COUNTING GENUS ONE PARTITIONS AND PERMUTATIONS

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Abstract. We prove the conjecture by M. Yip stating that counting genus one partitions by the number of their elements and parts yields, up to a shift of indices, the same array of numbers as counting genus one rooted hypermonopoles. Our proof involves representing each genus one permutation by a four-colored noncrossing partition. This representation may be selected in a unique way for permutations containing no trivial cycles. The conclusion follows from a general generating function formula that holds for any class of permutations that is closed under the removal and reinsertion of trivial cycles. Our method also provides a new way to count rooted hypermonopoles of genus one, and puts the spotlight on a class of genus one permutations that is invariant under an obvious extension of the Kreweras duality map to genus one permutations.

Introduction

Noncrossing partitions, first defined in G. Kreweras' seminal paper [10], have a vast literature in areas ranging from probability theory through polyhedral geometry to the study of Coxeter groups. Noncrossing partitions on a given number of elements are counted by the Catalan numbers, if we also fix the number of parts, the answer to the resulting counting problem is given by the Narayana numbers.

A natural generalization of the problem of counting noncrossing partitions is to count partitions of a given genus. The genus of a partition may be defined in terms of a topological representation (see [1] or [17] for example), but there exists also a purely combinatorial definition of the genus of a hypermap (thought of as a pair of permutations, generating a transitive permutation group) that can be specialized first to hypermonopoles, or permutations (that is, hypermaps whose first component is the circular permutation $(1, 2, \ldots, n)$) and then to partitions (that, is permutations whose cycles may be written as lists whose elements are in increasing order). Counting partitions of a given genus seems surprisingly hard, especially considering the fact that, for the closely related hypermonopoles, a general machinery was built by S. Cautis and D. M. Jackson [1] and explicit formulas were given by A. Goupil and G. Schaeffer [6]. It should be noted that for genus zero, i.e., noncrossing partitions, the notions of a hypermonopole (in our language: permutation) and of a partition coincide (see [5, Theorem 1]). Thus it seems hard to believe that the two notions would not only diverge but also give rise to counting problems of different difficulty in higher genus.

Concerning partitions of a fixed genus, a great deal of numerical evidence was collected in M. Yip's Master's thesis, who made the following conjecture [17, Conjecture 3.15]: the number of genus 1 partitions on n elements and k parts is the same as the

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number of genus one permutations of $n-1$ elements having $k-1$ cycles. In this paper we prove this conjecture and provide further insight into the structure of genus 1 partitions and permutations by representing them as four-colored noncrossing partitions.

Our paper is structured as follows. After reviewing some basic terminology and results on the genus of partitions and permutations in Section 1, in Section 2 we develop a theory of representing every permutation of genus 1 by a four-colored noncrossing partition. The four colors form consecutive arcs in the circular order and prescribe a relabeling that results in a permutation of genus at most one. The construction is not unique, but we show that every permutation of genus 1 may be represented in such a way. Moreover, as we show it in Section 3, if the permutation of genus 1 is *reduced* in the sense that it contains no cycle consisting of consecutive elements in the circular order (we call these *trivial cycles*) then we may select a unique four-colored noncrossing partition representation of our permutation which we call the canonical representation. This unicity enables us to count reduced permutations and partitions of genus 1 in Section 4. We only need to account for the possibility of having trivial cycles. In Section 5 we show how to do this, at the level of ordinary generating functions, for any class of permutations that is closed under the removal and reinsertion of trivial cycles. Since genus one permutations and partitions form such classes, we may combine the formula stated in Theorem 5.3 with the generating function formulas stated in Section 4 and obtain the generating function formulas counting genus 1 permutations and partitions with given number of permuted elements and cycles. Since the resulting formulas stated in Theorems 6.1 and 6.5 differ only by a factor of xy, the validity of M. Yip's conjecture is at this point verified. In Section 7 we show how to extract the coefficients from our generating functions to find the number of partitions of genus 1. It should be noted, that our paper thus also provides a new method to count permutations of genus 1, whose number was first found by A. Goupil and G. Schaeffer [6]. The generalized formula stated in Section 7 links the problem of counting genus 1 permutations and partitions to the problem of counting type B noncrossing partitions (at least numerically). The explanation of this connection, together with ideas of possible simplifications and further questions are collected in the concluding Section 8.

1. On the genus of permutations and partitions

1.1. **Hypermaps and permutations.** Since the sixties combinatorialists considered permutations as a useful tool for representing graphs embedded in a topological surface. One of the main objects in this representation is the notion of a hypermap.

A hypermap is a pair of permutations (σ, α) on a set of points $\{1, 2, \ldots, n\}$, such that the group they generate is transitive, meaning that the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{i, \alpha(i)\}, \{i, \sigma(i)\}$ is connected.

It was proved (in [9]) that the number $g(\sigma, \alpha)$ associated to a hypermap and defined by:

$$n + 2 - 2g(\sigma, \alpha) = z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma), \tag{1.1}$$

where $z(\alpha)$ denotes the number of cycles of the permutation α, is a non-negative integer. It is called the genus of the hypermap.

Taking for σ the circular permutation ζ_n such that for all i, $\zeta_n(i) = i + 1$ (where $n + 1$ means 1) allows to define the genus of a permutation $\alpha \in \mathrm{Sym}\,(n)$ as that of the hypermap (ζ_n, α). Notice that the pair (ζ_n, α) generates a transitive group for any α since $z(\zeta_n) = 1$; so that we may use the following definition:

Definition 1.1. *The genus of a permutation α is the non-negative integer $g(\alpha)$ given by:*

$$n + 1 - 2g(\alpha) = z(\alpha) + z(\alpha^{-1}\zeta_n).$$

Notice that hypermaps of the form (ζ_n, α) are often called hypermonopoles (for instance in [1] or [17]). A different definition of the genus was given in [4], where the genus $h(\alpha)$ of the permutation α is defined as the genus of the hypermap $(\zeta_n, \alpha^{-1}\zeta_n\alpha)$. In this definition a permutation is of genus 0 if and only if it is a power of ζ_n; in ours permutations of genus 0 correspond to noncrossing partitions, a central object in combinatorics.

1.2. **Partitions of the set $\{1, 2, \ldots, n\}$.** To a partition $P = (P_i)_{i=1,k}$ of the set $\{1, 2, \ldots n\}$ is associated the permutation α_P which has k cycles, each one corresponding to one of the P_i written with the elements in increasing order. This allows to define the genus of the partition P as that of the permutation α_P.

It was shown in [5, Theorem 1] that a permutation α is of genus 0, if and only if there exists a noncrossing partition P such that $\alpha = \alpha_P$.

A noncrossing partition may be drawn as a circle on which we put the points $1, 2, \ldots, n$ in clockwise order and parts of size $p > 2$ are represented with p-gons inscribed in the circle, parts of size 2 by segments, and parts of size 1 by isolated points.

The partition $P = (\{1, 5, 7, 8\}, \{2, 4\}, \{3\}, \{6\})$ is represented in Figure 1 below.

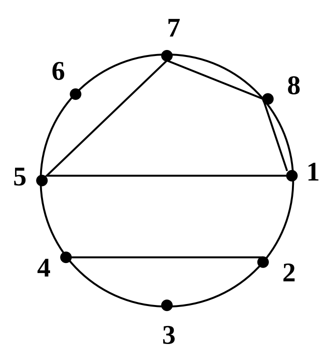

FIGURE 1. The noncrossing partition P

1.3. **The genus and the cycle structure.** Since the genus of a permutation α is a function of $z(\alpha)$, the number of its cycles, in the sequel we will consider permutations as products of their cycles, study their structure, and the effect of minor changes on the cycle structure. In particular, we will be interested in the change of the genus when we compose a permutation with a single *transposition*. A transposition $\tau \in \mathrm{Sym}\,(n)$, exchanging the two points i, j, will be denoted by $\tau = (i, j)$. It has $n-2$ cycles of length 1 and one of length 2, hence $z(\tau) = n - 1$. Note that we compose permutations right to left, i.e., we define the product $\alpha\beta$ of two permutations as the permutation which sends i into $\alpha(\beta(i))$.

We will often use the following Lemma:

Lemma 1.2. *The number of cycles of the products $\tau\alpha$ and $\alpha\tau$ of a permutation α and a transposition $\tau = (i, j)$ differs from the number of cycles of α by 1. The sign of the change depends on whether i and j belong to the same cycle of α or not. We have*

$$z(\tau\alpha) = z(\alpha\tau) = \begin{cases} z(\alpha)+1 & \text{if } i \text{ and } j \text{ belong to the same cycle of } \alpha; \\ z(\alpha)-1 & \text{if } i \text{ and } j \text{ belong to different cycles of } \alpha. \end{cases}$$

Definition 1.3. *Two cycles in a permutation α are* crossing *if there exists two elements a, a' in one of them and b, b' in the other such that $a < b < a' < b'$.*

Observe that if such elements exist they may be taken such that $a' = \alpha(a)$ and $b' = \alpha(b)$.

An element i of $1, 2, \ldots, n$ is a *back point* of the permutation α if $\alpha(i) < i$ and $\alpha(i)$ is not the smallest element in its cycle (i. e. there exist $k > 1$ such that $\alpha^k(i) < \alpha(i)$).

Definition 1.4. *A* twisted cycle *in a permutation α is a cycle $(b_1, b_2, \ldots, b_p)$ containing a back point.*

The genus of a permutation may be determined by counting back points as the following variant of [2, Lemma 5] shows.

Lemma 1.5. *For any permutation $\alpha \in \mathrm{Sym}\,(n)$, the sum of the number of back points of the permutation α and the number of those of $\alpha^{-1}\zeta_n$ is equal to $2g(\alpha)$.*

Proof. As usual, for a permutation $\alpha \in \mathrm{Sym}\,(n)$, let $\mathrm{EXC}(\alpha)$ denote the set of *excedances* of α, i.e., the set of elements i such that $\alpha(i) > i$. The number of back points of α is then $n - |\mathrm{EXC}(\alpha)| - z(\alpha)$. After replacing $2g(\alpha)$ with its expression in Definition 1.1, our lemma is equivalent to

$$|\mathrm{EXC}(\alpha)| + |\mathrm{EXC}(\alpha^{-1}\zeta_n)| = n - 1.$$

To prove this equation observe first that, for all i satisfying $i \neq \alpha^{-1}(1)$, the relation $i \in \mathrm{EXC}(\alpha)$ is equivalent to $\alpha(i) - 1 \notin \mathrm{EXC}(\alpha^{-1}\zeta_n)$. Thus the number of excedances of α in the set $\{\alpha^{-1}(2), \ldots, \alpha^{-1}(n)\}$ plus the number of excedances of $\alpha^{-1}\zeta_n$ in the set $\{1, \ldots, n-1\}$ is $n - 1$. Finally $\alpha^{-1}(1)$ is not an excedance of α and n is not an excedance of any permutation in $\mathrm{Sym}\,(n)$. $\square$

Notice that a permutation is associated to a partition if and only if it contains no twisted cycle, moreover the partition and the associated permutation are of genus 0 if and only if there are no crossing cycles. Noncrossing partitions were extensively studied (see for instance [13]).

