A RANDOM HIERARCHICAL LAPLACIAN

by

Elijah Everette Ray

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Approved by:

Dr. Stanislav A. Molchanov

Dr. Boris Vainberg

Dr. Yuri Godin

Dr. Harish Cherukuri

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ABSTRACT

ELIJAH EVERETTE RAY. A random hierarchical laplacian. (Under the direction of DR. STANISLAV A. MOLCHANOV)

The self-similar Hierarchical Laplacian, essentially proposed by Dyson [4] in his theory of 1-D ferromagnetic phase transitions, has a discrete spectrum with each eigenvalue having infinite multiplicity [9]. As a result, the integrated density of states is piecewise constant and the density of states is a sum of point-masses located on its spectrum. To correct these "defects", we present a modification of the Hierarchical Laplacian obtained by allowing its deterministic coefficients to instead vary randomly. In this way, the spectrum remains deterministic but the eigenvalues become random with finite multiplicity and we will obtain a continuous density of states. In the last section, we will examine the eigenvalue statistics near an individual point of the spectrum and show that, locally, the spectrum is approximately a Poisson point process.

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CHAPTER 1: INTRODUCTION

The self-similar Hierarchical Laplacian, essentially proposed by Dyson [4] in his theory of 1-D ferromagnetic phase transitions, has a discrete spectrum with each eigenvalue having infinite multiplicity [9]. As a result, the integrated density of states $N(\lambda)$ is piecewise constant and the density of states does not exist—or more precisely, it is a sum of point-masses located on the spectrum of $-\Delta$. When the probabilistic weights for the Hierarchical Laplacian are given by a geometric progression, the Hierarchical Laplacian can have an arbitrary spectral dimension s_h and as a result it is similar to the classical fractals, e.g., the Sierpinskii Lattice.

Usually in Mathematical Physics, after considering the Laplacian, we move on to consider the Schrödinger operator—in two different directions.

First in the classical spectral theory, the negative Laplacian typically has discrete non-negative spectrum which accumulates to the point zero. When we add a negative decreasing potential (potential well), the spectrum below zero will be discrete. The central questions are: under what conditions are there only finitely many negative eigenvalues and how can we estimate the number of negative eigenvalues [10], [11]. Let's formulate several classical results. Consider in $\mathbb{R}^d$, $d \geq 3$, the Schrödinger operator

$$H = -\Delta - V(x),$$

where $V(x) \geq 0$ and $V(x) \to 0$ as $|x| \to \infty$ in some sense. In this situation, the spectrum of H covers the half axis $[0, \infty)$ but for negative energies the spectrum is discrete. Letting $N_0(V) = \#\{\lambda_i < 0\}$, we have the *Lieb-Thirring* (*LT*) *Estimate*:

$$\sum_{i:\lambda_i<0} |\lambda_i|^\gamma \leq C_{d,\gamma} \int_{\mathbb{R}^d} V^{d/2+\gamma}(x)\, dx \tag{1.1}$$

and taking $\gamma = 0$ in (1.1), we have the *Cwikel-Lieb-Rozenblum* (*CLR*) *Estimate*:

$$N_0(V) \leq C_d \int_{\mathbb{R}^d} V^{d/2}(x)\, dx. \tag{1.2}$$

In particular, the CLR estimate implies that the operator $H = -\Delta + \sigma V(x)$ has non-negative spectrum whenever the coupling constant σ is small and $V \in L^{d/2}(\mathbb{R}^d)$. For small dimension, we have $N_0(\sigma V) > 0$ for any non-vanishing V and any $\sigma > 0$.

Another direction is the spectral theory of the random Schrödinger operator, i.e., $H = -\Delta + \sigma V_\omega(x)$, σ is a coupling constant, $V(x)$ i.i.d. One might conjecture that, in this case, classical Anderson phase-type transitions would be observed for small σ and $s_h > 2$, and that together with pure point spectrum, there exists some kind of continuous spectrum, i.e., Anderson delocalization [1]. Unfortunately, in [9], this natural conjecture appeared to be wrong. In [5], it is shown that for more or less general distributions, for arbitrary spectral dimension s_h and arbitrary σ, the spectrum of the random Schrödinger operator is pure point. One can propose the following physical explanation of this fact. It is well-known from the literature that the spectrum of the random Schrödinger operator on the lattice $\mathbb{Z}^d$ is pure point outside the spectrum of the Laplacian for arbitrarily small σ in any dimension [8]. Since the spectrum of the self-similar Hierarchical Laplacian consists of isolated points, all energies are outside the spectrum. Taking into account all these facts, it is important to modify the self-similar Hierarchical model in such a way that — instead of the isolated eigenvalues of infinite multiplicity — we will get spectrum which is dense on some interval and obtain a continuous density of states.

The goal of the thesis is the analysis of a random Hierarchical Laplacian obtained by allowing the deterministic eigenvalues of the Hierarchical Laplacian to instead vary randomly. The way in which we allow the eigenvalues to be random does not change which functions are eigenfunctions but it does have the effect of breaking each isolated (deterministic) eigenvalue of infinite multiplicity into a countable dense set of eigenvalues each having (the same) finite multiplicity. The spectrum remains deterministic but the isolated points of spectrum become widened into spectral bands supporting a continuous density of states. These spectral bands may or may not overlap depending on the value of a parameter $0 < \sigma < 1$ — for values of σ closer to one, the spectrum will be an interval while for $\sigma = 0$ we obtain the original (deterministic) Hierarchical Laplacian. In the last section, we examine the eigenvalue statistics near an individual point of the spectrum and show that, locally, the spectrum is approximately a Poisson point process.

CHAPTER 2: HIERARCHICAL LATTICE

2.1 Definitions

A *hierarchical lattice* is an ultrametric space (X, d_h) where X is an infinite set and the *hierarchical distance* d_h is an integer-valued ultrametric with the property that for each integer $r \geq 1$, there exists an integer $\nu_r \geq 2$ such that every closed metric ball of radius r (which we refer to as a *cube of rank* r)

$$Q^{(r)}(x) = \overline{B}(x, r) = \{y \in X : d_h(x, y) \leq r\} \tag{2.1}$$

contains exactly ν_r balls of radius $r - 1$. We call a hierarchical lattice *self-similar* if each $\nu_r = \nu$ for some integer $\nu \geq 2$. To say d_h is an *ultrametric* means, instead of just the triangle inequality, d_h satisfies the stronger condition that for all $x, y, z \in X$,

$$d_h(x, y) \leq \max\big\{d_h(x, z), d_h(y, z)\big\}. \tag{2.2}$$

Because d_h is an ultrametric, each element of a cube can serve as its center. As a result, two cubes are either disjoint or one is a subset of the other. In particular, because two different cubes of the same rank/radius must be disjoint, the hierarchical distance can be expressed as

$$d_h(x, y) = \min\big\{r : Q^{(r)}(x) = Q^{(r)}(y)\big\} = \max\big\{r : Q^{(r-1)}(x) \cap Q^{(r-1)}(y) = \varnothing\big\}. \tag{2.3}$$

Note that the sequence $\left\{Q^{(r)}(x)\right\}_{r\geq 0}$ increases to X, i.e.,

$$x \in Q^{(0)}(x) \subseteq Q^{(1)}(x) \subseteq Q^{(2)}(x) \subseteq \cdots \subseteq \bigcup_{r=0}^{\infty} Q^{(r)}(x) = X. \tag{2.4}$$

It follows that for each $r \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the collection of all cubes of rank r

$$\Pi_r = \left\{Q^{(r)}(x) : x \in X\right\} \tag{2.5}$$

forms a partition of X into finite subsets where every cube belonging Π_r is a disjoint union of ν_r cubes belonging to Π_{r-1}. Then the cardinality or *volume* of a cube is given by

$$\left|Q^{(r)}(x)\right| = \nu_1 \nu_2 \cdots \nu_r$$

and since each $\nu_r \geq 2$, it follows that each inclusion in (2.4) is strict and $|Q^{(r)}(x)|$ is of at least exponential order as $r \to \infty$.

