0, \dots, \omega_l = i, \dots, \omega_{s+l} = j)}{\sum_{i_0=1}^{r} \dots \sum_{i_{l-1}=1}^{r} P(\omega_0 = i_0, \dots, \omega_l = i)}$$

$$= \frac{\sum_{i_0=1}^{r} \dots \sum_{i_{l-1}=1}^{r} \sum_{i_{l+1}=1}^{r} \dots \sum_{i_{s+l-1}=1}^{r} \mu_{i_0} p_{i_0 i_1} \dots p_{i_{l-1} i} p_{i i_{l+1}} \dots p_{i_{s+l-1} j}}{\sum_{i_0=1}^{r} \dots \sum_{i_{l-1}=1}^{r} \mu_{i_0} p_{i_0 i_1} \dots p_{i_{l-1} i}}$$

$$= \frac{\sum_{i_0=1}^{r} \dots \sum_{i_{l-1}=1}^{r} \mu_{i_0} p_{i_0 i_1} \dots p_{i_{l-1} i} \sum_{i_{l+1}=1}^{r} \dots \sum_{i_{s+l-1}=1}^{r} \mu_{i_0} p_{i_0 i_1} \dots p_{i_{l-1} i} p_{i i_{l+1}} \dots p_{i_{s+l-1} j}}{\sum_{i_0=1}^{r} \dots \sum_{i_{l-1}=1}^{r} \mu_{i_0} p_{i_0 i_1} \dots p_{i_{l-1} i}}$$

$$\sum_{i_{l+1}=1}^{r} \dots \sum_{i_{s+l-1}=1}^{r} p_{i i_{l+1}} \dots p_{i_{s+l-1} j} = p_{ij}^{(s)}$$

Thus the conditional probabilities $p_{ij}^{(s)} = P(\omega_{s+l} = j|\omega_l = i)$ do not depend on l. They are called s-step transition probabilities. A similar calculation shows that for a homogeneous Markov chain with initial distribution μ,

$$P(\omega_s = j) = (\mu P^s)_j = \sum_{i=1}^{r} \mu_i p_{ij}^{(s)}. \tag{3.2}$$

Note that by considering infinite stochastic matrices, Definition 3.1.2 and the argument leading to (3.2) can be generalized to the case of Markov chains with a countable number of states.

Definition 3.1.3. A stochastic matrix P is said to be ergodic if there exists s such that the s-step transition probabilities $p_{ij}^{(s)}$ are positive for all i and j. A homogeneous Markov Chain is said to be ergodic if it can be generated by some initial distribution and an ergodic stochastic matrix.

3.1.2 Definition of a Markov Process

Here we define a homogeneous Markov process with values in a finite state space. We can assume that the state space X is the set the first r positive integers, that is $X = \{1, 2, ...r\}$.

Let $P(t)$ be a family of $r \times r$ stochastic matrices indexed by the parameter $t \in [0, \infty)$. The element of $P(t)$ will be denoted by $P_{ij}(t), \quad 1 \leq i, j \leq r$. We assume that the family $P(t)$ forms a semi-group, that is $P(s)P(t) = P(s+t)$ for any $s, t \geq 0$. Since $P(t)$ are stochastic matrices, the semi-group property implies $P(0)$ is the identity matrix. Let μ be a distribution X.

Let $\tilde{\Omega}$ be the set of all functions $\tilde{\omega} : \mathbb{R}^+ \to X$ and $\mathbb{B}$ be the $\sigma-$algebra generated by all cylindrical sets. Define a family of finite-dimensional distributions $P_{t_o, t_1, ..., t_k}$. where $0 = t_o \leq t_1 \leq ... \leq t_k$, as follows

$$\begin{aligned} &P_{t_o, t_1, ..., t_k}(\tilde{\omega}(t_0) = i_0, \tilde{\omega}(t_1) = i_1, ..., \tilde{\omega}(t_k) = i_k) \\ &= \mu_{i_0} P_{i_0 i_1}(t_1) P_{i_1 i_2}(t_2 - t_1) ... P_{i_{k-1} i_k}(t_k - t_{k-1}). \end{aligned}$$

It can easily be seen that this family of finite-dimensional distribution satisfies the consistency conditions. By the Kolmogorov Consistency Theorem, there is a process X_t with values in X with these finite-dimensional distribution. Any such process will be called a homogeneous Markov process with the family of transition matrices $P(t)$ and the initial distribution μ. (Since we donot consider non-homogeneous Markov process in this section, we shall refer to X_t simply as a Markov process.)

Lemma 1. Let X_t be a Markov process with the family of transition matrices $P(t)$. Then, for $0 \leq s_1 \leq ... \leq s_k, \quad t \geq 0$, and $i_1, i_2, ...i_k, j \in X$, we have

$$P(X_{s_k+t} = j | X_{s_1} = i_1, ..., X_{s_k} = i_k) = P(X_{s_k+t} = j | X_{s_k} = i_k) = P_{i_k j}(t) \qquad (3.3)$$

if the conditional probability on the left-hand side is defined.

Proof. Assume $P(X_{s_1} = i_1, ..., X_{s_k} = i_k) > 0$. The conditional probability

$$
\begin{aligned}
P(X_{s_k+t} = j | X_{s_1} = i_1, ..., X_{s_k} = i_k) \\
&= \frac{P(X_{s_1} = i_1, ..., X_{s_{k-1}} = i_{k-1}, X_{s_k} = i_k, X_{s_k+t} = j)}{P(X_{s_1} = i_1, ..., X_{s_{k-1}} = i_{k-1}, X_{s_k} = i_k)} \\
&= \frac{\mu_{i_1} P_{i_1 i_2}(2) ... P_{i_{k-1} i_k}(k-1) P_{i_k j}(t)}{\mu_{i_1} P_{i_1 i_2}(2) ... P_{i_{k-1} i_k}(k-1)} \\
&= P_{i_k j}(t)
\end{aligned}
$$

□

Definition 3.1.4. A distribution π is said to be stationary for a semi-group of Markov transition matrices $P(t)$ if $\pi P(t) = \pi$ for all $t \geq 0$.

Theorem 3.1.1. Let $P(t)$ be a semi-group of Markow Transition matrices such that for some t all the matrix entries of $P(t)$ are positive. Then there is a unique stationary distribution π for the semi-group of transition matrices. Moreover, $\sup_{i,j \in X} |P_{i,j}(t) - \pi_j|$ converges to zero exponentially fast as $t \in \infty$.

Proof. For the sake of transparency we'll prove the theorem in the case of discrete time. Let $\mu' = (\mu'_1, ..., \mu'_r)$, $\mu'' = (\mu''_1, ..., \mu''_r)$ be two probability distributions on the space X. We set $d(\mu', \mu'') = \frac{1}{2} \sum_{i=1}^{r} |\mu'_i - \mu''_i|$. Then d can be viewed as a distance on the space of probability distribution on X, and the space of distributions with this distance is a complete metric space. We note that

$$0 = \sum_{i=1}^{r} \mu'_i - \sum_{i=1}^{r} \mu''_i = \sum_{i=1}^{r} (\mu'_i - \mu''_i) = \sum^{+} (\mu'_i - \mu''_i) - \sum^{+} (\mu''_i - \mu'_i),$$

where $\sum^{+}$ denotes the summation with respect to those indices i for which the terms are positive. Therefore,

$$d(\mu',\mu'')=\frac{1}{2}\sum_{i=1}^{r}|\mu'_1-\mu''_i|=\frac{1}{2}\sum^{+}(\mu'_1-\mu''_i)+\frac{1}{2}\sum^{+}(\mu''_i-\mu'_i)=\sum^{+}(\mu'_1-\mu''_i).$$

It is clear that $d(\mu',\mu'')\leq 1$.

Let μ' and μ'' be two probability distributions on X and $Q=(q_{ij})$ a stochastic matrix. This implies $\mu'Q$ and $\mu''Q$ are also probability distributions. Let us demonstrate that

$$d(\mu'Q,\mu''Q)\leq d(\mu',\mu''), \tag{3.4}$$

for all $q_{ij}\geq\alpha$, then

$$d(\mu'Q,\mu''Q)\leq(1-\alpha)d(\mu',\mu''). \tag{3.5}$$

Let J be the set of indices j for which $(\mu'Q)_j-(\mu''Q)_j>0$. Then

$$\begin{aligned} d(\mu'Q,\mu''Q) &= \sum_{j\in J}(\mu'Q-\mu''Q)_j=\sum_{j\in J}\sum_{i=1}^{r}(\mu'-\mu'')q_{ij} \\ &\leq \sum_{i}^{+}(\mu'-\mu'')\sum_{j\in J}q_{ij}\leq\sum_{i}^{+}(\mu'-\mu'')=d(\mu',\mu''), \end{aligned}$$

which proves (3.4). We now note that J can not contain all indices of j since both $\mu'Q$ and $\mu''Q$ are probability distributions. Therefore, at least one index j is missing in the sum $\sum_{j\in J}q_{ij}$. Thus, if all $q_{ij}>\alpha$, then $\sum_{j\in J}q_{ij}<1-\alpha$ for all i, and

$$d(\mu'Q,\mu''Q)\leq(1-\alpha)\sum_{i}^{+}(\mu'_i-\mu''_i)=(1-\alpha)d(\mu',\mu''),$$

which implies (3.5).

Let μ_0 be an arbitrary probability distribution on X and $\mu_n=\mu_0P^n$. We shall show that the sequence of probability distribution μ_n is a Cauchy sequence, that is for $\epsilon > 0$ there exists $n_0(\epsilon)$ such that for any $k \geq 0$ we kave $d(\mu_n, \mu_{n+k}) < \epsilon$ for $n_0(\epsilon)$.By (3.4) and (3.5),

$$d(\mu_n, \mu_{n+k}) = d(\mu_0 P^n, \mu_0 P^{n+k}) \leq (1-\alpha) d(\mu_0 P^{n-s}, \mu_0 P^{n+k-s}) \leq ...$$
$$\leq (1-\alpha)^m d(\mu_0 P^{n-ms}, \mu_0 P^{n+k-ms}) \leq (1-\alpha)^m,$$

where m is such that $0 \leq n - ms < s$. For sufficiently large n we have $(1-\alpha)^m < \epsilon$, which implies that μ_n is a cauchy sequence.