2. Genus one permutations and four-colored noncrossing partitions

We define a four-coloring of a noncrossing partition of the set $\{1, 2, \ldots, n\}$ as a partitioning of the n points on the circle into four arcs denoted A, B, C, D in clockwise order where A is the arc containing the point 1 and in which C is only arc allowed to contain no point. We will denote by $\gamma = (A, B, C, D)$ such a 4-coloring. Equivalently a four-coloring may be defined by 4 integers defining the numberings of the points in the four arcs. These are $1 \leq i < j \leq k < \ell \leq n$, giving

$$\begin{array}{ll} A = \{\ell+1, \ldots, n, 1, \ldots, i\}, & B = \{i+1, \ldots, j\}, \\ C = \{j+1, \ldots, k\}, & D = \{k+1, \ldots, \ell\}, \end{array} \tag{2.1}$$

where C is empty when $j = k$. In this notation, $A = \{1, \ldots, i\}$ holds when $\ell = n$.

Definition 2.1. *We call the sequence (i, j, k, ℓ), marking the right endpoints of the color sets in (2.1), a* sequence of coloring points *of the partition P.*

To any four-colored noncrossing partition (P, γ) (where $\gamma = (A, B, C, D)$) we associate a permutation $\alpha = \Phi(P, \gamma)$ in which cycles are obtained from the parts of P by renumbering the points in the following way:

We leave the numbering of the points in A unchanged and we continue labeling in such a way that the elements of A are followed by the points in D, then by the points in C, and finally by the points in B. Within each color set, points are numbered in clockwise order. Thus the elements of A are numbered with $\ell+1, \ell+2, \ldots, n, 1, 2, \ldots i$, the elements of D are numbered from $i+1$ to $i+\ell-k$, the elements of C are numbered from $i+\ell-k+1$ to $i+\ell-j$ an the elements of B are numbered from $i+\ell-j+1$ to ℓ. After introducing

$$a = i,\ b = i+\ell-k,\ c = i+\ell-j, \quad \text{and} \quad d = \ell, \tag{2.2}$$

we obtain that the color sets, in terms of the relabeled elements, are given by

$$\begin{aligned}
A &= \begin{cases} \{1, 2, \ldots a, d+1, \ldots, n\} & \text{if } d \neq n, \\ \{1, 2, \ldots, a\} & \text{otherwise;} \end{cases} \\
B &= \{c+1, c+2, \ldots, d\}; \quad D = \{a+1, a+2, \ldots, b\}; \\
C &= \begin{cases} \{b+1, b+2, \ldots, c\} & \text{if } c \neq b, \\ \emptyset & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.3}$$

Let us also note for future reference that the linear map taking (i, j, k, ℓ) into (a, b, c, d) is its own inverse, i.e., we have

$$i = a,\ j = a+d-c,\ k = a+d-b \quad \text{and} \quad \ell = d. \tag{2.4}$$

Once the points are renumbered, each cycle of α is obtained from a part $P_q = \{x_1, x_2, \ldots x_p\}$ of P by writing the numbering of the corresponding points $x_1, x_2, \ldots x_p$, where the x_i's are in clockwise order.

For the example shown in Figure 2 we obtain the following permutation of genus 1:

$$\alpha \;=\; \Phi(P, \gamma) = (1, 4, 3, 8)(2, 7)(5)(6)$$

In the sequel it will be convenient to say that a point p has color X for $X = A, B, C, D$ if $p \in X$, a part P_q will be unicolored, bicolored, three-colored or four-colored depending on the number of different colors its points have.

Remark 2.2. A unicolored part of a noncrossing partition P gives rise to a cycle in $\Phi(P, \gamma)$ which does not cross any other cycle and is not twisted. A bicolored part with points in two different colors X and Y is not twisted but it crosses any cycle coming from a part that has points of color X as well as at least one point whose color is neither X nor Y. A bicolored part with points of color X and Y does not cross a any part that is contained in or disjoint from $X \cup Y$. A three or four-colored part gives rise to a twisted cycle.

The main point in this section is the following characterization:

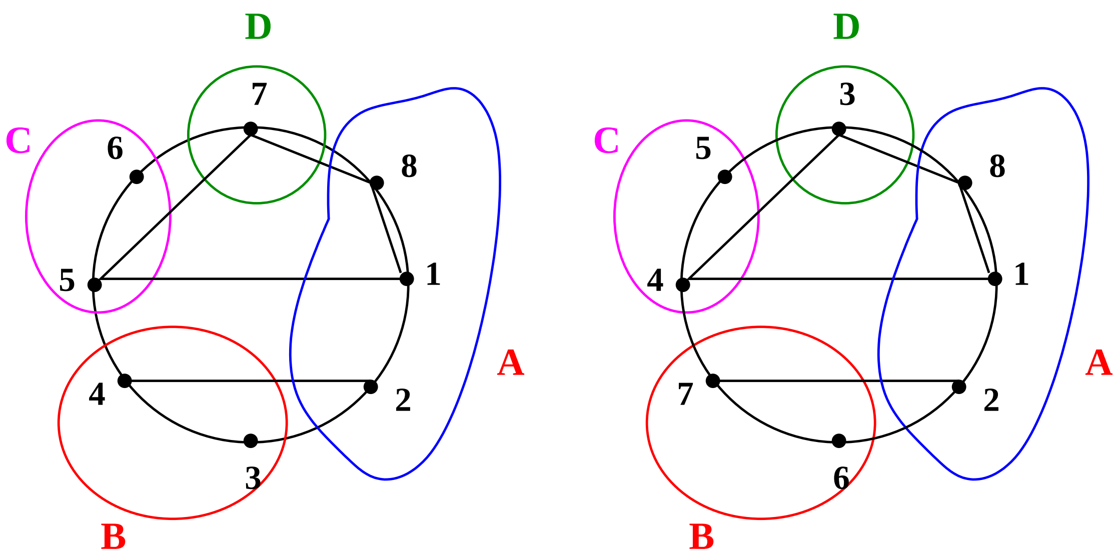

FIGURE 2. A four-coloring of P and the induced renumbering of points

Theorem 2.3. *If (P,γ) is a four-colored noncrossing partition then $\Phi(P,\gamma)$ is a permutation of genus 0 or 1. It is of genus 1 if and only if at least one of these two conditions is satisfied:*

(1) *There exists a part P_q which is three or four-colored.*
(2) *There exists two parts P_q, P_r which are two colored and share a common color, more precisely there are three different colors X, Y, Z such that*

$$P_q \cap X \neq \emptyset, \quad P_q \cap Y \neq \emptyset, \quad P_q \subseteq X \cup Y \quad \text{and} \quad P_r \cap X \neq \emptyset, \quad P_r \cap Z \neq \emptyset, \quad P_r \subseteq X \cup Z.$$

Proof. Let i, j, k, ℓ define the four-coloring γ and let β be the permutation associated to the partition P, set $\alpha = \Phi(P,\gamma)$. The renumbering of the points around the circle may be considered in two ways:

The first way is conjugation. Consider the permutation ϕ that takes each i into its new label after the renumbering operation. We then have $\alpha = \phi\beta\phi^{-1}$. Note that ϕ is given by the coloring points (i, j, k, ℓ) via the formula

$$\phi(x) = \begin{cases} x & \text{if } x \in A; \\ x + \ell - j & \text{if } x \in B; \\ x + i + \ell - j - k & \text{if } x \in C; \\ x + i - k & \text{if } x \in D. \end{cases} \tag{2.5}$$

Although this formula is unimportant for this proof, we will have good use of it later in the proof of the converse of our present statement. Now let $\theta = \phi\zeta_n\phi^{-1}$, since conjugation does not change the number of cycles we have:

$$g(\zeta_n, \beta) = g(\theta, \alpha) = 0. \tag{2.6}$$

Since θ has only one cycle, just like ζ_n, the above equation, together with formula (1.1) yields

$$n + 1 - z(\alpha) = z(\alpha^{-1}\theta). \tag{2.7}$$

The second way is multiplication by transpositions. It is easy to check that

$$\theta = (1, 2, \dots, a, c+1, \dots, d, b+1, \dots, c, a+1, \dots, b, d+1, \dots, n), \tag{2.8}$$

where $a = i$, $b = i + \ell - k$, $c = i + \ell - j$ and $d = \ell$, hence $\theta = \zeta_n(a, c)(b, d)$. We are now able to compute the genus of the permutation α. By Definition 1.1 we have

$$2g(\alpha) = n + 1 - z(\alpha) + z(\alpha^{-1}\zeta_n).$$

Using (2.7) we may rewrite the last equation as

$$2g(\alpha) = z(\alpha^{-1}\theta) - z(\alpha^{-1}\zeta_n).$$

But since $\alpha^{-1}\theta$ is obtained from $\alpha^{-1}\zeta_n$ by multiplying by two transpositions, by Lemma 1.2, the difference of their number of cycles is 0, 2 or -2. Since the genus is a non-negative integer we have that $g(\alpha)$ is 0 or 1. If any of the conditions given above are satisfied then α has a twisted cycle or two crossing cycles hence it cannot be of genus 0, ending the proof, if none of them is satisfied then α has no twisted cycle and no two crossing cycles, it is then of genus 0 (a permutation of a noncrossing partition). □

To state a converse of Theorem 2.3 we introduce the following notion:

Definition 2.4. *Let α be a permutation of genus 1. We say that the sequence of integers (a, b, c, d) is a* sequence of separating points *for α if the permutation $\theta = \zeta_n(a, c)(b, d)$ is such that the genus of the hypermap (θ, α) is zero and*

$$a < b \le c < d. \tag{2.9}$$

Notice that (2.9) implies that θ is a circular permutation. Equations (2.6) and (2.8) have the following consequence.

Remark 2.5. If a permutation α of genus 1 is represented as $\alpha = \Phi(P, \gamma)$ by a four-colored noncrossing partition (P, γ) then the sequence of coloring points (i, j, k, ℓ) gives rise to the sequence of separating points (a, b, c, d) given by (2.2).

Proposition 2.6. *Let α be a permutation of genus 1 on n elements that has a sequence of separating points (a, b, c, d). Then there is a noncrossing partition P and a four-coloring $\gamma = (A, B, C, D)$ representing α as $\alpha = \Phi(P, \gamma)$ whose sequence of coloring points (i, j, k, ℓ) is obtained from (a, b, c, d) via (2.4).*

Proof. Since $\theta = \zeta_n(a, c)(b, d)$ is circular, there is a permutation ϕ satisfying $\phi\zeta_n\phi^{-1} = \theta$. We make this map ϕ unique by requiring $\phi(1) = 1$. It is easy to verify that ϕ is given by (2.5) for the sequence (i, j, k, l) given by (2.4). The permutation $\beta = \phi^{-1}\alpha\phi$ satisfies

$$g(\zeta_n, \beta) = g(\phi^{-1}\theta\phi, \phi^{-1}\alpha\phi) = g(\theta, \alpha) = 0,$$

hence β determines a noncrossing partition P. As a consequence of (2.1) and (2.5), the four-coloring γ associated to (i, j, k, ℓ) satisfies $\alpha = \Phi(P, \gamma)$. □

Definition 2.7. *We call the representation described in Proposition 2.6* the four-colored noncrossing partition representation induced by the sequence of separating points *(a, b, c, d).*

Now we are ready to state the converse of Theorem 2.3.