The requirement that d_h be integer-valued implies X is discrete as a topological space. In fact, the definition implies the set X—being a countable union (2.4) of finite sets, must itself be countable. More generally, we could have simply required d_h to take as its values the terms of some strictly increasing sequence, $0 = t_0 < t_1 < t_2 < \cdots$. For any such sequence we can define a *renormalized* hierarchical distance by taking $\rho_h(x, y) = t_{d_h(x,y)}$. In this case, the cubes remain the same but the d_h-balls of radius r become ρ_h-balls of radius t_r. In a self-similar hierarchical lattice, taking $t_r = \beta^r$ for some $\beta > 1$ and all $r \geq 1$, the volume of a renormalized metric ball becomes, essentially as in $\mathbb{R}^d$, a power function of its radius, i.e., if $R = \beta^r$, we have

$$|\{y \in X : \rho_h(x, y) \leq R\}| = |Q^{(r)}(x)| = \nu^r = R^{\log_\beta \nu}.$$

For each $r \geq 0$, we denote the collection of all cubes of rank $\geq r$ by

$$\mathcal{V}_r = \bigcup_{k=r}^{\infty} \Pi_k.$$

Cubes belonging to $\mathcal{V}_1$ are said to be *non-degenerate*. For each $r \geq 0$, $\mathcal{V}_r$ forms a simple connected graph with edges

$$\mathcal{E}_r = \big\{\{Q, Q^+\} : Q \in \mathcal{V}_r\big\}$$

where we write $Q^+ = Q^{(r+1)}(x)$ whenever $Q = Q^{(r)}(x)$. The graph distance d_g between two cubes $Q \in \Pi_m$ and $Q' \in \Pi_n$ is given by

$$d_g(Q, Q') = \begin{cases} n - m & \text{if } Q \subseteq Q' \\ \\ 2r - m - n & \text{if } d_h(Q, Q') = r > 0 \end{cases}. \tag{2.6}$$

Note that for $y \notin Q$, the mapping $Q \ni x \mapsto d_h(x, y)$ is constant. Therefore, whenever Q and Q' are disjoint we have $d_h(Q, Q') = d_h(x, x')$ for all $x \in Q$ and $x' \in Q'$.

Equation (2.3) shows that the hierarchical distance can be recovered from a knowledge of the partitions (2.5). To see this, let's start from scratch and suppose we are given an abstract countably infinite set X and a sequence $\{\Pi_r\}_{r\geq 1}$ of partitions of X into finite subsets where every set belonging to Π_r is contained in some set belonging to Π_{r+1} and contains at exactly $\nu_r \geq 2$ subsets belonging to Π_{r-1}. Assume further that for each $x \in X$,

$$\bigcup_{r=0}^{\infty} Q^{(r)}(x) = X \tag{2.7}$$

where $Q^{(r)}(x)$ is the unique set from Π_r containing x and $Q^{(0)}(x) = \{x\}$. If $d_h(x, y)$ is defined by (2.3) then (X, d_h) is a hierarchical lattice. We assume (2.7) in order to

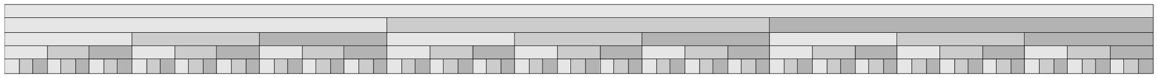


Figure 2.1: Cube of rank 4 in a self-similar hierarchical lattice where $\nu = 3$.

ensure that $d_h(x, y) < \infty$ for all $x, y \in X$.

The simplest example of a self-similar hierarchical lattice is given by $X = \mathbb{N}$ with

$$Q_i^{(r)} = \left\{n \in \mathbb{N} : i\nu^r \leq n < (i+1)\nu^r\right\} \quad \text{for all} \quad r \geq 0 \quad \text{and} \quad i \in \mathbb{N} \tag{2.8}$$

We denote the hierarchical distance on $\mathbb{N}$ by $\hat{d}_h(m, n)$.

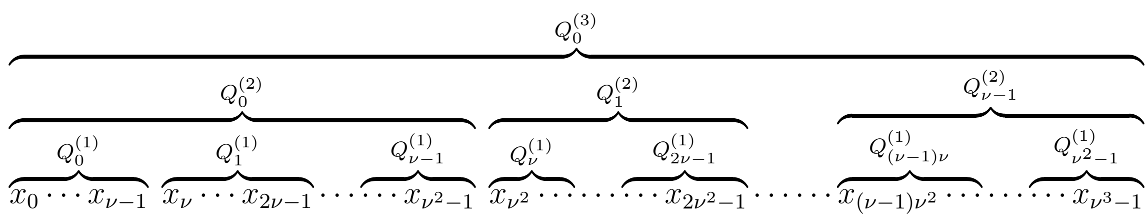
2.2 Enumeration of Self-similar Hierarchical Lattice

Proposition 2.1. *In a self-similar hierarchical lattice, we can enumerate the points* $X = \{x_0, x_1, \ldots\}$ *in such a way that* $d_h(x_m, x_n) = \hat{d}_h(m, n)$ *for all* $m, n \in \mathbb{N}$. *As a result, we can enumerate* Π_r

$$\Pi_r = \left\{Q_0^{(r)}, Q_1^{(r)}, Q_2^{(r)}, \ldots\right\}, \tag{2.9}$$

by defining for each $i = 0, 1, 2, 3, \ldots,$

$$Q_i^{(r)} = \left\{x_n : i\nu^r \leq n < (i+1)\nu^r\right\} \tag{2.10}$$



First we need a lemma.

Lemma 2.2. *In a self-similar hierarchical lattice, every cube can be enumerated*

$$Q = \{x_n : 0 \leq n < |Q|\}$$

in such a way that $d_h(x_m, x_n) = \hat{d}_h(m,n)$ *for all* $m, n < |Q|$.