Let $\pi = \lim_{n\to\infty} \mu_n$. Then

$$\pi P = \lim_{n\to\infty} \mu_n P = \lim_{n\to\infty} (\mu_0 P^n) P = \lim_{n\to\infty} (\mu_0 P^{n+1}) = \pi$$

We now show that the distribution π, such that $\pi P = \pi$, is unique. Let π_1 and π_2 be two distributions with $\pi_1 = \pi_1 P$ and $\pi_2 = \pi_2 P$. Then $\pi_1 = \pi_1 P^s$ and $\pi_2 = \pi_2 P^s$. Therefore, $d(\pi_1, \pi_2) = d(\pi_1 P^s, \pi_2 P^s) \leq (1-\alpha) d(\pi_1, \pi_2)$ by (3.4). If follows that $d(\pi_1, \pi_2) = 0$, that is $\pi_1 = \pi_2$.

We have proved that for any initial distribution μ_0 the limit

$$\lim_{n\to\infty} \mu_0 P^n = \pi$$

exists and does not depend on the choice of μ_0. Let us take μ_0 to be probability distribution which is concentrated at the point i. Then, for i fixed, $\mu_0 P^n$ is the probability distribution $\left(p_{ij}^{(n)}\right)$. Therefore, $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$. It is easy to show that $\pi_j > 0$ for $1 \leq j \leq r$.

$\square$

We now consider semi-groups of Markov transition matrices which are differentiable at zero. Namely, assume that there exists the following limits

$$Q_{i,j} = \lim_{t \downarrow o} \frac{P_{ij}(t) - I_{ij}}{t}, 1 \leq i, j \leq r \tag{3.6}$$

where I is the identity matrix.

Definition 3.1.5. If the limits in (3.6) exist for all $1 \leq i, j \leq r$, then the matrix Q is called the infinitesimal matrix of the semi-group $P(t)$.

Since $P_{ij}(t) \geq 0$ and $I_{ij} = 0$ for $i \neq j$, the off-diagonal elements of Q are non-negative. Moreover,

$$\sum_{j=1}^{r} Q_{ij} = \sum_{j=1}^{r} \lim_{t \downarrow 0} \frac{P_{ij}(t) - I_{ij}}{t} = \lim_{t \downarrow 0} \frac{\sum_{j=1}^{r} P_{ij}(t) - 1}{t} = 0$$

or equivalently,

$$Q_{ij} = -\sum_{i \neq j} Q_{ij}.$$

Lemma 2. If the limits in (3.6) exist, then the transition matrices are differentiable for $t \in \mathbb{R}^+$ and satisfy the following system of ordinary differential equations.

$$\frac{dP(t)}{dt} = P(t)Q \qquad \text{(forward system)}.$$

$$\frac{dP(t)}{dt} = QP(t) \qquad \text{(backward system)}.$$

The derivative at $t = 0$ should be understood as one-sided derivatives.

Proof. Due to the semi-group property of $P(t)$

$$\lim_{h \downarrow 0} \frac{P(t+h) - P(t)}{h} = P(t) \lim_{h \downarrow 0} \frac{P(h) - I}{h} = P(t)Q \tag{3.7}$$

This shows, in particular, that $P(t)$ is right-differentiable. Let us prove that $P(t)$

is left-continuous. For $t > 0$ and $0 \leq h < t$,

$$P(t) - P(t-h) = P(t-h)(P(h) - I).$$

All the elements of $P(t-h)$ are bounded, while all elements of $(P(h) - I) \to 0$ as $h \downarrow 0$. This establishes the continuity of $P(t)$.

For $t > 0$,

$$\lim_{h\downarrow 0} \frac{P(t) - P(t-h)}{h} = \lim_{h\downarrow 0} P(t-h) \lim_{h\downarrow 0} \frac{P(h) - I}{h} = P(t)Q \tag{3.8}$$

combining (3.7) and (3.8), we obtain the forward system of equations.

Due to the semi-group property of $P(t)$, for $t \geq 0$,

$$\lim_{h\downarrow 0} \frac{P(t+h) - P(t)}{h} = P(t) \lim_{h\downarrow 0} \frac{P(h) - I}{h} = P(t)Q$$

and similarly, for $t > 0$

$$\lim_{h\downarrow 0} \frac{P(t) - P(t-h)}{h} = \lim_{h\downarrow 0} P(t-h) \lim_{h\downarrow 0} \frac{P(h) - I}{h} = P(t)Q$$

This justifies the backward systems of equations. □

The systems $\frac{dP(t)}{dt} = P(t)Q$ with initial conditions $P_0 = I$ has the unique solution $P(t) = e^{tQ}$. Thus, the transition matrices can be uniquely expressed in terms of the infinitesimal matrix.

Let us note another property of the infinitesimal matrix. If π is a stationary distribution for the semi-group of transition matrices, then

$$\pi Q = \lim_{t\downarrow 0} \frac{\pi P(t) - \pi}{t} = 0.$$

Conversely, if $\pi Q = 0$ for some distribution π, then

$$\pi P(t) = \pi e^{tQ} = \pi(I + tQ + \frac{t^2Q^2}{2!} + \frac{t^3Q^3}{3!} + ...) = \pi$$

Thus, π is a stationary distribution for the family $P(t)$.

3.1.3 Construction of a Markov Process

Let μ ba a probability distribution on X and $P(t)$ be differentiable semi-group of transition matrices with the infinitesimal matrix Q. Assuming that $Q_{ii} < 0$ for all i.

On an intuitive level, a Markov process with the family of transition matrices $P(t)$ and initial distribution μ can be discribed as follows. At time $t = 0$ the process is distributed according to μ. If at time t the process is in a state i, then it will remain in the same state for time τ, where τ is a random variable with exponential distribution. The parameter of the distribution depends on i, but does not depend on t. After time τ the process goes to another state, where it remains for exponential time, and so on. The transition probability depends on i, but not on the moment of time t.

Now let us justify the above description and relate the transition times and transition probabilities to the infinitesimal matrix. Let Q be an $r \times r$ matrix with $Q_{ii} < 0$ for all i. Assume that there are random variables ξ, τ_i^n,$1 \leq i \leq r$, $n \in \mathbb{N}$, and η_i^n, $1 \leq i \leq r$, $n \in \mathbb{N}$, defined on a common probability space, with the following properties:

1. The random variable η takes values in X and has distribution μ.

2. For any $1 \leq i \leq r$, the random variable τ_i^n, $n \in \mathbb{N}$, are identically distributed according to the exponential distribution with parameter $r_i = -Q_{ii}$.

3. For any $1 \leq i \leq r$, the random variable η_i^n, $n \in \mathbb{N}$, takes values in $X\{i\}$ and are identically distributed with $P(\eta_i^n = j) = -Q_{ij}/Q_{ii}$ for $j \neq i$.

4. The random variable ξ, τ_i^n, η_i^n,$1 \le i \le r$,$n \in \mathbb{N}$, are independent.

We inductively define two sequences of random variables: σ^n, $n \ge 0$, with values in $\mathbb{R}^+$, and η^n, $n \ge 0$, with values in X. Let $\sigma^0 = 0$ and $\xi^0 = \xi$. Assume that σ^m and ξ^m have been defined for all $m < n$, where $n \ge 1$, and set

$$\sigma^n = \sigma^{n-1} + \tau^n_{\xi^{n-1}}.$$

$$\xi^n = \eta^n_{\xi^{n-1}}.$$

We shall treat σ^n as the time till the n^{th} transition takes place, and ξ^n as the n^{th} state visited by the process. Thus, define

$$X_t = \xi^n \quad \text{for} \quad \sigma^n \le t < \sigma^{n+1} \tag{3.9}$$

Lemma 3. Assume that the random variable ξ, τ_i^n, $1 \le i \le r$,$n \in \mathbb{N}$, and η_i^n, $1 \le i \le r$,$n \in \mathbb{N}$, are defined on a common probability space and satisfy assumptions $1 - 4$ above. Then the process X_t defined by (3.9) is a Markov process with the family of transition matrices $P(t) = \pi e^{tQ}$ and initial distribution μ.

Proof. It is clear from (3.9) that the initial distribution of X_t is μ. Using properties τ_i^n and η_i^n it is possible to show that, for $k \neq j$,

$$\begin{aligned} P(X_0 = i, X_t = k, X_{t+h} = j) &= P(X_0 = i, X_t = k)(P(\tau_k^1 < h)P(\xi_k^1 = j) + o(h)) \\ &= P(X_0 = i, X_t = k)(Q_{kj}h + o(h)) \quad \text{as} \quad h \downarrow 0. \end{aligned}$$

In other words, the main distribution to the probability on the left-hand side comes from the event that there is exactly one transition between the states k and j during the time interval $[t, t+h)$.

Similarly,

$$\begin{aligned} P(X_0 = i, X_t = k, X_{t+h} = j) &= P(X_0 = i, X_t = k)(P(\tau_k^1 \geq h)P(\xi_k^1 = j) + o(h)) \\ &= P(X_0 = i, X_t = j)(1 + Q_{jj}h + o(h)) \quad \text{as} \quad h \downarrow 0, \end{aligned}$$

that is, the main contribution to the probability on the left-hand side comes from the event that there are no transitions during the time interval $[t, t+h]$.