Theorem 2.8. *For any permutation α of genus 1, there exists a noncrossing partition P and a four-coloring γ such that $\alpha = \Phi(P, \gamma)$.*

Proof. By Proposition 2.6 it suffices to show that every permutation of genus 1 has a sequence (a, b, c, d) of separating points. Let α be a permutation of genus 1, then the permutation $\alpha' = \alpha^{-1}\zeta_n$ is also of genus 1, thus α' has two crossing cycles or a twisted cycle or both.

(1) If α' has two crossing cycles then one of these cycle contains two points a, c and the other one two points b, d such that $a < b < c < d$.
By (2.9), $\theta = \zeta_n(a, c)(b, d)$ is circular. Moreover $\alpha^{-1}\theta$ is obtained from $\alpha^{-1}\zeta_n$ by multiplying it by two transpositions exchanging elements belonging to the same cycle, hence $z(\alpha^{-1}\theta) = z(\alpha^{-1}\zeta_n) + 2$. By the definition of the genus, since $z(\theta) = z(\zeta_n)$, we get $g(\theta, \alpha) = g(\alpha) - 1 = 0$.

(2) If α' has a twisted cycle, this can be written $(a, x_1, \cdots, x_p, d, b, y_1, \cdots y_q)$, where a is the smallest element of the cycle and $d > b$, giving $a < b < d$. Consider the two transpositions (a, b) and (b, d) It easy to check that the product $\theta = \zeta_n(a, b)(b, d)$ is equal to: $(1, 2, \cdots a, b+1, \cdots d, a+1, \cdots b, d+1, \cdots n)$. Moreover, the permutation $\alpha'(a, b)(b, d)$ has the same cycles as α' except the one containing a, b, d which is broken into three cycles:
$$(a, y_1, \cdots y_q)\ (b)\ (d, x_1, \cdots, x_p),$$
showing that again
$$z(\alpha^{-1}\theta) = z(\alpha^{-1}\zeta_n) + 2$$
and $g(\theta, \alpha) = g(\zeta_n, \alpha) - 1 = 0$ hold.

We obtained that, in the first case (a, b, c, d), and in the second case (a, b, b, d), is a sequence of separating points for α. □

It is easy to detect in a four-colored noncrossing partition representation of a permutation of genus 1 whether it is a partition, or whether it has twisted cycles, as we will see in the following observations.

Corollary 2.9. *A permutation α of genus 1 is a partition if and only if it may be represented by a four-colored noncrossing partition (Q, γ) that has no three or four-colored part and has at least two two-colored parts.*

Indeed, a three or four-colored part would give rise to a twisted cycle which partition can not have. Without twisted cycles, a permutation of genus 1 must have a pair of crossing cycles which can only be represented by two-colored parts. To state our next observation, we introduce the notion of simply and doubly twisted cycles.

Remark 2.10. For future reference we also note that every genus 1 partition $\alpha \in \mathrm{Sym}(n)$ has a three-colored non-crossing partition representation, that is, a four-colored representation with $C = \emptyset$. Indeed, since α does not have any back point, by Lemma 1.5, $\alpha^{-1}\zeta_n$ must have two back points. We may use the construction presented in the second case of the proof of Theorem 2.8 to construct a three-colored noncrossing partition. A variant of this observation was also made in [17, p. 63].

Definition 2.11. *A cycle of α is simply twisted if contains exactly one back point and it is doubly twisted if it has two back points.*

Remark 2.12. In a four-colored noncrossing partition representation of a permutation of genus 1, three colored parts correspond to simply twisted cycles and four-colored parts correspond to doubly twisted cycles.

Proposition 2.13. *In a permutation α of genus 1, all cycles are either not twisted or simply or doubly twisted. Moreover, exactly one of the following assertions is satisfied:*

(1) *α has no twisted cycle, hence it corresponds to a partition;*
(2) *α has a unique simply twisted cycle;*
(3) *α has a unique doubly twisted cycle;*
(4) *α has two simply twisted cycles.*

There is an example of a permutation of genus 1 of each of the above four types.

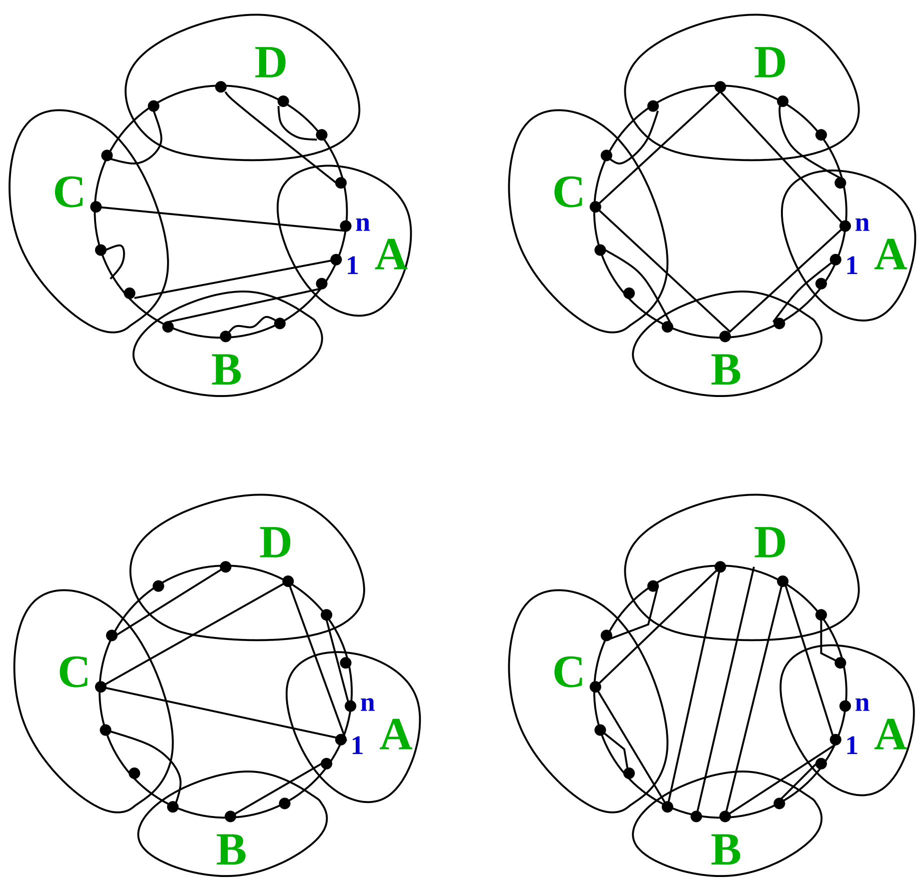

FIGURE 3. The four types of genus 1 permutations

Proof. By Lemma 1.5, a permutation of genus 1 may have at most two back points. If α has no back points then it is a partition. If it has one back point then it has a unique simply twisted cycle. If it has two back points, then these are either on the same (doubly twisted) cycle, or on two separate (simply twisted cycle). An example of a permutation of each type is sketched using a four-colored noncrossing partition representation in Figure 3. □

3. Reduced permutations and partitions

Definition 3.1. *A* trivial *cycle in a permutation is a cycle consisting of consecutive points on the circle,* i. e. *a cycle $C_i = (i, i+1, \ldots, i+p)$ where sums are taken modulo n. A permutation is* reduced *if it contains no trivial cycle.*

Lemma 3.2. *Let θ and α be two permutations in* $\mathrm{Sym}\,(n)$ *such that θ is circular and $g(\theta, \alpha) = 0$. If an integer x satisfies*

$$\alpha(x) = \theta^k(x) \qquad \text{for} \quad 1 < k < n$$

then there exists a cycle of α consisting of consecutive points in the sequence

$$\theta(x), \theta^2(x), \ldots, \theta^{k-1}(x)$$

Proof. Use conjugation by a permutation ϕ such that $\phi\theta\phi^{-1} = \zeta_n$. Then the statement follows by repeated use of the following, trivial observation: if a noncrossing partition contains a part $a_1 < a_2 < \cdots < a_p$ such that one of the a_i's satisfies $a_{i+1} > a_i + 1$ then there is another part contained in the set $\{a_i + 1, a_i + 2, \ldots, a_{i+1} - 1\}$. Applying the same observation repeatedly, we end up with a part consisting of consecutive integers greater than a_i and less than a_{i+1}. □

As a consequence of Lemma 3.2, a permutation α of genus 1 is reduced if and only if each of its cycles either crosses another one or it is twisted. Indeed, by Remark 2.2, a cycle that does not cross any other cycle and is not twisted corresponds to a unicolored part in a four-colored noncrossing partition representing α and, by Lemma 3.2, the same color set contains a part consisting of consecutive points, which represents a trivial cycle. Thus the representation of a reduced α can not have unicolored parts.

We now define for a reduced permutation α of genus 1 a canonical sequence of separating points and the canonical representation of it as a four-colored noncrossing partition.

Definition 3.3. *Let α be a reduced permutation of genus* 1*. The* canonical sequence of separating points (a, b, c, d) *of α is defined as follows:*

1. *a is the smallest integer such that $\alpha(a) \neq a + 1$;*
2. *b is the smallest integer satisfying $b > a$ and such that either $\alpha(b) > \alpha(a)$ or $\alpha(b) \leq a$ holds;*
3. *$c = \alpha(a) - 1$;*
4. *$d = n$ if $\alpha(b) = 1$ and $d = \alpha(b) - 1$ otherwise.*

We call the four-colored noncrossing partition representation induced by the canonical sequence of separating points the canonical representation *of α.*

In the proof of Proposition 3.5 below we will show that the canonical sequence of separating points exists, it is unique, and it is indeed a sequence of separating points, giving rise to a four-colored noncrossing partition representation. Our proof relies on the following lemma.