Proof. For $m < n$, we have $\hat{d}_h(m,n) = r$ if and only if there exist $i, j \in \mathbb{N}$ with

$$i\nu^r \le m < j\nu^{r-1} \le n < (i+1)\nu^r. \tag{2.11}$$

But if (2.11) holds we will also have

$$(i+1)\nu^r \le m + \nu^r < (j+\nu)\nu^{r-1} \le n + \nu^r < (i+2)\nu^r$$

hence $\hat{d}_h(m+\nu^r, n+\nu^r) = \hat{d}_h(m,n)$. It follows by induction that

$$\hat{d}_h\big(m + k\nu^{\hat{d}_h(m,n)}, n + k\nu^{\hat{d}_h(m,n)}\big) = \hat{d}_h(m,n) \quad \text{for all} \quad k \ge 1 \tag{2.12}$$

Now, let $Q \in \Pi_{r+1}$ and assume the result holds for all cubes of smaller rank contained in Q. Then there exist $Q_0^{(r)}, \ldots, Q_{\nu-1}^{(r)} \in \Pi_r$ with $Q = \bigcup_{i=0}^{\nu-1} Q_i^{(r)}$ and for $0 \le i < \nu$, we have

$$Q_i^{(r)} = \left\{x_n^{(i)} : 0 \le n < \nu^r\right\} \quad \text{with} \quad d_h(x_m^{(i)}, x_n^{(i)}) = \hat{d}_h(m,n).$$

Now, for each $N = i\nu^r + n \in \{0, 1, 2, \ldots, \nu^{r+1} - 1\}$, we define $x_N = x_n^{(i)}$. Then

$$Q = \left\{x_N : 0 \le N < \nu^{r+1}\right\} \quad \text{and each} \quad Q_i^{(r)} = \{x_N : i\nu^r \le N < (i+1)\nu^r\}.$$

Furthermore, for each $M = i\nu^r + m$ and $N = j\nu^r + n$ with $0 \le i, j < \nu$ and $0 \le m, n < \nu^r$,

$$d_h(x_M, x_N) = d_h(x_m^{(i)}, x_n^{(j)}) = \begin{cases} \hat{d}_h(m,n) & \text{if } i = j \\ r+1 & \text{if } i \ne j \end{cases}.$$

If $i = j$ then by (2.12), we have $\hat{d}_h(M,N) = \hat{d}_h(m,n) = d_h(x_M, x_N)$. If $i < j$ then $0 \leq M < i\nu^r \leq N < \nu^{r+1}$ hence $\hat{d}_h(M,N) = r + 1 = d_h(x_M, x_N)$. ■

Proof of Proposition 2.1. It is clear from the construction in Lemma 2.2 that we may recursively construct an infinite sequence $\{x_n\}_{n\geq 0}$ with $d_h(x_m, x_n) = \hat{d}_h(m,n)$ for all $m, n \geq 0$ and with the first ν^r terms of this sequence enumerating $Q^{(r)}(x_0)$

$$Q^{(r)}(x_0) = \{x_n : 0 \leq n < \nu^r\} \quad \text{for each} \quad r.$$

For the first step of the recursion, we may choose $x_0 \in X$ (*the origin*) arbitrarily. At the r^{th} step, we generate the next $\nu^r - \nu^{r-1}$ terms of the sequence which enumerate $Q^{(r)}(x_0) \diagdown Q^{(r-1)}(x_0)$. By (2.7), this sequence must enumerate all of X. ■

2.3 Hierarchical Addition

Enumerating each partition Π_r as in (2.9–2.10) we have

$$Q_i^{(m+r)} = \bigcup_{k=0}^{\nu^r - 1} Q_{i\nu^r + k}^{(m)} = \bigcup_{k=0}^{\nu^m - 1} Q_{i\nu^m + k}^{(r)} \tag{2.13}$$

and in particular, taking $m = 0$, we have $Q^{(r)}(x_n) = Q_i^{(r)}$ where $i = \lfloor n/\nu^r \rfloor$.

Now let's define a mapping $n : X \to \mathbb{N}$ by putting $n(x) = n$ if $x = x_n$ in the enumeration of X. Let's further define, for each $r \geq 0$, the r^{th}-*coordinate mapping* $n_r : X \to \{0, 1, \ldots, \nu - 1\}$ by putting $n_r(x) = n_r$ if, in the enumeration of Π_r, $Q^{(r)}(x)$ is the $(n_r + 1)^{\text{th}}$ cube of rank r contained in $Q^{(r+1)}(x)$. Then $n_r(x)$ is the $(r+1)^{\text{th}}$ digit of the base-ν representation of $n(x)$, i.e.,

$$n(x) = \sum_{r=0}^{\infty} n_r(x)\nu^r = \sum_{r=0}^{|x|_h - 1} n_r(x)\nu^r$$

where $|x|_h = d_h(x_0, x)$. Notice that $|x|_h = r$ if and only if $\nu^{r-1} \le n(x) < \nu^r$. We also have $n_k(x_{i\nu^r}) = 0$ for $k < r$ and $n_k(x_{i\nu^r}) = n_{k-r}(x_i)$ for $k \ge r$, hence $|x_{i\nu^r}|_h = r + |x_i|_h$.

Define an additive group (*hierarchical addition*) on X by putting for each $r \ge 0$

$$n_r(x \dot{+} y) = n_r(x) + n_r(y) \mod \nu.$$

It means that we add the indices for x and y in base-ν except that we forget to carry the "tens" over to the next digit whenever $n_r(x) + n_r(y) \ge \nu$. Proposition 2.1 says that no matter how (X, d_h) has been constructed, we may as well assume $(X, d_h) = (\mathbb{N}, \hat{d}_h)$. Accordingly, we will identify $x_n \in X$ with $n \in \mathbb{N}$ and write $x \dot{+} n$ instead of $x \dot{+} x_n$.

The first cube $Q_0^{(r)}$ of each rank is a subgroup of $(X, \dot{+})$ whose cosets are given by $Q^{(r)}(x) = x \dot{+} Q_0^{(r)}$. As a result, we have

$$Q^{(r)}(x) \dot{+} Q^{(r)}(y) = Q^{(r)}(x \dot{+} y).$$

Furthermore, since $Q_0^{(m)}$ is a subgroup of $Q_0^{(m+r)}$, we have

$$Q_0^{(m)} \dot{+} Q_0^{(m+r)} = Q_0^{(m+r)}$$

so that

$$Q^{(m)}(x) \dot{+} Q^{(m+r)}(y) = Q^{(m+r)}(x \dot{+} y).$$

Similarly, because $i\nu^r \dot{+} j\nu^r = (i \dot{+} j)\nu^r$ and $Q_i^{(r)} = Q^{(r)}(i\nu^r)$, we have

$$Q_i^{(r)} \dot{+} Q_j^{(r)} = Q_{i \dot{+} j}^{(r)}$$

and it follows from (2.13) that

$$Q_{i\nu^r+k}^{(m)} \dot{+} Q_j^{(m+r)} = Q_{i \dot{+} j}^{(m+r)} \quad \text{for} \quad 0 \le k < \nu^r.$$

Now (2.13) becomes

$$Q^{(m+r)}(x) = \bigcup_{k=0}^{\nu^r-1} Q^{(m)}(x \dot{+} k\nu^m) = \bigcup_{k=0}^{\nu^m-1} Q^{(r)}(x \dot{+} k\nu^r). \tag{2.14}$$