Therefore,

$$\sum_{k=1}^{r} P(X_0 = i, X_t = k, X_{t+h} = j) = P(X_0 = i, X_t = k) + \\ h \sum_{k=1}^{r} P(X_0 = i, X_t = k) Q_{kj} + o(h).$$

Let $R_{ij} = P(X_0 = i, X_t = k)$. The last equality can be written as

$$R_{ij}(t+h) = R_{ij}(t) + h \sum_{k=1}^{r} R_{ik}(t) Q_{kj} + o(h).$$

Using Matrix notation,

$$\lim_{h \downarrow 0} \frac{R(t+h) - R(t)}{h} = R(t)Q.$$

The existence of the left derivative is justified similarly. Therefore,

$$\frac{dP(t)}{dt} = P(t)Q \quad \text{for} \quad t \geq 0.$$

Note that $R_{ij(0)=\mu_i}$ for $i = j$, and $R_{ij}(0) = 0$ for $i \neq j$. These are the same equation and initial condition that are satisfied by the matrix-valued function $\mu_i P_{ij}(t)$. Therefore,

$$R_{ij} = P(X_0 = i, X_t = j) = \mu_i P_{ij}(t). \tag{3.10}$$

In order to prove that X_t is a Markov process with the family of transition matrices $P(t)$, it is sufficient to demonstrate that

$$P(X_{t_0} = i_0, X_{t_1} = i_1, ..., X_{t_k} = i_k)$$
$$= \mu_{i_0} P_{i_0 i_1}(t_1) P_{i_1 i_2}(t_2 - t_1) ... P_{i_{k-1} i_k}(t_k - t_{k-1}).$$

for $0 = t_0 \leq t_1 \leq ... \leq t_k$. The case $k = 1$ has been covered by (3.10). The proof for $k > 1$ is similar and is based on induction on k. □

$$\gamma(i) = \begin{cases} \lambda & \text{if} \quad i = 0 \\ \lambda + i\mu & \text{if} \quad 1 \leq i \leq n-1, \\ i\mu & \text{if} \quad i = n. \end{cases}$$

If the process is in the state $i = 0$, it can only make a transition to the state $i = 1$, which corresponds to an arrival of a request. From a state $1 \leq i \leq n-1$ the process can make a transition either to state $i - 1$ or to state $i + 1$. The former corresponds to completion of one i requests being serviced before the arrival of a new request. Therefore the probability of transition from i to $i-1$ is equal to the probability that the smallest of the i exponential random variable with parameter μ is less than an exponential random variable with parameter λ(all random variable are independent). This probability is equal to $\frac{i\mu}{i\mu+\lambda}$. Consequently, the transition probability from i to $i + 1$ is equal to $\frac{\lambda}{i\mu+\lambda}$. Finally, if the process is in the state n, it can only make a transition to the state $n - 1$.

Let the initial state of the process X_t be independent of the arrival times of the requests and the times it takes to service the requests. Then the process X_t satisfies the assumptions of Lemma 3. The matrix Q is the $(r + 1) \times (r + 1)$ tri-diagonal matrix with the vectors $\gamma(i)$, $0 \leq i \leq r$, on the diagonal, and $u(i) := \lambda$,$0 \leq i \leq r$ above the diagonal, and $l(i) = i\mu$, $0 \leq i \leq r$, below diagonal. By Lemma 3, the process X_t is Markov with the family of transition matrix $P(t) = e^{tQ}$.

It is not difficult to prove that all the entries of e^{tQ} are positive for some t, and therefore Ergodic Theorem is applicable. Let us find the stationary distribution for the family of transition matrices $P(t)$. As noted in a previous section, a distribution π is stationary for $P(t)$ if and only if $\pi Q = 0$. It is easy to verify that the solution of this linear system,subject to the conditions $\pi(i) \geq 0, \quad 0 \leq i \leq r$, and $\sum_{i=0}^{r} = 1$, is

$$\pi(i) = \frac{(\lambda/\mu)^i/i!}{\sum_{j=0}^{r}(\lambda/\mu)^j/j!}, \quad 0 \leq i \leq r.$$

Chapter 4. Random Walk with Heavy tails

4.1 Transition Probability

Let $\mathbb{Z}^d$ be the cubic lattice in $\mathbb{R}^d$, $d \geq 1$, equipped with l_1 norm $||x||_1 = \sum_{i=1}^{d} |x_i|, \quad x = (x_1, ..., x_d) \in \mathbb{Z}^d$. Each point $x \in \mathbb{Z}^d$ has $2d$ nearest neighbours of $x : x' : ||x' - x|| = 1$. The symmetric random walk $x(t), \quad t \geq 0$ is the Markov process with continuous time and the generator $\kappa\triangle$. Here

$$\triangle\Phi(x) = \sum_{x':||x'-x||_1=1} (\Phi(x') - \Phi(x))$$

and $\kappa > 0$ is a constant. It means that

$$P\left(x(t+dt) = x|x(t) = x\right) = \kappa dt,$$

and

$$P\left(x(t+dt) = x'|x(t) = x\right) = 1 - 2\mathrm{d}\kappa dt, \quad \text{where} \quad ||x' - x|| = 1.$$

We call $\kappa > 0$ the diffusion coefficient or diffusivity. The random walk spends in each site $x \in \mathbb{Z}^d$ the exponentially distributed time τ with parameter $2\mathrm{d}\kappa$ and jumps at moment $\tau + 0$ to one of the nearest neighbours $x' : ||x' - x|| = 1$ with equal probability $\frac{1}{2\mathrm{d}}$. The transition probability $p(t, x, y) = P\left(x(t) = y|x(0) = x\right)$ satisfies the heat equation

$$\begin{cases} \frac{dp(t,x,y)}{dt} & = \kappa\triangle_x p(t,x,y) = \kappa\triangle_y p(t,x,y) \\ p(0,x,y) & = \delta_y(x) \end{cases} \tag{4.1}$$

The symmetric random walk is transient in dimensions $d \geq 3$ and recurrent for $d \le 2$. In this section we consider the processes for which the transition rates are non-local and have heavy tails.

Let's define the operator

$$\begin{aligned} \mathcal{L}f(x) &= k\sum_{z\neq 0}[f(x+z)-f(x)]q(z) \\ q(z) &= q(-z) > 0, \\ \sum_{z\neq 0} q(z) &= 1 \end{aligned} \tag{4.2}$$

It is clear that the generator of simple symmetric random walk is a particular case of such an operator with

$$q(z) = \begin{cases} \frac{1}{2d} & \text{if } |z| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We'll however consider q which may be positive everywhere. The precise conditions on q will be provided below;

Let p be the solution of $\frac{dp}{dt} = \mathcal{L}_x p$ with initial condition $p(0,x,y) = \delta_y(x)$. As discussed in section 3.1.3, we can define the Markov process with the generator $\mathcal{L}$. Its transition density is $p(t,x,y)$ and the process satisfies

$$x_{t+dt} = \begin{cases} x_t & \text{with prob. } 1-kdt, \\ x_t + z & \text{with prob. } kq(z)dt. \end{cases} \tag{4.3}$$

As discussed in section 3.1.2 this process spends an exponentially distributed time in each state x before jumping to a new site $x+z$ with probability $q(z)$.

Let us note again that the generator of the process is defined by (4.2). Indeed since

$$p(t,x,y) = P_x(x_t = y) = P(x_t = y | x_0 = x),$$

we have

$$\begin{aligned} p(t+dt,x,y) &= P_x(x_{t+dt} = y) \\ &= \sum_{z \in Z^d} p(dt,x,x+z)p(t,x+z,y) \\ &= p(dt,x,x)p(t,x,y) + \sum_{z \neq 0} p(dt,x,x+z)p(t,x+z,y) \\ &= (1-kdt)p(t,x,y) + \sum_{z \neq 0} kq(z)p(t,x+z,y)dt. \end{aligned}$$

Hence

$$\frac{dp}{dt} = -kp(t,x,y) + \sum_{z \neq 0} kq(z)p(t,x+z,y) = \mathcal{L}_x p.$$

This equation can be solved using the Fourier transform. Define

$$\hat{p}(t,\varphi,y) = \sum_{x \in Z^d} p(t,x,y)e^{i(\varphi,x)}.$$

Then

$$\frac{d\hat{p}}{dt} = -\phi(\varphi)\hat{p}(t,\varphi,y), \qquad \hat{p}(0,\varphi,y) = e^{i(\varphi,y)}, \tag{4.4}$$

where

$$\phi(\varphi) = k \sum_{z \in Z^d} (1 - e^{i(\varphi,z)})q(z), \quad d \geq 1.$$

Since $q(z) = q(-z)$, the latter formula in 1-dimensional case can be re-written in the form

$$\phi(\varphi) = 2k \sum_{z=1}^{\infty} (1 - \cos(\varphi z))q(z) \geq 0, \quad d = 1. \tag{4.5}$$

If $d \geq 1$ is arbitrary, then it follows from (4.4) that

$$\hat{p}(t,\varphi,y) = e^{i(\varphi,y)}e^{-t\phi(\varphi)},$$

and therefore

$$p(t,x,y)=\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{i(\varphi,(y-x))-t\phi(\varphi)}\,\mathrm{d}\varphi,\quad d\geq 1. \tag{4.6}$$

We are now going to study asymptotic of $p(t,x,y)$ for cases $d=1$ and $d>1$. Since p depends on the difference $x-y$, we can put $y=0$ and consider function $p(t,x)=p(t,x,0)$.

4.2 Asymptotic approximation of transition probability in the 1-Dimensional case

First of all note, that if q decays fast enough at infinity, so that $\sum q(z)z^2<\infty$, then ϕ is twice differentiable and $\phi(\varphi)\sim\kappa\phi^2,\quad \varphi\to 0$.
After that, one can apply stationary phase method and prove that

$$p(t,x-y)\sim\frac{1}{\sqrt{4\pi\kappa t}}e^{-\frac{(x-y)^2}{4\kappa t}}\ \text{for}\quad |x-y|\leq A\sqrt{t},\quad t\to\infty. \tag{4.7}$$

We will assume below that q decays much slower (heavy tails):

$$q(z)\sim\frac{q_0}{|z|^{1+\alpha}},\quad z\to\infty,\quad 0<\alpha<2. \tag{4.8}$$

This section contains two parts. First, we establish an asymptotic behavior of the function ϕ as $\varphi\to 0$, and then, using the asymptotics of ϕ, we will find behavior of p. We will determine the behavior of φ for a specific q first:

Lemma 4. Let $q(z)=\frac{1}{|z|^{1+\alpha}}$. Then the following relation holds for function (4.5):

$$\phi(\varphi)=2k\sum_{z=1}^{\infty}(1-\cos(\varphi z))q(z)\sim c_0|\varphi|^{\alpha}+O(|\varphi|^{\gamma}),\ \varphi\to 0,$$

where $\gamma=\min\{2,1+\alpha\}$, c_0 and a_0 are constants.