Lemma 3.4. *Let α be a permutation of* $\mathrm{Sym}\,(n)$ *such that for some a satisfying $a+1 < \alpha(a)$, the set $X_1 = \{a+1, a+2, \dots, \alpha(a)-1\}$ is a union of cycles of α. Then α may be split into two permutations α_1 acting on X_1 and α_2 acting on $X_2 = \{1, 2, \dots, n\} \setminus X_1$ such that*

$$g(\alpha) = g(\alpha_1) + g(\alpha_2)$$

Proof. Let n_1 be the number of elements of X_1 and n_2 be that of X_2. Consider the transposition τ exchanging a and $c = \alpha(a)-1$, then $\zeta_n\tau$ has two cycles of lengths n_1 and n_2 respectively, permuting the elements of X_1 and X_2 respectively. Since $\alpha^{-1}\zeta_n(c) = a$, we have:

$$z(\alpha^{-1}\zeta_n\tau) = z(\alpha^{-1}\zeta_n) + 1$$

Moreover

$$z(\alpha) = z(\alpha_1) + z(\alpha_2) \text{ and } z(\alpha^{-1}\zeta_n\tau) = z(\alpha_1^{-1}\zeta_{n_1}) + z(\alpha_2^{-1}\zeta_{n_2})$$

where $\zeta_{n_1} = (a+1, a+2, \dots, \alpha(a)-1)$ and ζ_{n_2} is the analogous circular permutation on X_2. Computing the genus of α_1 and α_2 we get:

$$2g(\alpha_1) = n_1 - z(\alpha_1) - z(\alpha_1^{-1}\zeta_{n_1}) - 1 \text{ and } 2g(\alpha_2) = n_2 - z(\alpha_2) - z(\alpha_2^{-1}\zeta_{n_2}) - 1$$

Adding the two equations and using the preceding relations we get:

$$2(g(\alpha_1)) + g(\alpha_2)) = n_1 + n_2 - z(\alpha) - z(\alpha^{-1}\zeta_n\tau) - 2$$

Since $n_1 + n_2 = n$ and $z(\alpha^{-1}\zeta_n\tau) = z(\alpha^{-1}\zeta_n) + 1$ we obtain the expected relation between the genuses of $\alpha, \alpha_1, \alpha_2$. □

Proposition 3.5. *Every reduced permutation of genus* 1 *of n elements has a unique canonical sequence (a, b, c, d) of separating points, that induces a four-colored noncrossing partition representation.*

Proof. It is easy to see that an element a as defined above exists since if $\alpha(i) = i+1$ for all $i < n$ then α is of genus 0. An element $b > a$ such that $\alpha(b) > \alpha(a)$ or $\alpha(b) \le a$ exists also since there is at least an element $j > a$ such that $\alpha(j) = 1$. The minimality requirement stated in conditions (1) and (2) guarantees the uniqueness of a and b. Afterward, c and d are given by modulo n subtractions that can be performed in exactly one way. It remains to show that (a, b, c, d) is a sequence of separating points. To show that $a < b \le c < d$ holds, notice that if for all i such that $a < i < \alpha(a)$ we have $\alpha(i) < \alpha(a)$ then one of α_1 or α_2 given in Lemma 3.4 will have genus 0 and hence contain a trivial cycle, contradicting the fact that α is reduced. To show that $g(\zeta_n(a, c)(b, d), \alpha) = 0$, observe first that $a = \alpha^{-1}\zeta_n(c)$ and c belong to the same cycle of $\alpha^{-1}\zeta_n$, similarly $b = \alpha^{-1}\zeta_n(d)$ and d belong to the same cycle of $\alpha^{-1}\zeta_n$. Moreover, by $a = \alpha^{-1}\zeta_n(c)$, the cycle decomposition of $\alpha^{-1}\zeta_n(a, c)$ is obtained by deleting a from the cycle of $\alpha^{-1}\zeta_n$ containing it and turning it into a fixed point. Thus b and d are also on the same cycle of $\alpha^{-1}\zeta_n(a, c)$. Using Lemma 1.2 twice we obtain that $z(\alpha^{-1}\zeta_n(a, c)(b, d)) = z(\alpha^{-1}\zeta_n) + 2$. □

Proposition 3.6. *Let $\alpha = \Phi(\beta, \gamma)$ be the representation of the reduced permutation α of genus* 1 *induced by its canonical sequence of separating points (a, b, c, d). This representation has the following properties:*

(1) *$a < b \le c < d$ and $\alpha(a) \equiv c+1, \alpha(b) \equiv d+1 \mod n$.*

(2) *If x and $\alpha(x)$ are in the same subset A, B, C, or D then $\alpha(x) \equiv x + 1 (\mathrm{mod}\ n)$.*
(3) *There is no cycle of α containing elements in both A and D except the one containing b and $d + 1$.*
(4) *There is no cycle of α containing elements in both B and D except if this cycle is twisted and contains $b \in D, d + 1 \in A$ and an element $x \in B$.*

Proof. (1) is a direct consequence of Definition 3.3 and Proposition 3.5.

(2) Comes from the fact that if x and $\alpha(x)$ are in the same color class X, and $\alpha(x) \not\equiv x+1$ then, by Lemma 3.2, there is a trivial cycle of β which contains consecutive points in X, giving rise to a trivial cycle of α thus contradicting the fact that the permutation α is reduced.

To prove (3) observe that if there is a cycle bicolored by A and D then there is an element x of this cycle such that $x \in D$ and $\alpha(x)$ in A. But all elements in D are less than or equal to b, so that $x \neq b$ would contradict the fact that b was chosen as the smallest such that $\alpha(b) \in A \cup B$.

For (4), if there is a cycle containing elements in B and D this implies that there is an element x in D such that $\alpha(x) \in A \cup B$. As above $x = b$. And the cycle contains elements in A, B, D hence it is twisted. □

Proposition 3.7. *Let α be a reduced permutation of genus 1, represented as $\alpha = \Phi(\beta, \gamma)$ by a four-colored noncrossing partition. If this representation satisfies the properties stated in Proposition 3.6 then it is the representation induced by the canonical sequence of separating points.*

Proof. Let (A, B, C, D) denote the sequence of color sets of the coloring induced by the canonical sequence (a, b, c, d) of separating points via (2.3). Suppose that there exists another representation induced by the the sequence of separating points (a', b', c', d') satisfying the properties stated in in Proposition 3.6, and let (A', B', C', D') denote the the sequence of sets of colors in the induced coloring. Then $a = a'$ since both are the smallest x such that $\alpha(x) \neq x + 1$, this gives also $c = c' = \alpha(a) - 1$. Observe next that $b \leq b'$ since $\alpha(b') \equiv d' + 1 \mod n$ satisfies $\alpha(b') = 1$ or $\alpha(b') > \alpha(a)$, thus it is not in the interval $[a + 1, \alpha(a)]$ and b is the smallest integer with this property. It suffices to show that b can not be strictly less than b', afterward $d = d'$ follows from the fact that both are congruent to $\alpha(b) - 1$ modulo n.

Assume, by way of contradiction, that $b < b'$. As a consequence of $a < b < b'$, we must have $b \in D'$ since all the elements in B' and C' are greater than b' and those in $A' \cap [1, b']$ are less than a hence satisfy $\alpha(x) = x + 1$. By property (3) in Proposition 3.6 we can not have $\alpha(b) \in D'$ and by property (4) we can not have $\alpha(b) \in A'$ either. If $\alpha(b) \in D'$ then, by property (2), we must have $d + 1 = \alpha(b) = b + 1$, in contradiction with $b < d$. Finally, if $\alpha(b) \in C' = [b' + 1, c'] \subseteq [a' + 1, \alpha(a') - 1]$ then $\alpha(b)$ is not outside the interval $[a + 1, \alpha(a)]$, in contradiction with the definition of a canonical sequence of separating points. □

Corollary 3.8. *The canonical four-colored noncrossing partition representation of a reduced permutation α of genus 1 may be equivalently defined by requiring that the sequence of separating points inducing it must satisfy the four conditions of Proposition 3.6.*

4. Counting reduced partitions and permutations

4.1. Counting reduced partitions of genus 1.

Lemma 4.1. *A reduced partition of genus* 1 *having* k *parts is determined by a subset of* $2k$ *integers in* $\{1, 2, \ldots, n\}$ *and a sequence of four non-negative integers whose sum is* $k-2$.

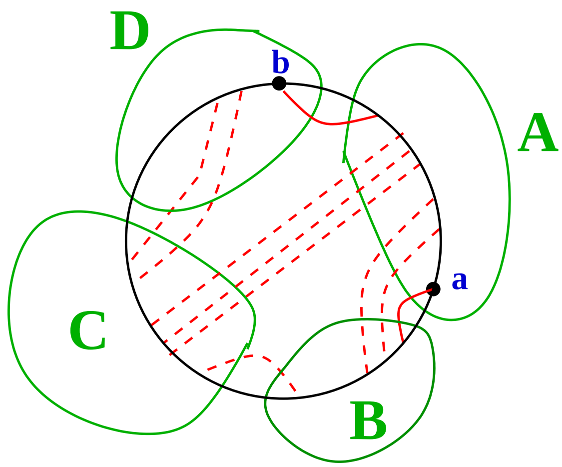

Figure 4. A reduced partition

Proof. By Corollary 2.9 and as a consequence of Lemma 3.2, in the canonical representation of a reduced partition each part is bicolored and contains exactly two points x_i and y_i such that $\alpha(x_i) \neq x_i + 1$ and $\alpha(y_i) \neq y_i + 1$. There is exactly one part bicolored by A and B that contains $a, c+1$ and exactly one part bicolored A, D that contains $b, d+1$. There is no other part bicolored by A, D and there is no part bicolored by D, B. The partition is determined by the elements x_i, y_i and by the numbers of the parts bicolored by (A, B), (A, C), (B, C), or (C, D), respectively, see Figure 4. □

Theorem 4.2. *The number* $r_0(n, k)$ *of reduced partitions of genus* 1*, of the set* $\{1, \ldots, n\}$*, having* k *blocks is*

$$r_0(n, k) = \binom{n}{2k}\binom{k+1}{3}.$$

Moreover, the ordinary generating function of these partitions is given by

$$R_0(x, y) = \sum_{n,k \geq 0} r_0(n, k) x^n y^k \;=\; \frac{y^2 x^4 (1-x)^3}{\left((1-x)^2 - yx^2\right)^4}. \tag{4.1}$$

Proof. To obtain the first part, observe that there are $\binom{n}{2k}$ ways to select the $2k$ integers and that the number $k-2$ may be written in $\binom{k+1}{3}$ ways as the sum of four non-negative integers.

To obtain a formula for the generating function, we will use the following variant of the binomial series formula for $(1-u)^{-m-1}$:

$$\sum_{n=m}^{\infty}\binom{n}{m}u^n = \frac{u^m}{(1-u)^{m+1}} \quad \text{holds for all } m\in\mathbb{N}. \tag{4.2}$$

Using this formula first for $u=x$ and $m=2k$ we obtain

$$\begin{aligned} R_0(x,y) &= \sum_{k\geq 2}\binom{k+1}{3}y^k\sum_{n\geq 2k}\binom{n}{2k}x^n = \sum_{k\geq 2}\binom{k+1}{3}y^k\frac{x^{2k}}{(1-x)^{2k+1}}\\ &= \frac{(1-x)}{yx^2}\sum_{k\geq 2}\binom{k+1}{3}\left(\frac{yx^2}{(1-x)^2}\right)^{k+1}. \end{aligned}$$

(The last part is a product of formal Laurent series.) Substituting now $u=yx^2/(1-x)^2$ and $m=3$ into (4.2) yields

$$R_0(x,y) = \frac{(1-x)}{yx^2}\cdot\frac{\left(\frac{yx^2}{(1-x)^2}\right)^2}{\left(1-\frac{yx^2}{(1-x)^2}\right)^4}.$$

Simplifying by the factors of $(1-x)$ yields the stated formula. □

Substituting $y=1$ in (4.1) allows us to find the ordinary generating function of all reduced genus one partitions of a given size, regardless of the number of blocks.