CHAPTER 3: HEIRARCHICAL LAPLACIANS

3.1 Averaging Operators and Associated Subspaces of $\mathbb{C}^X$

We define an operator $A_r : \mathbb{C}^X \to \mathbb{C}^X$, the r^{th}-*rank averaging operator*, in the space $\mathbb{C}^X$ of complex-valued functions defined on X by putting

$$A_r f(x) = \frac{1}{\nu^r} \sum_{z \in Q^{(r)}(x)} f(z) \tag{3.1}$$

for $f : X \to \mathbb{C}$. Equivalently,

$$A_r f = \sum_{Q \in \Pi_r} f_Q \mathbf{1}_Q \tag{3.2}$$

where $\mathbf{1}_Q : X \to \{0,1\}$ is the indicator function of a set $Q \subseteq X$ and $f_Q = f_i^{(r)}$ is the average value of f on the cube $Q = Q_i^{(r)}$, i.e.,

$$f_Q = \frac{1}{|Q|} \sum_{x \in Q} f(x).$$

Then $A_r : \mathbb{C}^X \to \mathcal{M}_r$ where $\mathcal{M}_r$ is the subspace of functions which are constant on cubes of rank r. Note that $f \in \mathcal{M}_r$ if and only if $A_r f = f$. Since every cube of rank $k < r$ is contained in a cube of rank r, we see that

$$k < r \quad \text{implies} \quad \mathcal{M}_r \subseteq \mathcal{M}_k. \tag{3.3}$$

Since every cube of rank r is a disjoint union of exactly ν^{r-k} cubes of rank $k < r$, considering the average of averages, we see that

$$k < r \quad \text{implies} \quad A_k A_r = A_r A_k = A_r. \tag{3.4}$$

Notice that whenever $z \in Q_0^{(r)}$, since $Q^{(r)}(x \dotplus z) = Q^{(r)}(x)$, we have

$$A_r f(x \dotplus z) = A_r f(x)$$

for all $x \in X$, i.e., each $z \in Q_0^{(r)}$ is a "period" for $A_r f$. Similarly, we have

$$f(x \dotplus z) \equiv f(x) \quad \text{for all } f \in \mathcal{M}_r \text{ and } z \in Q_0^{(r)}.$$

It means we may think of $\mathcal{M}_r$ as the space of $Q_0^{(r)}$-periodic functions defined on the group $(X, \dotplus)$.

Proposition 3.1. *For $1 \le k \le r$, if $f \in \mathcal{M}_r$ then for every $g \in \mathbb{C}^X$, we have*

$$A_k f g = f A_k g. \tag{3.5}$$

In other words, A_k treats functions $f \in \mathcal{M}_r$ like constants.

Proof. For $1 \le k \le r$, if $f \in \mathcal{M}_r$ then by (3.3), f is constant on cubes of rank k hence for every $g \in \mathbb{C}^X$, we have

$$(A_k f g)(x) = \frac{1}{\nu^k} \sum_{y \in Q^{(k)}(x)} f(y) g(y) = \frac{1}{\nu^k} \sum_{y \in Q^{(k)}(x)} f(x) g(y) = f(x)(A_k g)(x)$$

which proves (3.5). ∎

Now we mention two subspaces related to $\mathcal{M}_r$ and corresponding operators related to A_r which are needed in the sequel. First, the subspace $\mathcal{L}_r$ consists of all $f \in \mathbb{C}^X$

which are constant on cubes of rank $< r$ with $\sum f = 0$ on cubes of rank $\geq r$, i.e.,

$$A_k f = f \quad \text{for} \quad 1 \leq k < r \qquad \text{and} \qquad A_k f = 0 \quad \text{for all} \quad k \geq r. \tag{3.6}$$

We have $f \in \mathcal{L}_r$ if and only if $E_r f = f$ where $E_r : \mathbb{C}^X \to \mathcal{L}_r$ is defined by

$$E_r = A_{r-1} - A_r. \tag{3.7}$$

It follows from (3.4) that

$$k < r \quad \text{implies} \quad E_k E_r = E_r E_k = 0. \tag{3.8}$$

Next, for each $Q \in \Pi_r$ (each cube of rank r), the subspace $\mathcal{L}_Q$ consists of all $f \in \mathcal{L}_r$ which vanish outside of Q. Then $f \in \mathcal{L}_Q$ if and only if $E_Q f = f$ where $E_Q : \mathbb{C}^X \to \mathcal{L}_Q$ is given by

$$E_Q = \mathbf{1}_Q E_r. \tag{3.9}$$

Since $\mathbf{1}_X = \sum_{Q \in \Pi_r} \mathbf{1}_Q$, it follows that

$$E_r = \sum_{Q \in \Pi_r} E_Q. \tag{3.10}$$

If $Q = \bigcup_{i=1}^{\nu} Q_i^{(r-1)}$, then $\mathcal{L}_Q$ consists of all functions which vanish outside of Q and are constant on each subcube $Q_k^{(r-1)}$ of preceeding rank with the sum of those constants being zero, i.e., functions of the form

$$f = E_Q f = \sum_{i=1}^{\nu} c_i \mathbf{1}_{Q_i^{(r-1)}} \quad \text{with} \quad \sum_{i=1}^{\nu} c_i = 0. \tag{3.11}$$

Corollary 3.2. *For $1 \leq k \leq r$, if $f \in \mathcal{M}_r$ and $g \in \mathcal{L}_k$ then fg is constant on cubes of rank $k-1$ and $\sum fg = 0$ on cubes of rank k hence $fg \in \mathcal{L}_k$. In other words, $\mathcal{L}_k$*

absorbs multiplication from functions in $\mathcal{M}_r$.

From (2.14) we obtain

$$A_{m+r}f(x) = \frac{1}{\nu^r}\sum_{k=0}^{\nu^r-1} A_m f(x \dot{+} k\nu^m) = \frac{1}{\nu^m}\sum_{k=0}^{\nu^m-1} A_r f(x \dot{+} k\nu^r).$$

In particular,

$$A_r f(x) = \frac{1}{\nu}\sum_{k=0}^{\nu-1} A_{r-1} f(x \dot{+} k\nu^{r-1})$$

and therefore

$$-E_r f(x) = A_r f(x) - A_{r-1}f(x) = \frac{1}{\nu}\sum_{k=1}^{\nu-1} A_{r-1} f(x \dot{+} k\nu^{r-1}).$$

Next we consider the subspace of functions $f \in \mathbb{C}^X$ which have a limit as x approaches the point at infinity in the one-point compactification of X — Proposition 3.3 below shows that A_r is invariant on this subspace. Whenever we write $\lim_{x\to\infty} f(x) = c$ or $f(x) \to c$ as $x \to \infty$, it is equivalent to saying that for every $\varepsilon > 0$, there exists n such that $|x|_h > n$ implies $|f(x) - c| < \varepsilon$.

Proposition 3.3. *If* $\lim_{x\to\infty} f(x) = c$ *then* $\lim_{x\to\infty} A_r f(x) = c$.