In order to prove this statement, we will need the following lemma:

Lemma 5. Let $J(\varphi) = \int_1^\infty \frac{1-\cos\varphi z}{z^{1+\alpha}}\,\mathrm{d}z$.

Then

$$J(\varphi) = f(\varphi^2) + c|\varphi|^\alpha,$$

where c is a constant and $f(\cdot) \in C^\infty$.

Proof. If we let $x = |\varphi|z$, then

$$\begin{aligned} J &:= |\varphi|^\alpha \int_{|\varphi|}^\infty \frac{1-\cos x}{x^{1+\alpha}}\,\mathrm{d}x \\ &= |\varphi|^\alpha \left[\underbrace{\int_{|\varphi|}^1 \frac{1-\cos x}{x^{1+\alpha}}\,\mathrm{d}x}_{\text{I}} + \underbrace{\int_1^\infty \frac{1-\cos x}{x^{1+\alpha}}\,\mathrm{d}x}_{\text{II}} \right]. \end{aligned}$$

Since the integrand in (II) is continuous within the domain of integration, and therefore integrable, then $\int_1^\infty \frac{1-\cos x}{x^{1+\alpha}}\,\mathrm{d}x = c_1$. Evaluating (I) we obtain,

$$\begin{aligned} I &:= \int_{|\varphi|}^1 \frac{1-\cos x}{x^{1+\alpha}}\,\mathrm{d}x \\ &= \int_{|\varphi|}^1 \frac{\frac{x^2}{2} - \frac{x^4}{24} + \ldots}{x^{1+\alpha}}\,\mathrm{d}x + \int_{|\varphi|}^1 \frac{O(x^{2N})}{x^{1+\alpha}}\,\mathrm{d}x \\ &= c_2 + a_0|\varphi|^{2-\alpha} + a_1|\varphi|^{4-\alpha} + a_2|\varphi|^{6-\alpha} + \ldots O(|\varphi|^{2N-\alpha}). \end{aligned}$$

Thus

$$\begin{aligned} J &= |\varphi|^\alpha \left[\int_{|\varphi|}^1 \frac{1-\cos x}{x^{1+\alpha}}\,\mathrm{d}x + \int_1^\infty \frac{1-\cos x}{x^{1+\alpha}}\,\mathrm{d}x \right] \\ &= |\varphi|^\alpha \left[c_1 + a_0|\varphi|^{2-\alpha} + a_1|\varphi|^{4-\alpha} + a_2|\varphi|^{6-\alpha} + \ldots O(|\varphi|^{2N-\alpha}) + c_2 \right] \\ &= c|\varphi|^\alpha + f(\varphi^2), \end{aligned}$$

where

$$f(\varphi^2) = a_0\varphi^2 + a_1\varphi^4 + ... + O(\varphi^{2N}).$$

□

We now proceed to prove Lemma 4 using the results in Lemma 5.

Proof of Lemma 4. We have

$$\begin{aligned}\Psi(\varphi) &:= \sum_{n=1}^{\infty} \frac{1-e^{i\varphi n}}{n^{1+\alpha}} \\ &= k_0 - \frac{\sum_{n=1}^{\infty}\int_0^1 \frac{e^{i\varphi(n+\tau)}}{n^{1+\alpha}}\,\mathrm{d}\tau}{\int_0^1 e^{i\varphi\tau}\,\mathrm{d}\tau} = k_0 - g(\varphi)\int_1^{\infty} \frac{e^{i\varphi z}}{[z]^{1+\alpha}}\,\mathrm{d}z,\end{aligned}$$

where $g(\varphi) = \frac{i\varphi}{e^{i\varphi}-1} \in C^\infty$ when φ is small, and $k_0 = \sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}$. Note that $\lim_{\varphi\to 0} g(\varphi) = 1$. Thus

$$\begin{aligned}\Psi(\varphi) &= \sum_{n=1}^{\infty} \frac{1-e^{i\varphi n}}{n^{1+\alpha}} \\ &= k_1 + g(\varphi)\int_1^{\infty} \frac{1-e^{i\varphi z}}{[z]^{1+\alpha}}\,\mathrm{d}z. \qquad (4.9)\end{aligned}$$

If $z \in [n, n+1]$, i.e., $z = n+\tau$, for $\tau \in [0,1]$, then (using the Taylor's series)

$$\begin{aligned}\frac{1}{[z]^{1+\alpha}} &= \frac{1}{(n+\tau)^{1+\alpha}} \\ &= \frac{1}{n^{1+\alpha}} + \frac{a_0}{n^{2+\alpha}} + \frac{a_1}{n^{3+\alpha}} + ... \qquad (4.10)\end{aligned}$$

Similarly,

$$\frac{1}{n^{1+\alpha}} = \frac{1}{(z-\tau)^{1+\alpha}} = \frac{1}{z^{1+\alpha}} + \frac{c_0}{z^{2+\alpha}} + \frac{c_1}{z^{3+\alpha}} + ... \qquad (4.11)$$

Substituting (4.10) and (4.11) in (4.9) we have

$$\Psi(\varphi) = k_1 + g(\varphi) \int_1^\infty (1 - e^{i\varphi z}) \left[\frac{1}{z^{1+\alpha}} + \frac{c_0}{z^{2+\alpha}} + h(z) \right] \mathrm{d}z, \tag{4.12}$$

where $h(z) = O(\frac{1}{z^{3+\alpha}})$ as $z \to \infty$. Obviously,

$$\int_1^\infty (1 - e^{i\varphi z}) h(z) \, \mathrm{d}z = O(\varphi^2), \quad \varphi \to 0,$$

since the integral converges after the integrand is differentiated twice. Hence

$$\Psi(\varphi) = k_1 + g(\varphi) \int_1^\infty (1 - e^{i\varphi z}) \left[\frac{1}{z^{1+\alpha}} + \frac{c_0}{z^{2+\alpha}} \right] \mathrm{d}z + O(\varphi^2), \quad \varphi \to 0.$$

Note that Lemma 5 remains valid if the cosine function there is replaced by the exponential function. However, the function f in this case will depend on φ, not φ^2. It also remains valid if $z^{1+\alpha}$ is replaced by $z^{2+\alpha}$. Therefore,

$$\Psi(\varphi) = A_1 + A_2|\varphi| + A_3|\varphi|^\alpha + O(\varphi^2), \tag{4.13}$$

where $A_j \in \mathbb{C}$ are some constants. Since $\phi(\varphi) = \mathrm{Re}\ 2\mathrm{k}\Psi(\varphi)$, the same relation (4.13) is valid for $\phi(\varphi)$. It remains to note that $\phi(0) = 0$, and that $\phi(\varphi)$ is an even function. The latter two properties immediately imply the statement of Lemma 4. □

Lemma 6. Let

$$q(z) = \frac{q_0}{|z|^{1+\alpha}} + \frac{q_1}{|z|^{2+\alpha}} + \frac{q_2}{|z|^{3+\alpha}} + O(\frac{1}{|z|^{4+\alpha}}).$$

Then the following relation holds for function (4.5):

$$\phi(\varphi) = 2k\sum_{z=1}^{\infty}(1-\cos(\varphi z))q(z) \sim c_0|\varphi|^\alpha + O(|\varphi|^\gamma),\ \varphi \to 0,$$

where $\gamma = \min\{2, \alpha+1\}$, c_0 and a_0 are constants.

Proof. Since ϕ depends on q linearly, one can prove the statement of Lemma 6 for each term of q separately. The validity of the statement of the Lemma for the first three terms of q is proved in Lemma 4. If $q = O(\frac{1}{|z|^{4+\alpha}})$, one can differentiate formula (4.5) three times, which shows that $\phi = O(\varphi^3)$ as $\varphi \to 0$. Thus the statement of Lemma 6 is valid in this case also. $\square$

Now we will pass to the second part of this section. Namely, we will use the asymptotic formula for ϕ in order to obtain the asymptotic behavior of $p(t,x)$. Without loss of generality, we can replace c_0 in Lemma 6 by $c_0 = 1$ (since one can make the change of the variable $c_0 t \to t$) and write

$$p(t,x) = \int_{-\pi}^{\pi} e^{ix\varphi - t\phi(\varphi)}\,\mathrm{d}\varphi, \tag{4.14}$$

where ϕ has the following properties:

$$\begin{cases} \text{If } \varphi \neq 0, \text{ then } \ \phi(\varphi) \in C^\infty \ \text{ and } \ \phi(\varphi) > 0, \\ \phi(\varphi) = |\varphi|^\alpha + O(|\varphi|^\gamma), \quad \gamma = \min\{2, 1+\alpha\}, \quad \varphi \to 0, \\ \phi \quad \text{is } 2\pi \text{ - periodic.} \end{cases}$$

Lemma 7. Let $F(\sigma) = \int_{-\infty}^{\infty} e^{i\sigma\varphi - |\varphi|^\alpha}\,\mathrm{d}\varphi$.

Then F is analytic in σ and does not vanish

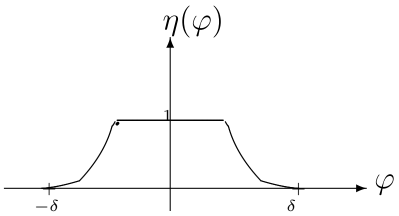
$$F(\sigma) \sim \frac{C_\pm}{|\sigma|^{\alpha+1}}\ ,\sigma \to \pm\infty, \quad C_\pm \neq 0. \tag{4.15}$$

The proof of this lemma is on page 51.

Theorem 4.2.1. The following relation holds for the transition probability $p(t,x) = p(t,x,0)$:

$$p(t,x) = \frac{1}{t^{\frac{1}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right)(1+o(1)), \quad \text{as} \quad x^2+t^2 \to \infty.$$

Proof. Let $\eta(\varphi) \in C_c^\infty$, and $\eta(\varphi) = 1$ when $|\varphi| < \delta,\ \delta > 0$.