Corollary 4.3. *Let $r_0(n)$ be the number of all reduced genus 1 partitions on $\{1,\dots,n\}$. Then the generating function $R_0(x)=\sum_{n\geq 4} r_0(n)x^n$ is given by*

$$R_0(x) = x^4\frac{(1-x)^3}{(1-2x)^4}.$$

As a consequence, the ordinary generating function of the sequence $r_0(4), r_0(5),\dots$ is $(1-x)^3/(1-2x)^4$. This sequence is listed as sequence A049612 in the Encyclopedia of Integer Sequences [12]. It is noted in [12] that the same numbers appear as the third row of the array given as sequence A049600. Essentially the same array is called the array of *asymmetric Delannoy numbers* $\widetilde{d}_{m,n}$ in [7] where they are defined as the number of lattice paths from $(0,0)$ to $(m,n+1)$ having steps $(x,y)\in\mathbb{N}\times\mathbb{P}$. (Here $\mathbb{P}$ denotes the set of positive integers.) Using [7, Lemma 3.2], it is easy to show the following formula:

$$r_0(n) = \widetilde{d}_{3,n-4} = 2^{n-4}+3\binom{n-4}{1}2^{n-5}+3\binom{n-4}{2}2^{n-6}+\binom{n-4}{3}2^{n-7}. \tag{4.3}$$

4.2. **Counting reduced permutations of genus 1.**

Theorem 4.4. *The number of reduced permutations of genus 1 of* Sym (n) *with k cycles is equal to:*

$$r_*(n,k) = \binom{n+2}{2k+2}\binom{k+1}{3}+\binom{n+1}{2k+2}\binom{k+1}{2}.$$

More precisely, for $j = 0, 1, 2$, the number $r_j(n,k)$ of reduced permutations of genus 1 *of* Sym (n) *with j back points and k cycles is given by the following formulas:*

$$r_0(n,k) = \binom{n}{2k}\binom{k+1}{3}, \quad r_2(n,k) = \binom{n}{2k+2}\binom{k+2}{3} \quad \textit{and}$$

$$r_1(n,k) = \binom{n}{2k+1}\left(\binom{k+2}{3} + \binom{k+1}{3}\right).$$

Proof. We count the four types of permutations listed in Proposition 2.13, in similar manner as we counted the partitions of genus 1.

(1) The reduced permutations with no twisted cycles. These correspond to the partitions, their number is given in Theorem 4.2.

(2) The reduced permutations with two back points. These may belong to the same doubly twisted cycle, or on two separate simply twisted cycles. Let us count first the permutations with one doubly twisted cycle.

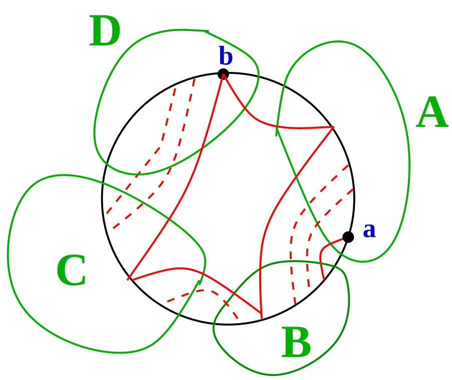

FIGURE 5. Reduced permutation with one doubly twisted cycle

The general shape of such a permutation is represented in Figure 5. Note that the number of points i such that $\alpha(i) \neq i+1$ is 4 for the doubly twisted cycle and 2 for each of the $k-1$ non-twisted cycles, giving a total number of $2k+2$ cycles. Moreover knowing these points the permutation is completely determined by the number of bicolored cycles having points in $(A,B), (B,C)(C,D)$ so that a sequence of three non-negative integers with sum equal to $k-1$. Since the number of such sequences is $\binom{k+1}{2}$, the number of such permutations is:

$$\binom{n}{2k+2}\binom{k+1}{2}.$$

Next we count the reduced permutations with two simply twisted cycles. The general shape of such a permutation is represented in Figure 6. Note that the number of points i such that $\alpha(i) \neq i+1$ is 3 for each of the two simply twisted cycles and 2 for each of the $k-2$ non twisted cycles giving a total number of $2k+2$ such points. Moreover, if we know these points then the permutation is completely determined by the number of bicolored cycles having points in $(A,B), (B,C)(A,C), (C,D)$ so that a sequence of four non-negative

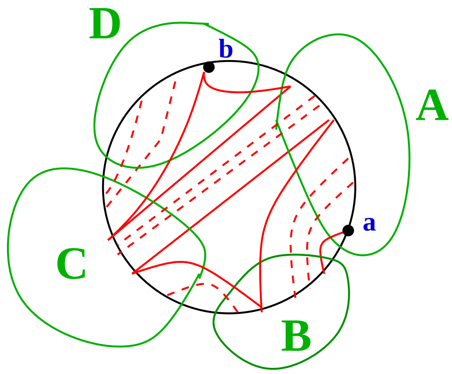

FIGURE 6. Reduced permutation with two simply twisted cycles

integers with sum equal to $k-2$. Since the number of such sequences is $\binom{k+1}{3}$ the number of such permutations is:

$$\binom{n}{2k+2}\binom{k+1}{3}.$$

We obtained that the number of all reduced permutations with two back points is

$$r_2(n,k) = \binom{n}{2k+2}\binom{k+1}{2} + \binom{n}{2k+2}\binom{k+1}{3},$$

and the stated equality follows from Pascal's formula.

(3) The reduced permutations with only one simply twisted cycle.

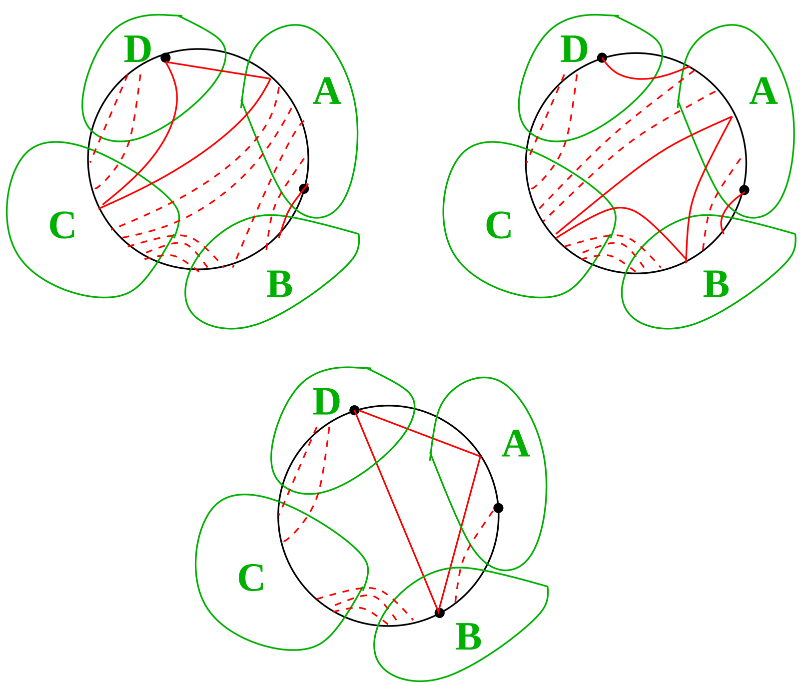

FIGURE 7. Three reduced permutations with one simply twisted cycle

The general shape of such a permutation is represented in Figure 7. There are three different cases depending on whether the twisted cycle is colored by A, B, C, or A, C, D or A, B, D. Note that the number of points i such that $\alpha(i) \neq i+1$ is 3 for the simply twisted cycle and 2 for each of the the $k-1$ non-twisted cycles, giving a total number of $2k+1$ such points. Moreover, if we know these points then the permutation is completely determined by the number of bicolored cycles having points in $(A, B), (B, C)(A, C), (C, D)$ in the two first situations so that a sequence of four non-negative integers with sum equal to $k-2$, is necessary since in the first case there is one cycle colored (A, D) and in the second one a cycle colored A, B. In the third situation there are no cycles with elements colored A, C so that there only 3 non-negative integers need to be known. So that the number of such sequences is $\binom{k+1}{3}$ in the first two cases and $\binom{k+1}{2}$ in the third one. in the and the number of such permutations is:

$$r_1(n,k) = \binom{n}{2k+1}\left(2\binom{k+1}{3} + \binom{k+1}{2}\right)$$

The stated equality follows by Pascal's formula.

Finally, adding the equations for the $r_j(n,k)$ yields

$$r_*(n,k) = \binom{n}{2k}\binom{k+1}{3} + \binom{n}{2k+1}\left(\binom{k+2}{3} + \binom{k+1}{3}\right) + \binom{n}{2k+2}\binom{k+2}{3}.$$

Using Pascal's formula two more times yields the stated result. $\square$

Proposition 4.5. *The ordinary generating function for the reduced permutations of genus* 1*, counting the number of points and cycles, is given by:*

$$R_*(x,y) = \frac{yx^3(1-x)^2(1-x+xy)}{((1-x)^2 - yx^2)^4}.$$

More precisely, for $j = 0, 1, 2$, the ordinary generating function for the reduced permutations of genus 1 *with j back points, counting the number of points and cycles, is given by:*

$$R_0(x,y) = \frac{y^2x^4(1-x)^3}{((1-x)^2 - yx^2)^4}, \quad R_2(x,y) = \frac{yx^4(1-x)^3}{((1-x)^2 - yx^2)^4} \quad \textit{and}$$

$$R_1(x,y) = \frac{yx^3(1-x)^2((1-x)^2 + yx^2)}{((1-x)^2 - yx^2)^4}.$$

Proof. We derive our formulas from the expressions for the numbers $r_j(n,k)$ stated in Theorem 4.4. The formula for $R_0(x,y)$ was shown in the proof of Theorem 4.2. Comparing the expressions for $r_0(n,k)$ and $r_2(n,k)$ yields $r_2(n,k) = r_0(n,k+1)$, implying $yR_2(x,y) = R_0(x,y)$. We are left to show the formula for $R_1(x,y)$, the formula for $R_*(x,y)$ may then be obtained by taking the sum of the equations for $R_j(x,y)$ where $j = 0, 1, 2$.

We may derive the formula for $R_1(x,y)$ in a way that is completely analogous to the computation $R_0(x,y)$ given in the proof of Theorem 4.2, using (4.2) several times, as outlined below:

$$
\begin{aligned}
R_1(x,y) &= \sum_{k\geq 1}\left(\binom{k+2}{3}+\binom{k+1}{3}\right)y^k\sum_{n\geq 2k+1}\binom{n}{2k+1}x^n\\
&= \sum_{k\geq 1}\left(\binom{k+2}{3}+\binom{k+1}{3}\right)y^k\frac{x^{2k+1}}{(1-x)^{2k+2}}\\
&= \frac{(1-x)^2}{y^2x^3}\sum_{k\geq 1}\binom{k+2}{3}\left(\frac{yx^2}{(1-x)^2}\right)^{k+2}+\frac{1}{yx}\sum_{k\geq 2}\binom{k+1}{3}\left(\frac{yx^2}{(1-x)^2}\right)^{k+1}\\
&= \left(\frac{(1-x)^2}{y^2x^3}+\frac{1}{yx}\right)\cdot\frac{\left(\frac{yx^2}{(1-x)^2}\right)^3}{\left(1-\frac{yx^2}{(1-x)^2}\right)^4}.
\end{aligned}
$$

Simplifying by the factors of $(1-x)$ yields the stated formula. □

5. Reducing permutations and reinserting trivial cycles

To count all partitions and permutations of genus 1 we first count the reduced objects in each class, and then count all objects obtained by inserting trivial cycles (see Definition 3.1) in all possible ways. In this section we describe in general how such a counting process may be performed.