Proof. Because $A_r(f - c) \equiv A_r f - c$ we may assume $c = 0$. Let $\varepsilon > 0$. Then there exists $n > r$ such that $|x|_h > n$ implies $|f(x)| < \varepsilon$. If $|x|_h > n$ then $|y|_h > n$ for every $y \in Q^{(r)}(x)$ hence

$$|A_r f(x)| \le \frac{1}{\nu^r}\sum_{y\in Q^{(r)}(x)} |f(y)| < \varepsilon.$$

Therefore, $\lim_{x\to\infty} A_r f(x) = 0$. ∎

Lemma 3.4. *If* $\lim_{x\to\infty} f(x) = c$ *then* $\lim_{r\to\infty} A_r f(x) = c$ *for every* $x \in X$.

Proof. Again we may assume $c = 0$. First observe that for $r \geq m$ we have

$$A_r f(x) = \frac{1}{\nu^{r-m}} \sum_{k=0}^{\nu^{r-m}-1} A_m f(x \dot{+} k\nu^m).$$

If $m > |x|_h$ and $k > 0$ then we have $x \dot{+} k\nu^m \geq \nu^{m-1}$ so that $|x \dot{+} k\nu^m|_h \geq m$ hence

$$|A_m f(x \dot{+} k\nu^m)| \leq \max_{|y|_h \geq m} |f(y)|.$$

Now let $\varepsilon > 0$ and choose $m > |x|_h$ so large that $\max\limits_{|y|_h \geq m} |f(y)| < \frac{\varepsilon}{2}$. Then we have

$$|A_r f(x)| \leq \frac{|A_m f(x)|}{\nu^{r-m}} + \sum_{k=1}^{\nu^{r-m}-1} \frac{|A_m f(x \dot{+} k\nu^m)|}{\nu^{r-m}} \leq \frac{\|f\|_\infty}{\nu^{r-m}} + \frac{\varepsilon}{2}$$

so that $|A_r f(x)| < \varepsilon$ for all $r > m + \log_\nu(1 + \frac{2}{\varepsilon} \|f\|_\infty)$. ■

Proposition 3.5. *If* $\lim\limits_{x\to\infty} f(x) = c$ *then we have*

$$f(x) = c + \sum_{r=1}^{\infty} E_r f(x) = c + \sum_{Q \in \mathcal{V}_1} E_Q f(x)$$

for every $x \in X$.

3.2 Symmetric Random Walk on Cubes of Rank r

To motivate our definition of A_r consider a random walk $\{x_n\}_{n\geq 0}$ beginning at the point $x \in X$ which at each step, jumps with equal probabilities to another point $y \in Q^{(r)}(x) = x \dot{+} Q_0^{(r)}$. In other words,

$$x_n = x \dot{+} z_1 \dot{+} \cdots \dot{+} z_n$$

where $\{z_n\}_{n\geq 1}$ is an i.i.d. sequence of uniformly distributed random elements of $Q_0^{(r)}$. It means that, beginning at $x \in X$, the probability, at the n^{th} step, of arriving at

$y \in X$ is given by

$$\mathbf{P}^x(x_n = y) = \frac{\mathbf{1}_r(x,y)}{\nu^r}$$

where

$$\mathbf{1}_r(x,y) = \mathbf{1}_{Q^{(r)}(x)}(y) = \mathbf{1}_{Q^{(r)}(y)}(x) = \mathbf{1}_r(y,x).$$

Then $\mathbf{P}^x$-a.s., $\{x_n\}$ never leaves the cube $Q^{(r)}(x)$ hence for all $f \in \mathbb{C}^X$ and $n \geq 1$,

$$\mathbf{E}^x f(x_n) = \sum_{y \in Q^{(r)}(x)} f(y)\, \mathbf{P}^x(x_n = y) = A_r f(x).$$

We say that A_r *generates a symmetric random walk on cubes of rank* r.

3.3 Averaging Operators and Associated Subspaces of $\ell^2(X)$

Let $\ell^2(X)$ be the Hilbert space of square-summable functions on X with inner product and norm

$$\langle \psi, \varphi \rangle = \sum_{x \in X} \psi(x)\overline{\varphi(x)} \qquad \text{and} \qquad \|\psi\|^2 = \sum_{x \in X} |\psi(x)|^2.$$

The matrix element for A_r is given by

$$\langle A_r \delta_x, \delta_y \rangle = \nu^{-r} \mathbf{1}_r(x,y) = \langle \delta_x, A_r \delta_y \rangle. \tag{3.12}$$

hence A_r is self-adjoint. Because $A_r^2 = A_r$, it follows that A_r is the orthogonal projection onto the subspace $\mathcal{M}_r$ of $\ell^2(X)$. Similarly, E_r is the orthogonal projection onto $\mathcal{L}_r$ and it follows from (3.3) that

$$\mathcal{L}_r = \mathcal{M}_{r-1} \cap \mathcal{M}_r^{\perp}.$$

For $r < s$, since $\mathcal{M}_{s-1} \subseteq \mathcal{M}_r$, we have

$$\mathcal{L}_r \cap \mathcal{L}_s \subseteq \mathcal{M}_r \cap \mathcal{M}_r^{\perp} = \{0\}$$

hence

$$\mathcal{L}_r \perp \mathcal{L}_s \quad \text{for} \quad r \neq s. \tag{3.13}$$

For each cube $Q \in \Pi_r$, E_Q is the orthogonal projection onto $\mathcal{L}_Q$. Furthermore, (3.13) implies that $\mathcal{L}_Q$ is orthogonal to $\mathcal{L}_{Q'}$ for $Q, Q' \in \mathcal{V}_1$ with $Q \neq Q'$. It follows from (3.10) that

$$\mathcal{L}_r = \bigoplus_{Q \in \Pi_r} \mathcal{L}_Q. \tag{3.14}$$

It follows from (3.11) that $\mathcal{L}_Q$ is finite dimensional with

$$\dim \mathcal{L}_Q = \dim \{(c_1 \ldots, c_\nu) \in \mathbb{C}^\nu : c_1 + \cdots + c_\nu = 0\} = \nu - 1.$$

Together with the second equation in (3.14), this further implies that $\dim \mathcal{L}_r = \infty$. From (3.11), it is immediate that the orthogonal complement of $\mathcal{L}_Q$ consists of all $\psi \in \ell^2(X)$ which are constant on Q. If $\psi \in \left(\bigoplus_{Q \in \mathcal{V}_1} \mathcal{L}_Q\right)^{\perp}$ then ψ is constant on every cube $Q \in \mathcal{V}_1$ so by (2.4), ψ is constant on X which means $\psi \equiv 0$ on X. It follows that

$$\ell^2(X) = \bigoplus_{Q \in \mathcal{V}_1} \mathcal{L}_Q = \bigoplus_{r=1}^{\infty} \mathcal{L}_r \tag{3.15}$$

hence

$$I = \sum_{Q \in \mathcal{V}_1} E_Q = \sum_{r=1}^{\infty} E_r. \tag{3.16}$$

Alternately, for functions in $\ell^2(X)$, (3.16) follows from Proposition 3.5.