Let

$$\begin{aligned} p_1(t,x) &= \int_{-\pi}^{\pi} e^{ix\varphi - t\phi(\varphi)}(1-\eta(\varphi))\,\mathrm{d}\varphi \\ &= \int_{-\pi}^{-\delta} e^{ix\varphi - t\phi(\varphi)}(1-\eta(\varphi))\,\mathrm{d}\varphi + \int_{\delta}^{\pi} e^{ix\varphi - t\phi(\varphi)}(1-\eta(\varphi))\,\mathrm{d}\varphi. \end{aligned} \tag{4.16}$$

Estimating (4.16), we have

$$|p_1(t,x)| \le \int_{-\pi}^{-\delta} e^{-t\phi(\varphi)}\,\mathrm{d}\varphi + \int_{\delta}^{\pi} e^{-t\phi(\varphi)}\,\mathrm{d}\varphi \le 2\pi e^{-\epsilon t}, \tag{4.17}$$

where $\quad \epsilon = \min_{\delta \le |\varphi| \le \pi} \phi(\varphi)$.

The latter estimate is not effective when t is bounded. We can integrate (4.16) by parts to obtain a better estimate:

$$p_1(t,x) = -\frac{1}{ix}\int_{-\pi}^{\pi} e^{ix\varphi}\left((1-\eta)e^{-t\phi(\varphi)}\right)'\,\mathrm{d}\varphi,$$

which implies (similar to (4.17)) that

$$p_1(t,x) \le \frac{C}{|x|} e^{-\epsilon t}.$$

If we repeat the integration by parts N times, we will obtain that

$$|p_1(t,x)| \le C_N \left(\frac{1+t}{|x|}\right)^N e^{-\epsilon t} \le A_N \left(\frac{1}{|x|}\right)^N e^{-\epsilon t/2}.$$

By combining the latter formula with (4.17), we obtain

$$|p_1(t,x)| \le D_N \left(\frac{1}{1+|x|}\right)^N e^{-\epsilon t/2}, \quad x^2+t^2 \to \infty. \tag{4.18}$$

From Lemma 7 it follows that estimate (4.18) allows us to consider p_1 as a part of the remainder terms in the statement of Theorem 4.2.1.

Now we put

$$p(t,x) = p_1(t,x) + p_2(t,x), \quad \text{where} \quad p_2(t,x) = \int_{-\pi}^{\pi} e^{ix\varphi - t\phi(\varphi)} \eta(\varphi)\,\mathrm{d}\varphi,$$

and we introduce p_3, which is obtained from p_2 by leaving only the main term of the asymptotics of the function $\phi(\varphi)$ in the integral defining p_2:

$$\begin{aligned} p_3(t,x) &= \int_{-\pi}^{\pi} e^{ix\varphi - t|\varphi|^\alpha} \eta(\varphi)\,\mathrm{d}\varphi \\ &= \underbrace{\int_{-\infty}^{\infty} e^{ix\varphi - t|\varphi|^\alpha}\,\mathrm{d}\varphi}_{\mathrm{I}} - \underbrace{\int_{-\infty}^{\infty} e^{ix\varphi - t|\varphi|^\alpha}(1-\eta(\varphi))\,\mathrm{d}\varphi}_{\mathrm{II}} \end{aligned}$$

The first integral term of p_3, I, can be expressed through F, which is defined in Lemma 7, and the second term, II, can be evaluated similar to p_1. Thus

$$p_3(t,x) = \frac{1}{t^{\frac{1}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) + O\left[\left(\frac{1}{1+|x|}\right)^N e^{-\epsilon t/2}\right], \quad x^2+t^2 \to \infty.$$

It remains to justify that $|p_2 - p_3|$ can be estimated in such a way that allows one to consider it as a part of remainder terms in the statement of the Theorem 4.2.1. This property of $|p_2 - p_3|$ will be proved if we show that the following two relations hold:

$$1)\quad |p_2 - p_3| = o\left(\frac{t}{|x|^{\alpha+1}}\right), \quad \text{when} \quad x^2 + t^2 \to \infty, \quad \frac{|x|}{t^{1/\alpha}} \to \infty, \tag{4.19}$$

and 2) for any $A < \infty$,

$$|p_2 - p_3| = o\left(\frac{1}{t^{1/\alpha}}\right), \qquad \frac{|x|}{t^{1/\alpha}} < A. \tag{4.20}$$

Indeed, assume that (4.20) and (4.19) are proved. From Lemma 7 it follows that

$$\frac{1}{t^{\frac{1}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) \sim \frac{ct}{|x|^{\alpha+1}}, \quad \frac{|x|}{t^{1/\alpha}} \to \infty.$$

Let us fix an arbitrary $\varepsilon > 0$. From (4.19) it follows that there exists $A_0 = A_0(\varepsilon)$ such that

$$|p_2 - p_3| < \varepsilon \frac{1}{t^{\frac{1}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) \quad \text{when} \quad \frac{|x|}{t^{1/\alpha}} > A_0, \quad x^2 + t^2 \to \infty. \tag{4.21}$$

From (4.20) it follows that

$$|p_2 - p_3| < \varepsilon \frac{1}{t^{\frac{1}{\alpha}}} \quad \text{when} \quad \frac{|x|}{t^{1/\alpha}} \le A_0, \quad x^2 + t^2 \to \infty. \tag{4.22}$$

Thus, (4.21) and (4.22) with the fact that F does not vanish imply that $p_2 - p_3$ can be included into the remainder term in the statement in Theorem 4.2.1. Hence, it remains only to prove (4.19) and (4.20).

Let us prove (4.20). We have

$$p_2(t, x) = \int_{-\infty}^{\infty} e^{ix\varphi - t(|\varphi|^\alpha + O(|\varphi|^\gamma))} \eta(\varphi)\, \mathrm{d}\varphi.$$

$$p_3(t, x) = \int_{-\infty}^{\infty} e^{ix\varphi - t|\varphi|^\alpha} \eta(\varphi)\, \mathrm{d}\varphi.$$

In p_2, we use the substitution $|\psi|^\alpha = |\varphi|^\alpha + O(|\varphi|^\gamma)$, which implies

$$\varphi = f(\psi) = \psi + O(\psi^2),$$

$$\mathrm{d}\varphi = f^{'}(\psi)\,\mathrm{d}\psi = (1 + O(\psi))\,\mathrm{d}\psi.$$

Thus

$$\begin{aligned}
p_2(t,x) &= \int_{-\infty}^{\infty} e^{ixf(\psi)-t|\psi|^\alpha}\eta(f(\psi))f^{'}(\psi)\,\mathrm{d}\psi \\
&= \int_{-\infty}^{\infty} e^{ixf(\psi)-t|\psi|^\alpha}\eta(f(\psi))(1+O(\psi))\,\mathrm{d}\psi \\
&= \int_{-\infty}^{\infty} e^{ix(\psi+O(\psi^2))-t|\psi|^\alpha}\eta(f(\psi))(1+O(\psi))\,\mathrm{d}\psi.
\end{aligned}$$

Hence

$$\begin{aligned}
|p_2(t,x) - p_3(t,x)| &\le \int_{-\infty}^{\infty} e^{-t|\psi|^\alpha}|e^{ix(\psi+O(\psi^2))}\eta(f(\psi))(1+O(\psi)) - e^{ix\psi}\eta(\psi)|\,\mathrm{d}\psi \\
&= \int_{-\infty}^{\infty} e^{-t|\psi|^\alpha}|e^{ixO(\psi^2)}\eta(f(\psi))(1+O(\psi)) - \eta(\psi)|\,\mathrm{d}\psi \\
&= \int_{-\infty}^{\infty} e^{-t|\psi|^\alpha}|(1+O(x\psi^2))\eta(f(\psi))(1+O(\psi)) - \eta(\psi)|\,\mathrm{d}\psi \\
&= \int_{-\infty}^{\infty} e^{-t|\psi|^\alpha}|O(\psi) + O(x\psi^2)|\,\mathrm{d}\psi \\
&\le C\int_{-\infty}^{\infty} \psi e^{-t|\psi|^\alpha}\,\mathrm{d}\psi + C_1 x\int_{-\infty}^{\infty} \psi^2 e^{-t|\psi|^\alpha}\,\mathrm{d}\psi \\
&= \frac{C_2}{t^{\frac{2}{\alpha}}} + \frac{C_3 x}{t^{\frac{3}{\alpha}}},
\end{aligned} \tag{4.23}$$

which implies (4.20).

Let us prove (4.19). We have

$$\begin{aligned} p_2(t,x) &= \int_{-\infty}^{\infty} e^{ix\varphi - t(|\varphi|^\alpha + O(|\varphi|^\gamma))}\eta(\varphi)\,\mathrm{d}\varphi \\ &= \int_{-\infty}^{\infty} e^{ix\varphi}(1 - t(|\varphi|^\alpha + O(|\varphi|^\gamma)) + O(t^2|\varphi|^{2\alpha}))\eta(\varphi)\,\mathrm{d}\varphi. \\ p_3(t,x) &= \int_{-\infty}^{\infty} e^{ix\varphi - t|\varphi|^\alpha}\eta(\varphi)\,\mathrm{d}\varphi \\ &= \int_{-\infty}^{\infty} e^{ix\varphi}(1 - t|\varphi|^\alpha + O(t^2|\varphi|^{2\alpha}))\eta(\varphi)\,\mathrm{d}\varphi. \end{aligned}$$

Thus,

$$p_2 - p_3 = \int_{-\infty}^{\infty} e^{ix\varphi} O(t|\varphi|^\gamma + t^2|\varphi|^{2\alpha})\eta(\varphi)\,\mathrm{d}\varphi.$$

We assume that the asymptotic expansion of $\phi(\varphi)$ as $\varphi \to 0$ admits differentiation. Then, using stationary phase method, we obtain

$$\int_{-\infty}^{\infty} e^{ix\varphi} O(|\varphi|^\gamma)\eta(\varphi)\,\mathrm{d}\varphi \sim \frac{c_1}{|x|^{\gamma+1}}, \quad |x| \to \infty,$$

$$\int_{-\infty}^{\infty} e^{ix\varphi} O(|\varphi|^{2\alpha})\eta(\varphi)\,\mathrm{d}\varphi \sim \frac{c_2}{|x|^{2\alpha+1}}, \quad |x| \to \infty.$$

From the last three formulas it follows that

$$\begin{aligned} |p_2 - p_3| &< C\left(\frac{t}{|x|^{\gamma+1}} + \frac{t^2}{|x|^{2\alpha+1}}\right) \\ &= C\,\frac{t}{|x|^{\alpha+1}}\left(\frac{1}{|x|^{\gamma-\alpha}} + \frac{t}{|x|^\alpha}\right), \quad |x| \to \infty. \end{aligned}$$