Definition 5.1. *A* trivial reduction π' *of a permutation* π *of* $\{1,2,\ldots,n\}$ *is a permutation obtained from* π *by removing a trivial cycle* $(i,i+1,\ldots,j)$ *and decreasing all* $k\in\{j+1,j+1,\ldots,n\}$ *by* $j-\min(0,i-1)$ *in the cycle decomposition of* π*.*

Note that a trivial cycle may contain n, followed by 1, in the case when $i>j$, and that a trivial cycle may also consist of a single fixed point when $i=j$. Clearly π' is a permutation of $\{1,\ldots,n'\}$ for $n'=n-|\{i,i+1,\ldots,j\}|$ and has the same genus (if we replace ζ_n with $\zeta_{n'}$). Conversely we will say that π is a *trivial extension* (or an extension) of π'. For example, a trivial reduction of $(1,6)(2,3,4)(5,7)$ is $(1,3)(2,4)$. Clearly a permutation is *reduced* exactly when it has no trivial reduction. In order to avoid having to treat permutations of genus zero differently, we postulate that *the empty permutation is a reduced permutation of the empty set.*

Proposition 5.2. *For any permutation* π *of positive genus there is a unique reduced permutation* π' *that may be obtained by performing a sequence of reductions on* π*. If* π *has genus zero then this reduced permutation is the empty permutation on the empty set.*

Proof. There is at least one reduced permutation that we may reach by performing reductions until no reduction is possible. We only need to prove the uniqueness of the resulting permutation.

Let us call a cycle $(i_1,\ldots,i_k)$ of π *removable* if it has the following properties:

(1) the cyclic order of the elements $(i_1,\ldots,i_k)$ is the restriction of the cyclic order ζ to the set $\{i_1,\ldots,i_k\}$;
(2) no other cycle of π crosses $(i_1,\ldots,i_k)$;

(3) the cycles whose elements belong to one of the arcs $[i_1, i_2]$, $[i_2, i_3]$, $\ldots$, or $[i_{k-1}, i_k]$ are not twisted;
(4) no cycle whose elements belong to one of the arcs $[i_1, i_2]$, $[i_2, i_3]$, $\ldots$, or $[i_{k-1}, i_k]$ crosses any other cycle of π.

We claim that a cycle of π gets removed in any and every reduction process that leads to a reduced permutation, exactly when π is removable. On the one hand it is easy to see directly that any cycle that gets removed in the reduction process must be removable: assume after a certain number of reductions, the cycle $(i_1, \ldots, i_k)$ becomes the trivial cycle $(i, i+1, \ldots, j)$ where i_1 corresponds to i_1. Applying a reduction or an extension does not change the fact whether a cycle, present in both permutation is obtained by the restricting the cyclic order of all elements, this proves property (1). Neither the previously removed cycles, nor the cycles surviving after the removal of $(i_1, \ldots, i_k)$ can cross $(i_1, \ldots, i_k)$. The last two properties follow from the fact that the cycles whose elements belong to one of the arcs $[i_1, i_2]$, $[i_2, i_3]$, $\ldots$, or $[i_{k-1}, i_k]$ all become trivial cycles in the reduction process.

On the other hand, it is easy to show by induction on the number of cycles located on the arcs $[i_1, i_2]$, $[i_2, i_3]$, $\ldots$, $[i_{k-1}, i_k]$ of a removable cycle that every removable cycle ends up being removed in the reduction process. The basis of this induction is that a removable cycle containing no other cycles on its arcs is trivial. Any other removable cycle becomes trivial after the removal of all cycles contained on the arcs $[i_1, i_2]$, $[i_2, i_3]$, $\ldots$, $[i_{k-1}, i_k]$: these cycles are easily seen to be removable due to properties (3) and (4) and, if we list the elements of each such cycle $(j_1, \ldots, j_l)$ in the order they appear on the respective arc $[i_s, i_{s+1}]$, then the set of cycles contained on the arcs $[j_1, j_2]$, $\ldots$, $[j_{l-1}, j_l]$ is a proper subset of the cycles contained on the arc $[i_s, i_{s+1}]$. The induction hypothesis thus becomes applicable.

We found that the exact same cycles get removed in every reduction process that yields a reduced permutation, even if the order of the reduction steps may vary. After each reduction step, the surviving elements get relabeled, and the new label depends on the actual reduction step. However, it is easy to find the final label of each element i located in a cycle that "survives" the entire reduction process: i gets decreased exactly by the number of all elements of $\{1, \ldots, i-1\}$ that belong to a removable cycle.

Clearly a permutation has genus zero exactly when all of its cycles are removable. □

As a consequence of Proposition 5.2, if a class of permutations is closed under reductions and extensions then we are able to describe this class reasonably well by describing the reduced permutations in the class. Examples of such permutation classes include:

– the class of all partitions;
– the class of all permutations of a given genus;
– the class of all partitions of a given genus.

The main result of this section shows that knowing the reduced permutations allows not only to *describe* but also to *count* the permutation in the class closed under reductions and extensions that they generate. To state our main result we will need to use the generating function

$$D(x,y) = \frac{1 - x - xy - \sqrt{(x + xy - 1)^2 - 4x^2y}}{2 \cdot x} + 1 \tag{5.1}$$

of noncrossing partitions. This function is the formal power series solution of the quadratic equation

$$D(x,y) = 1 + xy \cdot D(x,y) + x \cdot (D(x,y)-1)D(x,y), \tag{5.2}$$

whose other solution is only a formal Laurent series. As it is well-known [12, sequence A001263], $[x^n y^k]D(x,y)$ is the number of noncrossing partitions of the set $\{1,\ldots,n\}$ having k parts. Note that we deviate from the usual conventions by defining the constant term to be 1, i.e. we consider that there is one noncrossing partition on the empty set and it has zero blocks. Our main result is the following.

Theorem 5.3. *Consider a class $\mathcal{C}$ of permutations that is closed under trivial reductions and extensions. Let $p(n,k)$ and $r(n,k)$ respectively be the number of all, respectively all reduced permutations of $\{1,\ldots,n\}$ in the class having k cycles. Then the generating functions $P(x,y) = \sum_{n,k} p(n,k)x^n y^k$ and $R(x,y) = \sum_{n,k} r(n,k)x^n y^k$ satisfy the equation*

$$P(x,y) = R(x \cdot D(x,y), y) \cdot \left(1 + x \cdot \frac{\frac{\partial}{\partial x}D(x,y)}{D(x,y)}\right).$$

Here $D(x,y)$ is the generating function of noncrossing partitions given in (5.1).

Proof. Consider an arbitrary permutation π of $\{1,\ldots,n\}$ in the class having k cycles. We distinguish two cases, and describe the generating function of the permutations belonging to each case. The term "removable cycle" we use here is the one that was defined in the proof of Proposition 5.2.

Case 1 *The element* 1 *does not belong to a removable cycle.* After reducing the permutation to the reduced permutation π', we obtain a reduced permutation on the set $\{1,\ldots,n_1\}$ having k_1 blocks for some $n_1 \le n$ and $k_1 \le k$. The cycles of π permutation that were removed have $n-n_1$ elements, and they form n_1 noncrossing partitions on the arcs created by the elements appearing in π'. They also have $k-k_1$ blocks. Thus there are exactly $[x^{n-n_1}y^{k-k_1}]D(x,y)^{n_1}$ permutations that may be reduced to the same reduced partition. The number of permutations counted in this case is

$$\sum_{n_1 \ge 4} \sum_{k_1 \ge 2} r(n_1,k_1)[x^{n-n_1}y^{k-k_1}]D(x,y)^{n_1}$$

Using the fact that, for any formal power series $f(x,y)$, $[x^{n-n1}y^{k-k_1}]f(x,y)$ is the same as $[x^n y^k]x^{n_1}y^{k_1}f(x),y$, we see that the above sum is exactly the coefficient of $x^n y^k$ in $R(x \cdot D(x,y), y)$.

Case 2 *The element* 1 *belongs to a removable cycle.* Let $j+1$, respectively $i-1$ be the smallest, respectively largest element that does not belong to a removable cycle. The arc $\{i, i+1, \ldots, n, 1, \ldots, j\}$ is then a union of elements of removable cycles. (Here we allow $i-1=n$, then $i=1$ and n does not belong to the arc). Let us denote the number of elements of this arc by n_2 and assume that the noncrossing partition formed by the removable cycles whose elements belong to this arc has k_2 blocks. As in the previous case, let n_1 be the number of elements belonging to not removable cycles, and assume that there are k_1 not removable cycles. There are $r(n_1,k_1)$ ways to select the reduced permutation, $[x^{n_2}y^{k_2}]D(x,y)$ ways to select the noncrossing partition on the arc $\{i, i+1, \ldots, n, 1, \ldots, j\}$ containing 1, and n_2 ways to select the position of 1 in its arc. We need to fill in the remaining $n - n_1 - n_2$ elements of removable cycles and group them into noncrossing partitions on the $n_1 - 1$ other arcs created by the n_1 elements of not removable cycles. We also need to make sure that the number of these other removable cycles is $k - k_1 - k_2$. The number of permutations counted in this case is

$$\sum_{n_1 \geq 4} \sum_{k_1 \geq 2} \sum_{n_2 \geq 1} \sum_{k_2 \geq 1} r(n_1, k_1) \left(n_2 [x^{n_2} y^{k_2}] D(x,y)\right) \cdot \left([x^{n-n_1-n_2} y^{k-k_1-k_2}] D(x,y)^{n_1-1}\right).$$

Note that $n_2 [x^{n_2} y^{k_2}] D(x,y)$ in the above sum is the coefficient of $x^{n_2} y^{k_2}$ in $x \cdot \frac{\partial}{\partial x} D(x,y)$. Using the same observation as at the end of the previous case, we obtain that the number of partitions counted in this case is

$$[x^n y^k] \left(R(x \cdot D(x,y), y) \cdot x \cdot \frac{\frac{\partial}{\partial x} D(x,y)}{D(x,y)} \right).$$

□

We conclude this section with rewriting the factor $1 + x \cdot \frac{\partial}{\partial x} D(x,y)/D(x,y)$, appearing in Theorem 5.3, in an equivalent form.