3.4 Heirarchical Random Walk and Laplacian

The hierarchical laplacian is defined for each $\psi \in \ell^2(X)$ by

$$\Delta\psi(x) = \sum_{y\in X} p(x,y)\big(\psi(y) - \psi(x)\big) \tag{3.17}$$

where $p(x,y)$ are the transition probabilities for the discrete time hierarchical random walk $\{x_n\}_{n\geq 0}$ whose probability matrix is given by $I + \Delta = [p(x,y)]_{X\times X}$ where I is the identity operator on $\ell^2(X)$, i.e., for each $\psi \in \ell^2(X)$,

$$(I+\Delta)\psi(x) = \sum_{y\in X} p(x,y)\psi(y). \tag{3.18}$$

It means that Δ generates the semigroup $e^{t\Delta} = [p(t,x,y)]_{X\times X}$ for the continuous time random walk, $x_t = x_{N(t)}$, where $N(t)$ is a Poisson process independent of $\{x_n\}_{n\geq 0}$ with intensity equal to one. Our definition of the hierarchical Laplacian follows [10, 11] but sometimes [9, 6], $I + \Delta$ is referred to as the hierarchical Laplacian.

To define the discrete time hierarchical random walk, we fix an i.i.d. sequence $\{\rho_n\}_{n\geq 1}$ of random variables supported on the positive integers and we assume there exist constants $p \in (0,1)$ and $\alpha > 0$ such that for every $r \in \mathbb{Z}^+$,

$$(1/p - 1)p^{r+\alpha} \leq \mathbf{P}(\rho = r) \leq (1/p-1)p^{r-\alpha}. \tag{3.19}$$

In (3.19), we always keep in mind the case where ρ is geometrically distributed, i.e., where $\alpha = 0$. Now, at each time n, the random-walking particle jumps to the site x_n which is uniformly distributed within the cube of rank ρ_n containing x_{n-1}, i.e.,

$$\mathbf{P}\big(x_n = y \,|\, x_{n-1} = x \;\&\; \rho_n = r\big) = \frac{\mathbf{1}_r(x,y)}{\nu^r} = \langle A_r\delta_x, \delta_y\rangle \tag{3.20}$$

Since ρ_n is independent of x_{n-1} the transition probabilities are easily computed:

$$p(x,y) = \mathbf{P}\big(x_n = y \,|\, x_{n-1} = x\big) = \sum_{r=1}^{\infty} \frac{\mathbf{P}(\rho = r)\mathbf{1}_r(x,y)}{\nu^r}. \tag{3.21}$$

In particular, $p(x,y)$ depends only on $d_h(x,y)$, i.e., $p(x,x) = a_1$ and $p(x,y) = a_r$ for $d_h(x,y) = r \geq 1$ where $a_r = \sum_{k=r}^{\infty} \frac{\mathbf{P}(\rho=k)}{\nu^k}$. This allows us to diagonalize Δ. We do this by summing first, for each individual rank r, the terms in (3.18) with $d_h(x,y) = r$, i.e., we first sum over each sphere $Q^{(r)}(x) \setminus Q^{(r-1)}(x)$ of radius r centered around x. We have

$$\sum_{y:d_h(x,y)=r} \psi(y) = \big\langle \psi, \mathbf{1}_{Q^{(r)}(x) \setminus Q^{(r-1)}(x)} \big\rangle = \big\langle \psi, \mathbf{1}_{Q^{(r)}(x)} \big\rangle - \big\langle \psi, \mathbf{1}_{Q^{(r-1)}(x)} \big\rangle.$$

Therefore, since $a_r - a_{r+1} = \frac{\mathbf{P}(\rho=r)}{\nu^r}$, summation by parts gives us

$$\sum_{y:y\neq x} p(x,y)\psi(y) = \sum_{r=1}^{\infty} a_r \sum_{y:d_h(x,y)=r} \psi(y) = -a_1\psi(x) + \sum_{r=1}^{\infty} \frac{\mathbf{P}\big(\rho = r\big)\big\langle \psi, \mathbf{1}_{Q^{(r)}(x)} \big\rangle}{\nu^r} \tag{3.22}$$

so that

$$(I + \Delta)\psi(x) = \sum_{r=1}^{\infty} \frac{\mathbf{P}\big(\rho = r\big)\big\langle \psi, \mathbf{1}_{Q^{(r)}(x)} \big\rangle}{\nu^r}. \tag{3.23}$$

Equation (3.23) now becomes

$$I + \Delta = \sum_{r=1}^{\infty} \mathbf{P}(\rho = r) A_r \qquad \text{or} \qquad \Delta = \sum_{r=1}^{\infty} \mathbf{P}(\rho = r)(A_r - I) \tag{3.24}$$

which implies Δ is self-adjoint. If we put $\lambda_r = \mathbf{P}\big(\rho \geq r\big)$, since $E_r = A_{r-1} - A_r$, another summation by parts gives us

$$-\Delta = \sum_{r=1}^{\infty} (\lambda_r - \lambda_{r+1})(I - A_r) = \sum_{r=1}^{\infty} \lambda_r E_r. \tag{3.25}$$

From (3.15), we see that (3.25) diagonalizes Δ. The functional calculus for Δ is given by

$$f(\Delta) = \sum_{r=1}^{\infty} f(-\lambda_r) E_r. \tag{3.26}$$

For any function f which is bounded on $\mathrm{Sp}(\Delta) = \{-\lambda_r : r \geq 1\} \cup \{0\}$, the operator $f(\Delta)$ is bounded with

$$\|f(\Delta)\| = \max \{|f(\lambda)| : \lambda \in \mathrm{Sp}(\Delta)\}.$$

Since $\langle A_r \delta_x, \delta_y \rangle = \nu^{-r} \mathbf{1}_r(x, y)$, the matrix element for $f(\Delta)$ is given by

$$f(\Delta)(x, y) = \langle f(\Delta)\delta_x, \delta_y \rangle = \sum_{k=1}^{\infty} f(-\lambda_k) \left(\frac{\mathbf{1}_{k-1}(x, y)}{\nu^{k-1}} - \frac{\mathbf{1}_k(x, y)}{\nu^k} \right) \tag{3.27}$$

The sum in (3.27) can be simplified in two ways depending on whether or not $d_h(x, y) = 0$. We have

$$f(\Delta)(x, x) = \left(1 - \frac{1}{\nu}\right) \sum_{k=0}^{\infty} \frac{f(-\lambda_{k+1})}{\nu^k} \tag{3.28}$$

and for $d_h(x, y) = r > 0$ we have

$$f(\Delta)(x, y) = -\frac{f(-\lambda_r)}{\nu^r} + \left(1 - \frac{1}{\nu}\right) \sum_{k=r}^{\infty} \frac{f(-\lambda_{k+1})}{\nu^k} \tag{3.29}$$