This inequality implies (4.20) since $|x| \to \infty$ when $x^2 + t^2 \to \infty$ and $\frac{|x|}{t^{1/\alpha}} \to \infty$. □

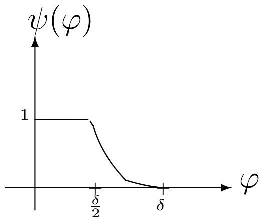
Proof of Lemma 7. We now determine the asymptotic estimate of

$$I(\sigma) = \int_0^{\infty} e^{i\sigma\varphi - \varphi^\alpha}\,\mathrm{d}\varphi.$$

We split $I(\sigma)$ in two terms $I_1(\sigma) + I_2(\sigma)$, where

$$I_1(\sigma) = \int_0^\infty e^{i\sigma\varphi - \varphi^\alpha} \psi(\varphi)\, \mathrm{d}\varphi, \quad I_2(\sigma) = \int_0^\infty e^{i\sigma\varphi}(1 - \psi(\varphi))e^{-\varphi^\alpha}\, \mathrm{d}\varphi,$$

and $\psi(\varphi)$ is defined by the following graph

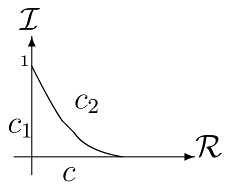


Integrating $I_2(\sigma)$ by parts N times we have

$$I_2(\sigma) = \int_0^\infty \frac{e^{i\sigma\varphi}}{(-i\sigma)^n} \left[(1 - \psi(\varphi))e^{-\varphi^\alpha}\right]^{(n)} \mathrm{d}\varphi = O(\frac{C_1}{\sigma^n}). \tag{4.24}$$

It remains to estimate $I_1(\sigma)$. In order to do this, we deform the contour $[0, 2]$ into contour $C = C_1 \cup C_2$ in the complex plane $z = \varphi + i\rho$, where C_1 is the segment $[0, \imath]$ and C_2 is an infinitely smooth contour in the first quadrant, which is given by an equation $\rho = f(\varphi)$. We assume that $f(\varphi) \in C^\infty[0, \infty)$, $f(\varphi) \geq 0$, $f(0) = 1$, $f(\varphi) = 0$ for $\varphi > 1$. Thus contour C_2 starts at $z = i$, comes to the point $z = 1$ and then goes to infinity along the real axis. Then

$$I_1(\sigma) = \int_C e^{i\sigma\varphi - \varphi^\alpha} \psi(\varphi)\, \mathrm{d}\varphi.$$



We now estimate the integral along the path C_1 and C_2 respectively.

$$\int_{C_1} e^{i\sigma\varphi-\varphi^\alpha}\psi(\varphi)\,\mathrm{d}\varphi = i\int_0^1 e^{-\sigma s}e^{-(is)^\alpha}\,\mathrm{d}s = i\int_0^1 e^{-\sigma s}\left[\sum_{n=0}^{N}\frac{(-is)^{n\alpha}}{n!}+h(is)\right]\mathrm{d}s,$$

where

$$h(is) = |e^{-(is)^\alpha} - \sum_{n=0}^{N}\frac{(-is)^{n\alpha}}{n!}| \sim (s)^{(N+1)\alpha}, \quad \text{as} \quad s\to 0.$$

Thus, after integrating by parts $N+1$ times, we will obtain that

$$i\int_0^1 e^{-\sigma s}h(is)\,\mathrm{d}s \simeq i\int_0^1 e^{-\sigma s}O(s^{(N+1)\alpha})\,\mathrm{d}s = O(\frac{1}{\sigma^{N+2}}).$$

Standard Laplace method implies that

$$\int_0^1 e^{-\sigma s}s^{n\alpha}\,\mathrm{d}s \sim \frac{c_n}{\sigma^{n\alpha+1}}, \quad \sigma\to\infty, \quad c_0 = 1.$$

Hence

$$\int_{C_1} e^{i\sigma\varphi-\varphi^\alpha}\psi(\varphi)\,\mathrm{d}\varphi = \frac{i}{\sigma}+\frac{c_1}{\sigma^{\alpha+1}}+O(\frac{1}{\sigma^{2\alpha+1}}).$$

The asymptotic integral of I_1 along C_2 is estimated by

$$\begin{aligned}\int_{C_2} e^{i\sigma\varphi-\varphi^\alpha}\psi(\varphi)\,\mathrm{d}\varphi &= \int_{C_2} e^{i\sigma z-z^\alpha}\psi(z)\,\mathrm{d}z\\ &= \frac{e^{i\sigma z}\psi(z)e^{-z^\alpha}}{i\sigma}\Big|_{z=i}^{z=2} - \frac{1}{i\sigma}\int_{C_2} e^{i\sigma z}(\psi e^{-z^\alpha})'\,\mathrm{d}z\\ &= O(\frac{1}{\sigma^{N+1}}), \quad \text{after integrating by parts N times.}\end{aligned}$$

Hence

$$I(\sigma) = I_1(\sigma)+I_2(\sigma) = \frac{i}{\sigma}+\frac{c_1}{\sigma^{\alpha+1}}+O(\frac{1}{\sigma^{2\alpha+1}}).$$

It remains to note that $F(\sigma) = I(\sigma)+\overline{I}(\sigma)$. □

4.3 Asymptotic Approximation of the Transition Probability in n-Dimension

We will find the asymptotics of the transition probability (4.6) in n-dimensional case. In order to distinguish from the one dimensional case (studied in the previous section) we will use another notation L for the symbol ϕ. Thus, $p(t,x)$ has the form

$$p(x,t) = \int_{(-\pi,\pi)^d} e^{i(\varphi,x)-L(\varphi)t}\,\mathrm{d}\varphi, \begin{cases} L > 0, & \text{for} \quad \varphi \neq 0 \\ L & \text{is} \quad 2\pi - \text{periodic} \\ L \simeq |\varphi|^\alpha h(\dot{\varphi}), & \varphi \to 0, \text{where} \quad \dot{\varphi} = \frac{\varphi}{|\varphi|} \\ 0 < \alpha < 2 & \end{cases}$$

We assume that L has an asymptotic behavior at zero similar to one that was established in the one dimensional case. Namely, we assume that

$$L(\varphi) = |\varphi|^\alpha h(\dot{\varphi}) + \sum_{i=1}^{M-1} |\varphi|^{\alpha+i} h_i(\dot{\varphi}) + O(|\varphi|^{\alpha+M}), \quad \varphi \to 0,$$

for some large enough M. Here, $\dot{\varphi} = \frac{\varphi}{|\varphi|}$, $h = h(\dot{\varphi})$ and h_i are smooth functions on the sphere. We wish to estimate $p(x,t)$ asymptotically as $|x|^2 + t^2 \to \infty$.

Let

$$F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) = \int_{R^d} e^{i\left(\varphi, \frac{x}{t^{\frac{1}{\alpha}}}\right) - |\varphi|^\alpha h(\dot{\varphi})}\,\mathrm{d}\varphi.$$

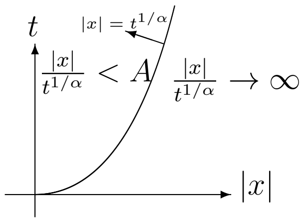
Theorem 4.3.1. The following asymptotic expansions hold when $|x|^2 + t^2 \to \infty$:

$$\text{a)}\quad p(x,t) = \frac{1}{t^{\frac{d}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) + O\left(\frac{1}{t^{\frac{d}{\alpha}+1}}\right), \quad \frac{|x|}{t^{\frac{1}{\alpha}}} \leq A,$$

where A is arbitrary

$$\text{b)}\quad p(x,t) = \frac{t}{|x|^{d+\alpha}} f(\dot{x}) + o\left(\frac{t}{|x|^{d+\alpha}}\right) \quad \text{if} \quad \frac{|x|}{t^{\frac{1}{\alpha}}} \to \infty$$

where the two regions in the domain of $p(x,t)$ is described by the figure below



and $f(\dot{x})$ is defined by $h(\dot{\varphi})$ as follows:

$$\int_{R^d} h(\dot{\varphi})|\varphi|^{\alpha} e^{i(\varphi,y)} d\varphi = -|y|^{-d-\alpha} f(\dot{y}).$$

The integral here is understood in the sense of the Fourier transform in the space of distributions.

$$\text{c)}\quad \frac{1}{t^{\frac{1}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) = \frac{t}{|x|^{d+\alpha}} f(\dot{x}) + o\left(\frac{t}{|x|^{d+\alpha}}\right) \quad \text{if} \quad \frac{|x|}{t^{\frac{1}{\alpha}}} \to \infty.$$

If $f(\dot{x}) \neq 0$, then statements a) - c) can be written in the following form. Define by $B(\epsilon)$ an ϵ-neighborhood of the set in $\mathbb{R}^d$ where $F(z) = 0$. Then

$$p(x,t) = \frac{1}{t^{\frac{d}{\alpha}}} F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right)(1 + o(1)), \quad |x|^2 + t^2 \to \infty, \quad \text{if} \quad \frac{x}{t^{\frac{d}{\alpha}}} \notin B(\epsilon),$$

and

$$p(x,t) = \frac{1}{t^{\frac{d}{\alpha}}} \left[F\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) + o(1)\right], \quad |x|^2 + t^2 \to \infty, \quad \text{if} \quad \frac{x}{t^{\frac{d}{\alpha}}} \in B(\epsilon).$$

The next lemma provides the asymptotics for $F(y)$ at infinity, which is equivalent to the statement c) above.