Proposition 5.4.

$$1 + x \cdot \frac{\frac{\partial}{\partial x} D(x,y)}{D(x,y)} = \frac{1 - xD(x,y)}{\sqrt{(x + xy - 1)^2 - 4x^2 y}}.$$

Proof. We may rewrite (5.2) as

$$x \cdot D(x,y)^2 + (xy - 1 - x) D(x,y) + 1 = 0.$$

Taking the partial derivative with respect to x on both sides we obtain

$$D(x,y)^2 - 2xD(x,y) \frac{\partial}{\partial x} D(x,y) - (1-y) D(x,y) - (1 + x - xy) \frac{\partial}{\partial x} D(x,y) = 0.$$

Using this equation we may express $\frac{\partial}{\partial x} D(x,y)$ as follows:

$$\frac{\partial}{\partial x} D(x,y) = \frac{D(x,y)(D(x,y) + y - 1)}{1 + x - xy - 2xD(x,y)}. \tag{5.3}$$

This equation directly implies

$$1 + \frac{x \frac{\partial}{\partial x} D(x,y)}{D(x,y)} = \frac{1 - xD(x,y)}{1 + x - xy - 2xD(x,y)}. \tag{5.4}$$

Finally, as a direct consequence of (5.1) we have

$$1 + x - xy - 2xD(x,y) = \sqrt{(x + xy - 1)^2 - 4x^2 y}. \tag{5.5}$$

Combining (5.4) and (5.5) yields the stated equality. □

Corollary 5.5. *The formula stated in Theorem 5.3 is equivalent to stating*

$$P(x,y) = R(x \cdot D(x,y), y) \cdot \frac{1 - xD(x,y)}{\sqrt{(x + xy - 1)^2 - 4x^2 y}}.$$

Remark 5.6. The numbers

$$J(n,k) = [x^n y^k]\left(x \cdot \frac{\frac{\partial}{\partial x} D(x,y)}{D(x,y)}\right)$$

are tabulated as entry A103371 in [12]. It is stated in the work of A. Laradji and A. Umar [11, Corollary 3.10] referenced therein, that

$$J(n,k) = \binom{n}{k}\binom{n-1}{k-1}.$$

6. Counting all partitions and permutations of genus one

In this section we find the ordinary generating function for the numbers $p_0(n,k)$ of all partitions of genus one the set $\{1,\ldots,n\}$, having k parts, and prove an analogous result for permutations of genus 1. Our main result is the following.

Theorem 6.1. *Let the number $p_0(n,k)$ of all partitions of $\{1,\ldots,n\}$ of genus one having k parts. Then the generating function*

$$P_0(x,y) = \sum_{n\geq 4}\sum_{k\geq 2} p_0(n,k)x^n y^k$$

is given by the equation

$$P_0(x,y) = \frac{x^4y^2}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}}.$$

We will see in Section 7 that Theorem 6.1 is equivalent to an explicit formula (7.2) for the numbers $p_0(n,k)$, originally conjectured by M. Yip [17, Conjecture 3.15]. We will prove Theorem 6.1 by combining Theorem 5.3 with the formula (4.1) for the generating function $R_0(x,y)$ of reduced partitions of genus one. We use the equivalent form of Theorem 5.3 stated in Corollary 5.5 and use Propositions 6.2 below to simplify $R_0(x\cdot D(x,y),y)$. Theorem 6.1 thus follows from Theorem 5.3, by multiplying the formulas given in Propositions 5.4 and 6.2.

Proposition 6.2. *The generating function $R_0(x,y)$ of reduced partitions of genus one satisfies the equality*

$$R_0(x\cdot D(x,y),y) = \frac{x^4y^2}{(1-xD(x,y))((x+xy-1)^2-4x^2y)^2}.$$

Proof. We will use D as a shorthand for $D(x,y)$. Using (4.1) we may write

$$R_0(x\cdot D,y) = \frac{y^2x^4D^4(1-xD)^3}{((1-xD)^2-yx^2D^2)^4} \tag{6.1}$$

An equivalent form of (5.2) is

$$xyD = (D-1)(1-xD), \tag{6.2}$$

which may be used to eliminate the variable y in the denominator on the right hand side of (6.1). Thus we obtain

$$R_0(x\cdot D,y) = \frac{y^2x^4D^4(1-xD)^3}{((1-xD)^2-(D-1)(1-xD)xD)^4} = \frac{y^2x^4D^4(1-xD)^3}{((1-xD)(1-xD^2))^4}.$$

Simplifying by the factors of $(1 - xD)$ yields

$$R_0(x \cdot D, y) = \frac{y^2x^4D}{(1 - xD)} \cdot \left(\frac{D}{1 - xD^2}\right)^4.$$

We are left to show that the second factor is $((x + xy - 1)^2 - 4x^2y)^{-2}$. By (5.5), this is equivalent to showing

$$1 + x - xy - 2xD = \frac{1 - xD^2}{D}$$

which is a rearranged version of (5.2). $\square$

Substituting $y = 1$ into the formula given in Theorem 6.1 has the following consequence.

Corollary 6.3. *The number of $p_0(n)$ all partitions of $\{1, \ldots, n\}$ of genus one has the ordinary generating function*

$$\sum_{n=4}^{\infty} p_0(n)x^n = \frac{x^4}{(1 - 4x)^{5/2}}.$$

The coefficient of x^n in the above formula is easily extracted:

Corollary 6.4. *The number of all genus one partitions on $\{1, \ldots, n\}$ is*

$$p_0(n) = \binom{-5/2}{n-4}(-1)^{n-4}4^{n-4} = \frac{(2n-5)!}{6 \cdot (n-4)!(n-3)!}.$$

The sequence $p_0(4), p_0(5), \ldots$ is listed as sequence A002802 in [12] and referred to (essentially) as the number of permutations of genus one. See also [16, formula (13)]. Now we see that partitions of genus one are counted by the same sequence, shifted by one.

Next we follow an analogous procedure to count all permutations of genus 1.

Theorem 6.5. *Let $p_*(n, k)$ be the number of all permutations in $\mathrm{Sym}\,(n)$ of genus one having k cycles. Then the generating function $P_*(x, y) = \sum_{n,k} p_*(n, k)x^ny^k$ is given by the equation*

$$P_*(x, y) = \frac{x^3y}{(1 - 2(1 + y)x + x^2(1 - y)^2)^{5/2}}.$$

More precisely, for $j = 0, 1, 2$, let $p_j(n, k)$ be the number of all permutations in $\mathrm{Sym}\,(n)$ of genus one having k cycles and j back points. Then the generating functions $P_j(x, y) = \sum_{n,k} p_j(n, k)x^ny^k$ are given by the formulas

$$P_0(x, y) = \frac{x^4y^2}{(1 - 2(1 + y)x + x^2(1 - y)^2)^{5/2}},$$

$$P_2(x, y) = \frac{x^4y}{(1 - 2(1 + y)x + x^2(1 - y)^2)^{5/2}} \quad \text{and}$$

$$P_1(x, y) = \frac{x^3y(1 - xy - x)}{(1 - 2(1 + y)x + x^2(1 - y)^2)^{5/2}}.$$

The formula for $P_0(x,y)$ was shown in Theorem 6.1 above. As noted in the proof of Proposition 4.5, the generating function $R_2(x,y)$ differs from $R_0(x,y)$ only by a factor of y. After reproducing the same calculation to obtain $P_2(x,y)$ from $R_2(x,y)$, we find that $P_0(x,y) = yP_2(x,y)$. Therefore, to prove Theorem 6.5 above, it suffices to show the formula for $P_1(x,y)$, the equation for $P_*(x,y)$ will then arise as the sum of the equations for the $P_j(x,y)$.

Similarly to the proof of Theorem 6.1, we may show this formula by combining Corollary 5.5 with the formula for $R_1(x,y)$ given in Proposition 4.5. We may use Propositions 6.6 below to simplify $R_1(x\cdot D(x,y),y)$.

Proposition 6.6. *The generating function $R_1(x,y)$ of reduced permutations of genus 1 having one back point satisfies the equality*

$$R_1(x\cdot D(x,y),y) = \frac{x^3y(1-xy-x)}{(1-xD(x,y))((x+xy-1)^2-4x^2y)^2}.$$

Proof. We will use D as a shorthand for $D(x,y)$. Using Proposition 4.5 we may write

$$R_1(x\cdot D,y) = \frac{yx^3D^3(1-xD)^2((1-xD)^2+yx^2D^2)}{((1-xD)^2-yx^2D^2)^4}$$

Just like in the proof of Proposition 6.2 we may use (6.2) to eliminate the variable y in the denominator and get

$$R_1(x\cdot D,y) = \frac{yx^3D^3(1-xD)^2((1-xD)^2+yx^2D^2)}{((1-xD)(1-xD^2))^4} = \frac{yx^3D^3((1-xD)^2+yx^2D^2)}{(1-xD)^2(1-xD^2)^4}.$$

We use (6.2) again to rewrite the factor $((1-xD)^2+yx^2D^2)$ in the numerator and get

$$R_1(x\cdot D,y) = \frac{yx^3D^3(1-2xD+xD^2)}{(1-xD)(1-xD^2)^4} = \frac{yx^3(1-2xD+xD^2)}{(1-xD)D}\cdot\left(\frac{D}{(1-xD^2)}\right)^4$$

We have seen at the end of the proof of Proposition 6.2 that the last factor is $(x+xy-1)^2-4x^2y)^{-2}$. Taking this fact into account, comparing the last equation with the proposed statement, we only need to show the following equality:

$$\frac{1-2xD+xD^2}{D} = 1-xy-x.$$

This last equation is a rearranged version of (5.2). □

7. Extracting the coefficients from our generating functions

In this section we will show a generalization of the following equation.

$$\frac{x^4y^2}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}} = \sum_{n\geq 4}\frac{1}{6}\binom{n}{2}x^n\sum_{k=2}^{n-2}\binom{n-2}{k}\binom{n-2}{k-2}y^k. \tag{7.1}$$

According to this equation, M. Yip's conjecture [17, Conjecture 3.15], stating

$$p_0(n,k) = \frac{1}{6}\binom{n}{2}\binom{n-2}{k}\binom{n-2}{k-2}. \tag{7.2}$$

is equivalent to our Theorem 6.1 and thus true. By showing a generalization of (7.1) we may view the problem of counting genus one partitions as part of a broader class of related problems. Since, by Theorem 6.5, the generating function of genus one permutations only differs by a factor of xy, the calculations of the present section also provide a new way to count these objects, thus providing a new proof of the result first stated by A. Goupil and G. Schaeffer [6].

After dividing both sides by x^4y^2 and shifting n and k down by two, we obtain the following equivalent form of equation (7.1).

$$\frac{1}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}} = \sum_{n\geq 2}\frac{1}{6}\binom{n+2}{2}x^{n-2}\sum_{k=0}^{n-2}\binom{n}{k+2}\binom{n}{k}y^k. \tag{7.3}$$

This equation may be obtained from Theorem 7.1 below as a special case, after substituting $m=2$.

Theorem 7.1. *For any nonnegative integer m*

$$\frac{1}{(1-2(1+y)x+x^2(1-y)^2)^{(2m+1)/2}} = \sum_{n\geq m}\sum_{k\geq 0}\frac{\binom{n+m}{m}\binom{n}{k}\binom{n}{m+k}}{\binom{2m}{m}}x^{n-m}y^k$$

holds.