In particular, for $\lambda > 0$, taking $f(x) = \mathbf{1}_{[0,\lambda)}(-x)$ in (3.28), we obtain an expression for the integrated density of states for $-\Delta$ (see [11]). We have

$$N(\lambda) = \left(1 - \frac{1}{\nu}\right) \sum_{k=0}^{\infty} \frac{\mathbf{1}_{[0,\lambda)}(\lambda_{k+1})}{\nu^k}. \tag{3.30}$$

It follows that the "density" of states for $-\Delta$ is simply a sum

$$n(\lambda) = \left(1 - \frac{1}{\nu}\right) \sum_{k=0}^{\infty} \frac{\delta_{\lambda_{k+1}}(\lambda)}{\nu^k}$$

of point masses along $\mathrm{Sp}(-\Delta)$. Our assumption (3.19) allows us to find the asymptotics of $N(\lambda)$ as $\lambda \to 0+$. Observe that for every $r \in \mathbb{Z}^+$, by (3.19), we have

$$p^{r+\alpha} \leq \lambda_{r+1} \leq p^{r-\alpha}. \tag{3.31}$$

Since $\mathbf{1}_{[0,\lambda)}$ is non-increasing on $[0, \infty)$, it follows that for every $k \geq 0$,

$$\frac{\mathbf{1}_{[0,\lambda)}(p^{k-\alpha})}{\nu^k} \leq \frac{\mathbf{1}_{[0,\lambda)}(\lambda_{k+1})}{\nu^k} \leq \frac{\mathbf{1}_{[0,\lambda)}(p^{k+\alpha})}{\nu^k}. \tag{3.32}$$

The left-hand side of (3.32) is non-zero if and only if $k > \alpha + \log_p \lambda$ and the right-hand side is non-zero if and only if $k > -\alpha + \log_p \lambda$. Summing the geometric series $\left(1 - \frac{1}{\nu}\right) \sum \frac{1}{\nu^k}$ over all $k > \pm\alpha + \log_p \lambda$ (separately), we obtain

$$\frac{\lambda^{s_h/2}}{\nu^{\alpha+1}} \leq N(\lambda) \leq \nu^{\alpha} \lambda^{s_h/2} \tag{3.33}$$

where $s_h = -2 \log_p \nu > 0$—we call s_h the *spectral dimension of* Δ. The first inequality in (3.33) is valid for $0 \leq \lambda \leq p^{-\alpha}$ and the second for $0 \leq \lambda \leq p^{\alpha}$. It immediately implies that

$$\lim_{\lambda \to 0+} \lambda^{-s_h/2} N(\lambda) = 1. \tag{3.34}$$

For the resolvent operator, $R_\lambda = (\lambda I - \Delta)^{-1}$, (3.27) gives us

$$R_\lambda(x, x) = \left(1 - \frac{1}{\nu}\right) \sum_{k=0}^{\infty} \frac{1}{\nu^k(\lambda + \lambda_{k+1})}$$

and applying (3.27) to the semigroup

$$e^{t\Delta} = \sum_{r=1}^{\infty} e^{-\lambda_r t} E_r = [p(t, x, y)]_{X \times X}, \tag{3.35}$$

we obtain the transition probabilities for the continuous-time hierarchical random walk. We want to find the asymptotics of

$$p(t,x,x) = \left(1 - \frac{1}{\nu}\right) \sum_{k=0}^{\infty} \frac{e^{-\lambda_{k+1} t}}{\nu^k} \tag{3.36}$$

as $t \to \infty$ and and of $R_\lambda(x,x)$ as $\lambda \to 0$. For that, we will first find the asymptotics of the function

$$\theta(t) = \left(1 - \frac{1}{\nu}\right) \sum_{k=0}^{\infty} \frac{e^{-p^k t}}{\nu^k} \tag{3.37}$$

and its Laplace transform

$$\Theta(\lambda) = \int_0^\infty e^{-\lambda t}\theta(t)\,dt = \left(1 - \frac{1}{\nu}\right) \sum_{k=0}^{\infty} \frac{1}{\nu^k(\lambda + p^k)}, \tag{3.38}$$

i.e., $p(t,x,x)$ and $R_\lambda(x,x)$ in the case where $\mathbf{P}\big(\rho = r\big) = (1/p - 1)p^r$. Considering the continuous analogue of $\theta(t)$, i.e., $\tilde{\theta}(t) = \log \nu \int_0^\infty \nu^{-x} e^{-p^x t}\,dx$, we see that $t^{s_h/2}\theta(t)$ is essentially the discrete analogue of an incomplete Gamma function—when we substitute $y = p^x t$, we obtain

$$t^{s_h/2}\tilde{\theta}(t) = \tfrac{s_h}{2} \int_0^t y^{s_h/2-1} e^{-y}\,dy \to \Gamma\big(1 + \tfrac{s_h}{2}\big) \quad \text{as} \quad t \to \infty.$$

Replacing $\Gamma(1 + \frac{s_h}{2})$ with a logarithmically periodic function of t, the same holds for $\theta(t)$.

Proposition 3.6. *For arbitrary spectral dimension s_h, there exists a periodic function $h(z) = \left(1 - \frac{1}{\nu}\right) \sum_{-\infty}^{\infty} \frac{e^{-p^{k+z}}}{\nu^{k+z}}$ such that $t^{s_h/2}\theta(t) \sim h\left(\log_p t\right)$ as $t \to \infty$.*

Proof. First, observe that

$$h(\log_p t) = h(z) = \left(1 - \frac{1}{\nu}\right) \sum_{k=-\infty}^{\infty} \frac{e^{-p^{k+z}}}{\nu^{k+z}} = \left(1 - \frac{1}{\nu}\right) \sum_{k=-\infty}^{\infty} \frac{e^{-p^{k+\{z\}}}}{\nu^{k+\{z\}}}$$

where $z = z(t) = \log_p t$ and the last equality is from replacing the index k with

$k - \lfloor z \rfloor$. Since $\{z+1\} \equiv \{z\}$, this also shows that $h(z) \equiv h(\{z\})$ is periodic with period one. Next, since $t = p^z$ implies $t^{s_h/2} = \nu^{-z}$ and $e^{-p^k t} = e^{-p^{k+z}}$, we have

$$t^{s_h/2}\theta(t) = \left(1 - \frac{1}{\nu}\right) \sum_{k=0}^{\infty} \frac{e^{-p^{k+z}}}{\nu^{k+z}} = \left(1 - \frac{1}{\nu}\right) \sum_{k=\lfloor z \rfloor}^{\infty} \frac{e^{-p^{k+\{z\}}}}{\nu^{k+\{z\}}}.$$

Therefore, since $\lfloor z(t) \rfloor \to -\infty$ as $t \to \infty$,

$$\frac{t^{s_h/2}\theta(t)}{h_1(\log_p t)} = \frac{\sum_{k=\lfloor z \rfloor}^{\infty} \frac{e^{-p^{k+\{z\}}}}{\nu^{k+\{z\}}}}{\sum_{k=-\infty}^{\infty} \frac{e^{-p^{k+\{z\}}}}{\nu^{k+\{z\}}}} \to 1 \quad \text{as} \quad t \to \infty$$

which completes the proof. ■

Proposition 3.7. *For arbitrary spectral dimension s_h, $p(t,x,x) \asymp t^{-s_h/2}$ as $t \to \infty$. More precisely, there exists a periodic function $h(z) = \left(1 - \frac{1}{\nu}\right) \sum_{-\infty}^{\infty} \frac{e^{-p^{k+z}}}{\nu^{k+z}}$ such that*

$$\frac{1}{\nu^{\alpha+1}} \le \frac{t^{s_h/2} p(t,x,x)}{h(\log_p t)} \le \nu^{\alpha+1} \quad \textit{as} \quad t \to \infty. \tag{3.39}$$