Lemma 8. Let $F(y) = \int_{R^d} e^{i(\varphi,y)-|\varphi|^\alpha h(\dot\varphi)}\,\mathrm{d}\varphi$, where $\dot\varphi = \frac{\varphi}{|\varphi|}$. Then

$$F(y) = |y|^{-d-\alpha} f(\dot y) + o(|y|^{-d-\alpha}), \quad |y| \to \infty,$$

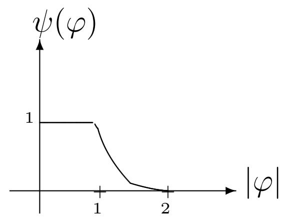
where $\dot y = \frac{y}{|y|}$ and $f(\dot y)$ is defined as follows:

$$\int_{R^d} h(\dot\varphi)|\varphi|^\alpha e^{iy\varphi} d\varphi = -|y|^{-d-\alpha} f(\dot y).$$

The integral here is understood in the sense of the Fourier transform in the space of distributions.

Remark: Function f is defined by h.

Proof. Let $\psi(\varphi)$ be defined by



Then

$$\begin{aligned} F_1(y) &= \int_{R^d} e^{i(\varphi,y)-|\varphi|^\alpha h(\dot\varphi)}(1-\psi(\varphi))\,\mathrm{d}\varphi \\ &= -\frac{1}{|y|^2}\int_{R^d} \left(\Delta e^{i(\varphi,y)}\right) e^{-|\varphi|^\alpha h(\dot\varphi)}(1-\psi(\varphi))\,\mathrm{d}\varphi \\ &= -\frac{1}{|y|^2}\int_{R^d} e^{i(\varphi,y)}\Delta\left(e^{-|\varphi|^\alpha h(\dot\varphi)}(1-\psi(\varphi))\right)\,\mathrm{d}\varphi. \end{aligned}$$

$$|F_1(y)| = \left| -\frac{1}{|y|^2} \int_{R^d} e^{i(\varphi,y)} \Delta \left(e^{-|\varphi|^\alpha h(\dot{\varphi})} (1 - \psi(\varphi)) \right) \mathrm{d}\varphi \right|$$
$$\leq \frac{1}{|y|^2} \int_{R^d} \left| \Delta \left(e^{-|\varphi|^\alpha h(\dot{\varphi})} (1 - \psi(\varphi)) \right) \right| \mathrm{d}\varphi$$
$$= \frac{C_0}{|y|^2}.$$

We can repeat the integration by parts N times to obtain $F_1(y) \sim \frac{C_N}{|y|^{2N}}$. If $F_2 = F - F_1$, then

$$F_2(y) = \int_{|\varphi|<2} e^{i(\varphi,y) - |\varphi|^\alpha h(\dot{\varphi})} \psi(\varphi) \, \mathrm{d}\varphi$$
$$= \int_{|\varphi|<2} e^{i(\varphi,y)} \left(e^{-|\varphi|^\alpha h(\dot{\varphi})} - \sum_{j=1}^{M} \frac{(-|\varphi|^\alpha h(\dot{\varphi}))^j}{j!} \right) \psi(\varphi) \, \mathrm{d}\varphi +$$
$$\int_{|\varphi|<2} e^{i(\varphi,y)} \sum_{j=1}^{M} \frac{(-|\varphi|^\alpha h(\dot{\varphi}))^j}{j!} \psi(\varphi) \, \mathrm{d}\varphi.$$

The first integrand can be made as smooth as we please if M is large enough. Thus we can integrate by parts and prove that the first integral has order $O\left(\frac{1}{|y|^N}\right)$ if $M = M(N)$ is large enough.

Hence,

$$F(y) = \int_{|\varphi|<2} e^{i(\varphi,y)} \sum_{j=1}^{M} \frac{(-|\varphi|^\alpha h(\dot{\varphi}))^j}{j!} \psi(\varphi) \, \mathrm{d}\varphi + O\left(\frac{1}{|y|^N}\right) \tag{4.25}$$

for $|y| \to \infty$, $M = M(N)$.

Denote by Φ the Fourier transform in the space of distributions $\mathcal{S}'$. Consider

$$F_s(y) = \int_{|\varphi|<2} e^{i(\varphi,y)} |\varphi|^s g(\dot{\varphi}) \psi(\varphi) \, \mathrm{d}\varphi.$$

This function can be written as

$$F_s(y) = \Phi(|\varphi|^s g(\dot{\varphi})) - \Phi(|\varphi|^s g(\dot{\varphi})(1 - \psi(\varphi))). \tag{4.26}$$

The first term on the right is a homogeneous function of order $-d-s$, i.e., $\Phi(|\varphi|^s g(\dot{\varphi})) = |y|^{-d-s}q(\dot{y})$. The second term can be written in the form

$$\Phi(|\varphi|^s g(\dot{\varphi})(1-\psi(\varphi))) = \frac{1}{(-|y|^2)^N}\Phi\left(\Delta_\varphi^N(|\varphi|^s g(\dot{\varphi})(1-\psi(\varphi)))\right).$$

If $N > s+d$, then the function $u = \Delta_\varphi^N(|\varphi|^s g(\dot{\varphi})(1-\psi(\varphi)))$ is integrable, and therefore $|\Phi u| < c_N$. Thus the second term does not exceed $\frac{c_N}{|y|^{2N}}$. Hence

$$F_s(y) = |y|^{-d-s}q(\dot{y}) + O\left(\frac{1}{|y|^{2N}}\right), \quad |y| \to \infty.$$

This and (4.25) proves the lemma. □

We will obtain the following two lemmas before we start proving Theorem 4.3.1. We need to study

$$p(x,t) = \int_{(-\pi,\pi)^d} e^{i(\varphi,x)-L(\varphi)t}\,\mathrm{d}\varphi, \ |\varphi| \to 0,$$

where $L > 0$ for $|\varphi| \neq 0$, L is 2π-periodic and

$$L(\varphi) = |\varphi|^\alpha h(\dot{\varphi}) + \sum_{i=1}^{M-1} |\varphi|^{\alpha+i}h_i(\dot{\varphi}) + O(|\varphi|^{\alpha+M}), \quad h > 0. \tag{4.27}$$

Recall that $D_\varphi^k = \frac{\partial^{|k|}}{\partial\varphi_1^{k_1}\partial\varphi_2^{k_2}...\partial\varphi_d^{k_d}}, \quad |k| = \sum_{i=1}^d k_i$. The next statement is obvious.

Lemma 9. There is $\gamma > 0$ such that for $\varphi \in (-\pi,\pi)^d$, and $k \leq M$,

$$i). \quad L > \gamma|\varphi|^\alpha,$$

$$ii). \quad |\bigtriangledown L| < c|\varphi|^{\alpha-1},$$

$$iii). \quad |D_\varphi^k L| \leq C_k|\varphi|^{\alpha-|k|}.$$

Lemma 10.

$$\left|D_\varphi^k e^{-L(\varphi)t}\right| \le c_k t^{\frac{|k|}{\alpha}} e^{-\frac{\gamma}{2}|\varphi|^\alpha t} \quad \text{when} \quad t \ge 1, \quad |\varphi| \ge \frac{1}{t^{\frac{1}{\alpha}}}.$$

Proof. Applying Lemma 9 i)& ii), and the fact that $x^\gamma e^{-x}$ is bounded for $x \ge 1$, we have

$$\begin{aligned}
\left|\frac{\partial}{\partial\varphi_i} e^{-L(\varphi)t}\right| &= \left|L_{\varphi_i} t e^{-L(\varphi)t}\right| \\
&\le c|\varphi|^{\alpha-1} t e^{-\gamma|\varphi|^\alpha t} \\
&\le c t^{1/\alpha} e^{-\frac{\gamma}{2}|\varphi|^\alpha t} \max\left[(|\varphi|^\alpha t)^{\frac{\alpha-1}{\alpha}} e^{-\frac{\gamma}{2}|\varphi|^\alpha t}\right] \\
&= c_1 t^{\frac{1}{\alpha}} e^{-\frac{\gamma}{2}|\varphi|^\alpha t}.
\end{aligned}$$

For the second derivative we'll have

$$\begin{aligned}
\left|\frac{\partial^2}{\partial\varphi_i\partial\varphi_j} e^{-L(\varphi)t}\right| &\le \left|L_{\varphi_i\varphi_j} t e^{-L(\varphi)t}\right| + \left|L_{\varphi_i} L_{\varphi_j} t^2 e^{-L(\varphi)t}\right| \\
&\le c_2|\varphi|^{\alpha-1} t e^{-\gamma|\varphi|^\alpha t} + c^2 t^2 |\varphi|^{2\alpha-2} e^{-\gamma|\varphi|^\alpha t} \\
&= c_2 t^{\frac{2}{\alpha}} e^{-\frac{\gamma}{2}|\varphi|^\alpha t} \max_{|\varphi|^\alpha t \ge 1}\left[(|\varphi|^\alpha t)^{\frac{\alpha-2}{\alpha}} e^{-\frac{\gamma}{2}|\varphi|^\alpha t}\right] \\
&+ c^2 t^{\frac{2}{\alpha}} e^{-\frac{\gamma}{2}|\varphi|^\alpha t} \max_{|\varphi|^\alpha t \ge 1}\left[(|\varphi|^\alpha t)^{\frac{2\alpha-2}{\alpha}} e^{-\frac{\gamma}{2}|\varphi|^\alpha t}\right] \\
&\le C t^{\frac{2}{\alpha}} e^{-\frac{\gamma}{2}|\varphi|^\alpha t}.
\end{aligned}$$

Other higher order derivatives can be estimated similarly, and this completes the proof of the lemma. □

Proof of Theorem 4.3.1. Statement c) was established in Lemma 8. Statements a) can be proved by the same arguments that were used to prove the similar statement in the one dimensional case. We are going to prove b) now.

Let us introduce functions

$$I_2(x,t) = \int_{(-\pi,\pi)^d} e^{i(\varphi,x)-L(\varphi)t}(1-\psi(|\varphi|t^{\frac{1}{\alpha}}))\,\mathrm{d}\varphi$$

and

$$I_3(x,t) = \int_{\left(|\varphi|t^{\frac{1}{\alpha}}\right)<2} e^{i(\varphi,x)-L(\varphi)t}\psi(|\varphi|t^{\frac{1}{\alpha}})\,\mathrm{d}\varphi,$$

where the cut off function ψ is defined on page 55. Then $p(t,x) = I_2 + I_3$. We will show that I_2 has an estimate that allows us to consider it as a part of the remainder terms in the statement b) of the theorem, and I_3 has the asymptotics that has to be proved in b) for p. Our next step is to prove the estimate on I_2.