Proof. Recall (see [3, Ch. V, (2.34)]), the following generating function of the Legendre polynomials:

$$\frac{1}{\sqrt{1-2ut+t^2}} = \sum_{n\geq 0}L_n(u)t^n.$$

Taking the mth derivative with respect to u on both sides yields

$$\frac{(-1/2)\cdots(-(2m-1)/2)\cdot(-2t)^m}{(1-2ut+t^2)^{(2m+1)/2}} = \sum_{n\geq m}\frac{d^m}{du^m}L_n(u)t^n.$$

Using the fact that $\binom{2m}{m}=2^m(2m-1)!!/m!$, after multiplying both sides by $2^m/(t^m m!)$, the above equation may be rewritten as

$$\frac{\binom{2m}{m}}{(1-2ut+t^2)^{(2m+1)/2}} = \sum_{n\geq m}t^{n-m}\frac{2^m}{m!}\frac{d^m}{du^m}L_n(u). \tag{7.4}$$

On the right hand side we use the following well-known formula (it may be found, for example, in [15, (4.21.2)], in a slightly more general form):

$$L_n(u) = \sum_{j=0}^{n}\binom{n+j}{j}\binom{n}{j}\left(\frac{u-1}{2}\right)^j. \tag{7.5}$$

As an immediate consequence of this equation we obtain

$$\frac{2^m}{m!}\frac{d^m}{du^m}L_n(u) = \sum_{j=m}^{n}\binom{n+j}{j}\binom{n}{j}\binom{j}{m}\left(\frac{u-1}{2}\right)^{j-m}.$$

Since $\binom{n}{j}\binom{j}{m}=\binom{n}{m}\binom{n-m}{j-m}$, after shifting the index j down by m, we may rewrite the last equation as

$$\frac{2^m}{m!}\frac{d^m}{du^m}L_n(u) = \binom{n}{m}\sum_{j=0}^{n-m}\binom{n+j+m}{j+m}\binom{n-m}{j}\left(\frac{u-1}{2}\right)^j.$$

Let us substitute this last equation on the right hand side of (7.4).

$$\frac{\binom{2m}{m}}{(1-2ut+t^2)^{(2m+1)/2}} = \sum_{n\geq m} t^{n-m}\binom{n}{m}\sum_{j=0}^{n-m}\binom{n+j+m}{j+m}\binom{n-m}{j}\left(\frac{u-1}{2}\right)^j.$$

Let us substitute $u=(1+y)/(1-y)$ and $t=x(1-y)$ in the above equation. Under this substitution $(u-1)/2$ becomes $y/(1-y)$, thus, after simplifying with factors of $(1-y)$, we obtain that the coefficient of x^{n-m} in $\binom{2m}{m}/(1-2x(1+y)+x^2(1-y)^2)^{(2m+1)/2}$ is

$$p_n(y) = \binom{n}{m}\sum_{j=0}^{n-m}\binom{n+j+m}{j+m}\binom{n-m}{j}y^j(1-y)^{n-m-j}.$$

Here, for any $k\leq n$,

$$[y^k]p_n(y) = \binom{n}{m}\sum_{j=0}^{k}\binom{n+j+m}{j+m}\binom{n-m}{j}(-1)^{k-j}\binom{n-m-j}{k-j}.$$

Using $\binom{n-m}{j}\binom{n-m-j}{k-j} = \binom{n-m}{k}\binom{k}{k-j}$ we may rewrite the above equation as

$$[y^k]p_n(y) = \binom{n}{m}\binom{n-m}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{n+j+m}{j+m}\binom{k}{k-j}.$$

Since $\binom{n+j+m}{j+m} = (-1)^{j+m}\binom{-n-1}{j+m}$, using the Chu-Vandermonde identity we may write

$$\begin{aligned}
[y^k]p_n(y) &= \binom{n}{m}\binom{n-m}{k}(-1)^{k+m}\sum_{j=0}^{k}\binom{-n-1}{j+m}\binom{k}{k-j}\\
&= \binom{n}{m}\binom{n-m}{k}(-1)^{k+m}\binom{-n-1+k}{m+k} = \binom{n}{m}\binom{n-m}{k}\binom{n+m}{m+k}\\
&= \binom{n+m}{m}\binom{n}{k}\binom{n}{m+k}.
\end{aligned}$$

The statement now follows after substituting the last obtained formula for $[y^k]p_n(y)$ into

$$\frac{\binom{2m}{m}}{(1-2x(1+y)+x^2(1-y)^2)^{(2m+1)/2}} = \sum_{n\geq m} x^{n-m}p_n(y)$$

and dividing both sides by $\binom{2m}{m}$. □

We conclude this section with providing an explicit formulas for the number of all permutations of genus 1, with a given numbers of points, cycles, and back points.

Theorem 7.2. *The number of all permutations of genus* 1 *of* Sym (n) *with* k *cycles is equal to:*

$$p_*(n,k) = \frac{1}{6}\binom{n+1}{2}\binom{n-1}{k+1}\binom{n-1}{k-1}$$

More precisely, for $j = 0, 1, 2$, the number $p_j(n,k)$ of permutations of genus 1 of $\mathrm{Sym}\,(n)$ *with j back points and k cycles is given by the following formulas:*

$$p_0(n,k) = \frac{1}{6}\binom{n}{2}\binom{n-2}{k}\binom{n-2}{k-2}, \quad p_2(n,k) = \frac{1}{6}\binom{n}{2}\binom{n-2}{k+1}\binom{n-2}{k-1} \quad \textit{and}$$

$$p_1(n,k) = \frac{1}{3}\binom{n}{2}\binom{n-2}{k}\binom{n-2}{k-1}.$$

Proof. The formulas for $p_0(n,k)$, $p_2(n,k)$ and $p_*(n,k)$ are all direct consequences of Theorems 6.5 and 7.1. Using the same Theorems to find $p_1(n,k)$ amounts to using the obvious equality

$$p_1(n,k) = p_*(n,k) - (p_0(n,k) + p_2(n,k)),$$

which is equivalent to showing that the sum of the stated values of the $p_j(n,k)$ gives the stated value of $p_*(n,k)$. For that purpose note that

$$p_0(n,k) + \frac{p_1(n,k)}{2} = \frac{1}{6}\binom{n}{2}\binom{n-2}{k}\left(\binom{n-2}{k-2} + \binom{n-2}{k-1}\right),$$

which, by Pascal's formula, gives

$$p_0(n,k) + \frac{p_1(n,k)}{2} = \frac{1}{6}\binom{n}{2}\binom{n-2}{k}\binom{n-1}{k-1} = \frac{n}{12}(k+1)\binom{n-1}{k+1}\binom{n-1}{k-1}. \tag{7.6}$$

A similar use of Pascal's formula yields

$$p_2(n,k) + \frac{p_1(n,k)}{2} = \frac{1}{6}\binom{n}{2}\binom{n-2}{k-1}\binom{n-1}{k+1} = \frac{n}{12}(n-k)\binom{n-1}{k-1}\binom{n-1}{k+1}. \tag{7.7}$$

The sum of (7.6) and (7.7) is

$$\sum_{j=0}^{2} p_j(n,k) = \frac{n(n+1)}{12}\binom{n-1}{k-1}\binom{n-1}{k+1},$$

as required. □

8. Concluding remarks

Our four-colored noncrossing partition representation of permutations of genus 1 is reminiscent of the use of three types of crossing hyperedges in the hypermonopole diagram representing a genus 1 partition in M. Yip's Master's thesis [17]. This analogy becomes even more explicit at the light of Remark 2.10 stating that, for partitions of genus 1, three colors suffice. Whereas the hypermonopole diagrams are of topological nature (parts are represented with "curvy lines") our representation is combinatorial (parts may be represented with polygons). By better understanding the relation between the two models, perhaps it is possible to show that every genus one partition has a hypermonopole diagram on a torus in such a way that boundaries of hyperedges are finite unions of "straight" (circular) arcs. In either case, non-uniqueness of the representation makes direct counting difficult.

Lemma 1.5 establishes a relationship between α and $\alpha^{-1}\zeta_n$. It is worth noting that, in the case when $g(\alpha) = 0$, the permutation $\alpha^{-1}\zeta_n$ is the permutation representing the *Kreweras dual* of the noncrossing partition represented by α. G. Kreweras [10] used this correspondence to show that the lattice of noncrossing partitions is self-dual. M. Yip has shown that the poset of genus 1 partitions is rank-symmetric [17, Proposition 4.5], but not self dual [17, Proposition 4.6] for $n \geq 6$. Lemma 1.5 suggests that maybe true duality could be found between genus 1 partitions and permutations with 2 back points, after defining the proper partial order on the set of all genus 1 permutations. In this setting, permutations with exactly one back point would form a self-dual subset. Their number $p_1(n,k)$, given in Theorem 7.2, may be rewritten as

$$p_1(n,k) = \binom{n}{3} N(n-2, k-1),$$

where $N(n-2,k-1)$ is a Narayana number. It is a tantalizing thought that this simple formula could have a very simple proof. If this is the case, then the formulas for $p_0(n,k)$ and $p_1(n,k)$ could be easily derived, using Lemma 1.5 and Yip's rank-symmetry result [17, Proposition 4.5] to establish $p_2(n,k) = p_0(n,k+1)$, and then the formula for $p_*(n,k)$ already stated by A. Goupil and Shaeffer [6] to complete a setting in which the formula for $p_0(n,k)$ may be shown by induction on k. A "numerically equivalent" conjecture (albeit for sets of partitions) was stated by M. Yip [17, Conjecture 4.10].

Theorem 7.1 naturally inspires the question: what other combinatorial objects are counted by the coefficients of $x^n y^k$ in the Taylor series of

$$(1-2(1+y)x + x^2(1-y)^2)^{-(2m+1)/2},$$

when m is some other nonnegative integer. For $m = 0$, we obtain

$$\frac{1}{(1-2(1+y)x+x^2(1-y)^2)^{1)/2}} = \sum_{n\geq m}\sum_{k\geq 0} \binom{n}{k}^2 x^n y^k.$$

These coefficients are listed as sequence A008459 in [12]. Among others, they count the *type B noncrossing partitions* of rank k of an n-element set. In [14], R. Simion constructed a simplicial polytope in each dimension whose h vector entries are the squares of the binomial coefficients. The number of j-element faces of the n-dimensional polytope is $f_{j-1} = \binom{n+j}{j}$. Another class of simplicial polytopes with the same face numbers was defined in [8] as the class of all simplicial polytopes arising by taking any pulling triangulation of the boundary complex of the Legendrotope. The Legendrotope is combinatorially equivalent to the intersection of a standard crosspolytope with any hyperplane passing through its center that does not contain any of its vertices. For all these examples the polynomial

$$F(u) = \sum_{j=0}^{n} f_{j-1} \left(\frac{u-1}{2}\right)^j$$

is a Legendre polynomial by (7.5), and the squares of the binomial coefficients are their h-vector entries. For higher values of m, taking the mth derivative of $F(u)$ (see the proof of Theorem 7.1) corresponds to summing over the links of all $(m-1)$ dimensional faces. It is not evident from this interpretation why we should get integer entries, even after dividing by $\binom{2m}{m}$, and it is an interesting question for future research to see whether for the type B simplicial associahedron or for some very regular triangulation of the Legendrotope, symmetry reasons would explain the integrality. Little seems to be known for these higher values of m. For $m = 1$, substituting $y = 1$ yields the sequence of coefficients $\{(2n+1)!/n!^2\}_{n\geq 0}$, listed as entry A002457 in [12]. None of the recorded interpretations seems to be geometric or connected to counting partitions subject to special noncrossing conditions.

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