Lemma 11. For each N, the following estimate is valid:

$$I_2(x,t) \le \frac{1}{t^{\frac{d}{\alpha}}} C_N \left(\frac{t^{\frac{1}{\alpha}}}{|x|}\right)^N.$$

Proof.

$$|I_2(x,t)| \le c\int_{(-\pi,\pi)^d} e^{-\gamma|\varphi|^\alpha t}\,\mathrm{d}\varphi \le c\int_{\Re^d} e^{-\gamma|\varphi|^\alpha t}\,\mathrm{d}\varphi = c_1\frac{1}{t^{\frac{d}{\alpha}}}$$

Further using Lemma 9, we obtain

$$\begin{aligned}
|I_2(x,t)| &= \frac{1}{|x|^2}\left|\int_{(-\pi,\pi)^d} \left(\Delta e^{i(\varphi,x)}\right) e^{-L(\varphi)t}\left(1-\psi(|\varphi|t^{\frac{1}{\alpha}})\right)\,\mathrm{d}\varphi\right| \\
&= \frac{1}{|x|^2}\left|\int_{(-\pi,\pi)^d} e^{i(\varphi,x)}\Delta\left(e^{-L(\varphi)t}(1-\psi(|\varphi|t^{\frac{1}{\alpha}}))\right)\,\mathrm{d}\varphi\right| \\
&\le \frac{Ct^{\frac{2}{\alpha}}}{|x|^2}\int_{(-\pi,\pi)^d} e^{-\gamma|\varphi|^\alpha t}\,\mathrm{d}\varphi \le \frac{Ct^{\frac{2}{\alpha}}}{|x|^2}\int_{\Re^d} e^{-\gamma|\varphi|^\alpha t}\,\mathrm{d}\varphi = c_1\frac{1}{t^{\frac{d}{\alpha}}}\frac{t^{\frac{2}{\alpha}}}{|x|^2}. \qquad (4.28)
\end{aligned}$$

We can apply Δ any number of times to prove the lemma. □

We represent

$$I_3(x,t) = \int_{|\varphi|<2\frac{1}{t^{\frac{1}{\alpha}}}} e^{i(\varphi,x)-L(\varphi)t}\psi(|\varphi|t^{\frac{1}{\alpha}})\,\mathrm{d}\varphi$$

in the form

$$I_3(x,t) = \int_{|\varphi|<2\frac{1}{t^{\frac{1}{\alpha}}}} e^{i(\varphi,x)}\left(e^{-L(\varphi)t} - \sum_{j=0}^{J}\frac{(-L(\varphi)t)^j}{j!}\right)\psi(|\varphi|t^{\frac{1}{\alpha}})\,\mathrm{d}\varphi$$
$$+ \int_{|\varphi|<2\frac{1}{t^{\frac{1}{\alpha}}}} e^{i(\varphi,x)}\sum_{j=0}^{J}\frac{(-L(\varphi)t)^j}{j!}\psi(|\varphi|t^{\frac{1}{\alpha}})\,\mathrm{d}\varphi. \qquad (4.29)$$

Function $f = \left(e^{-L(\varphi)t} - \sum_{j=0}^{J}\frac{(-L(\varphi)t)^j}{j!}\right)\psi(|\varphi|t^{\frac{1}{\alpha}})$ is smooth, and from Lemma 9 it follows that $|D_\varphi^k f| < c_k t^{\frac{|k|}{\alpha}}$ if $|\varphi| < 2t^{-\frac{1}{\alpha}}$. Thus, each integration by parts of the first integral term in (4.29) will provide the factor, which can be estimated by $\frac{t^{\frac{1}{\alpha}}}{|x|}$. We integrate by parts N times, and we choose $J > J_0(N)$ large enough. If we also take into account that the domain of integration does not exceed $Ct^{-d/\alpha}$, we will arrive to the following bound on the the first integral term in (4.29): this term does not exceed

$$C_1 t^{-\frac{d}{\alpha}}\left(\frac{t^{\frac{1}{\alpha}}}{|x|}\right)^N \quad \text{if} \quad J \geq J_0(N) \quad \text{is large enough.}$$

Thus

$$I_3(x,t) = \int_{|\varphi|<2t^{-\frac{1}{\alpha}}} e^{i(\varphi,x)}\sum_{j=0}^{J}\frac{(-L(\varphi)t)^j}{j!}\psi(|\varphi|t^{\frac{1}{\alpha}})\,\mathrm{d}\varphi + \frac{1}{t^{\frac{d}{\alpha}}}O\left(\frac{t^{\frac{1}{\alpha}}}{|x|}\right)^N.$$

We substitute here expression (4.27) for L with $M >> 1$ and add together terms with the same powers of t and $|\varphi|$. This implies that

$$I_3 = \int_{|\varphi|<2t^{-\frac{1}{\alpha}}} e^{i(\varphi,x)}\sum_{j=0}^{J} t^j\left(\sum_{s=0}^{l}|\varphi|^{\alpha j+s}q_{s,j}(\dot{\varphi}) + O\left(|\varphi|^{\alpha j+l+1}\right)\right)\psi\,\mathrm{d}\varphi$$

$$+\frac{1}{t^{\frac{d}{\alpha}}}O\left(\frac{t^{\frac{1}{\alpha}}}{|x|}\right)^{N}. \tag{4.30}$$

Here $q_{0,0} = 1$, $q_{1,0} = -h(\varphi)$, l will be chosen later. By integrating by parts $[\alpha j] + l + 1$ times (where $[\alpha j]$ is the integer part) we get that

$$\left|\int_{|\varphi|<2t^{-\frac{1}{\alpha}}} e^{i(\varphi,x)}t^{j}O(|\varphi|^{\alpha j+l+1})\psi\,\mathrm{d}\varphi\right| \leq C\frac{t^{j}}{|x|^{[\alpha j]+l+1}}\int_{|\varphi|<2t^{-\frac{1}{\alpha}}} \left|\sum D_{\varphi}^{k_1}O(|\varphi|^{\alpha j})D_{\varphi}^{k_2}\psi(|\varphi|t^{\frac{1}{\alpha}})\right|\mathrm{d}\varphi,$$

where summation is taken over all k_1, k_2, $|k_1| + |k_2| = [\alpha j] + l + 1$.
Since $D_{\varphi}^{k_2}\psi(|\varphi|t^{\frac{1}{\alpha}}) \leq ct^{\frac{|k_2|}{\alpha}}$, the latter expression does not exceed

$$C\frac{t^{j+\frac{|k_2|}{\alpha}}}{|x|^{[\alpha j]+l+1}}\int_{|\varphi|<2t^{-\frac{1}{\alpha}}}|\varphi|^{\alpha j+l+1-|k_1|}\,\mathrm{d}\varphi = C_1\frac{t^{j+\frac{|k_1|+|k_2|}{\alpha}-\frac{[\alpha j]+l+1+d}{\alpha}}}{|x|^{[\alpha j]+l+1}}$$
$$\leq C_1\left(\frac{t^{\frac{1}{\alpha}}}{|x|}\right)^{[\alpha j]+l+1}t^{-\frac{l+2}{\alpha}}. \tag{4.31}$$

We needed the integrability here, i.e. we needed $\alpha j + l + 1 - |k_1| > -d$, but this is always true since $\alpha j - [\alpha j] > -d$. We'll take $l = d$. Then (4.31) implies that the remainder terms in (4.30) contribute $o\left(\frac{t^{\frac{1}{\alpha}}}{|x|}\right)^{d+\alpha}\frac{1}{t^{\frac{d}{\alpha}}}$ in I_3.

Other terms in formula (4.30) for I_3 can be reduced to the Fourier transform Φ of homogeneous functions. The arguments similar to those used in the proof of lemma 8, lead the following formula for these terms $F_{s,j}$:

$$F_{s,j} = t^{j-\frac{d}{\alpha}-\frac{\alpha j+s}{\alpha}}\left(\frac{t^{1/\alpha}}{|x|}\right)^{d+\alpha j+s} f_{s,j}(\dot{x}) + t^{j-\frac{d}{\alpha}-\frac{\alpha j+s}{\alpha}}\left(\frac{t^{1/\alpha}}{|x|}\right)^{N},$$

where N is arbitrary large. The main contributions to I_3 comes from $s = 0, j = 1$

(since $f_{0,0} = 0$) and this gives

$$I_3 = \frac{1}{t^{\frac{d}{\alpha}}} \left(\frac{t^{1/\alpha}}{|x|} \right)^{d+\alpha} f(\dot{x}) + \frac{1}{t^{\frac{d}{\alpha}}} o \left(\frac{t^{1/\alpha}}{|x|} \right)^{d+\alpha}$$
$$= \frac{t}{|x|^{d+\alpha}} f(\dot{x}) + o \left(\frac{t}{|x|^{d+\alpha}} \right)$$

Since $p(x,t) = I_2 + I_3$ we now have the asymptotics of $p(x,t)$ in the case b), and this completes the proof of the theorem.

$\square$

Chapter 5. conclusion

This thesis is concerned with the asymptotics behavior of the probability of rare events related to large deviations of the trajectories of random walks, whose jump distributions decay not too fast at infinity and possesses some form of "regular behavior". Typically we consider regularly varying distribution. The first classical results in large deviation theory were obtained for the case of distributions decaying exponentially fast. However, this condition of fast (exponential) decay fails in many applied problems. Consequently, we presented the asymptotic approximation of the transition probability of random walks with heavy tails. We examine the asymptotic probability in both the 1-dimension and n-dimensional cases respectively.

The transition probability $p(t, x, y) = P(x(t) = y|x(0) = x)$ satisfies the heat equation (4.1). We obtained $p(t, x, y)$ as an integral function (4.6) from (4.1) by applying the Fourier transform. We used the asymptotic approximation of (4.5) as a power series to establish the asymptotic behavior of $p(t, x, y)$.

Random walks form a classical object of probability theory, the study of which is of tremendous theoretical interest. They constitute a mathematical model of great importance for applications in mathematical statistics. Computing large deviation probabilities enables one to find, for example, small error probabilities in mathematical statistics, small ruin probabilities in risk theory, small buffer overflow probabilities in queueing theory and so on.

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