Proof. As in (3.32), since the function $\lambda \mapsto e^{-\lambda t}$ is decreasing, by (3.31) we have

$$\theta(p^{-\lceil \alpha \rceil} t) \le \theta(p^{-\alpha} t) \le p(t,x,x) \le \theta(p^{\alpha} t) \le \theta(p^{\lceil \alpha \rceil} t) \tag{3.40}$$

where $\lceil \alpha \rceil = \min\{n \in \mathbb{Z} : n \ge \alpha\}$. Dividing through by $t^{-s_h/2} h(\log_p t)$ and observing that $h\big(\log_p(p^{\pm\lceil \alpha \rceil} t)\big) = h(\log_p t)$, we have

$$\frac{t^{s_h/2}\theta(p^{-\lceil \alpha \rceil} t)}{h\big(\log_p(p^{-\lceil \alpha \rceil} t)\big)} \le \frac{t^{s_h/2} p(t,x,x)}{h(\log_p t)} \le \frac{t^{s_h/2}\theta(p^{\lceil \alpha \rceil} t)}{h\big(\log_p(p^{\lceil \alpha \rceil} t)\big)}. \tag{3.41}$$

As $t \to \infty$, the left-hand side converges to $\nu^{-\lceil \alpha \rceil} \ge \nu^{-\alpha-1}$ while the right-hand side converges to $\nu^{\lceil \alpha \rceil} \le \nu^{\alpha+1}$. ■

The next statement provides the asymptotics of $\Theta(\lambda)$ as $\lambda \to +0$ in the case where

$\alpha = 0$ and $s_h < 2$.

Proposition 3.8.

$$\Theta(\lambda) = \frac{u(\log_p \lambda)}{\lambda^{1-s_h/2}} + c_0 + O(\lambda), \quad \textit{as} \quad \lambda \to 0+, \tag{3.42}$$

where $c_0 = \frac{p(\nu-1)}{\nu p - 1}$, *and* $u(z) = (\nu p)^z \left(1 - \frac{1}{\nu}\right) \sum_{-\infty}^{\infty} \frac{(\nu p)^k}{1+p^{k+z}}$ *is a positive periodic function with period one.*

Proof. Observe that $c_0 = -\left(1 - \frac{1}{\nu}\right) \sum_{k=1}^{\infty} (\nu p)^k$ and because $\nu p < 1$, the series

$$\frac{u(\log_p \lambda)}{\lambda^{1-s_h/2}} = \left(1 - \frac{1}{\nu}\right) \sum_{k=-\infty}^{\infty} \frac{(\nu p)^k}{1 + p^k \lambda}$$

converges for all complex $\lambda \notin \{0\} \cup \{-p^N : N \in \mathbb{Z}\}$. From the series representation (3.38),

$$\Theta(\lambda) = \left(1 - \frac{1}{\nu}\right) \sum_{k=-\infty}^{0} \frac{(\nu p)^k}{1 + p^k \lambda}.$$

Then

$$\Theta(\lambda) - \frac{u(\log_p \lambda)}{\lambda^{1-s_h/2}} - c_0 = \left(1 - \frac{1}{\nu}\right) \sum_{k=1}^{\infty} \left[(\nu p)^k - \frac{(\nu p)^k}{1 + p^k \lambda}\right] = \left(1 - \frac{1}{\nu}\right) \sum_{k=1}^{\infty} \frac{(\nu p^2)^k \lambda}{1 + p^k \lambda}$$

so that

$$\left|\Theta(\lambda) - \frac{u(\log_p \lambda)}{\lambda^{1-s_h/2}} - c_0\right| < \left(1 - \frac{1}{\nu}\right) \sum_{k=1}^{\infty} (\nu p^2)^k \lambda = \frac{p^2(\nu - 1)}{1 - \nu p^2} \lambda.$$

If $\lambda \to 0$ in the complex plane with $|\arg \lambda| \leq \pi - \delta$ for some $\delta > 0$, we obtain

$$\left|\Theta(\lambda) - \frac{u(\log_p \lambda)}{\lambda^{1-s_h/2}} - c_0\right| < \frac{p^2(\nu - 1)|\lambda|}{(1 - \nu p^2)\sqrt{1 - 2p|\lambda| \cos \delta}}$$

so that (3.42) remains valid. ∎

3.5 Heirarchical Laplacian with Variable Coefficients

The diagonalization of the hierarchical laplacian

$$-\Delta\psi(x) = \sum_{r=1}^{\infty} \lambda_r E_r \psi(x) \tag{3.43}$$

displays the fact that each eigenvalue $\lambda_r = \mathbf{P}(\rho \geq r)$ is an isolated point in $\mathrm{Sp}(-\Delta)$ and has multiplicity $\dim \mathcal{L}_r = \infty$. To correct these "defects", we first observe that by (3.14), we have the further diagonalization

$$-\Delta\psi(x) = \sum_{r=1}^{\infty} \lambda_r \left(\sum_{Q \in \Pi_r} E_Q \psi(x) \right) = \sum_{Q \in \mathcal{V}_1} \lambda_Q E_Q \psi(x) \tag{3.44}$$

where $\lambda_Q = \lambda_r$ for each $Q \in \Pi_r$. In essence, it seems that because the mapping $Q \mapsto \lambda_Q$ from $\mathcal{V}_1$ to $\mathrm{Sp}(-\Delta)$ is constant on each $\Pi_r \subseteq \mathcal{V}_1$, the finite dimensional subspaces, $\mathcal{L}_Q$ for $Q \in \Pi_r$, which should have been the eigenspaces, have instead been collapsed into the infinite dimensional eigenspace $\mathcal{L}_r$.

A *hierarchical Laplacian* $\widetilde{\Delta}$ *with variable coeficients* is a modification of Δ where, in (3.44), we instead allow λ_Q to vary for different $Q \in \Pi_r$. This amounts to replacing each constant λ_r in (3.43) with a function $\lambda^{(r)} : X \to \mathbb{R}$ which is single-valued on every cube of rank r, i.e., $\widetilde{\Delta}$ is an operator of the form

$$-\widetilde{\Delta}\psi(x) = \sum_{r=1}^{\infty} \lambda^{(r)}(x) E_r \psi(x) = \sum_{Q \in \mathcal{V}_1} \lambda_Q E_Q \psi(x) \tag{3.45}$$

where $\lambda_Q = \lambda_i^{(r)}$ is now the single value of the function $\lambda^{(r)}(x)$ on the cube $Q = Q_i^{(r)}$. Note that because $\lambda^{(r)} \in \mathcal{M}_r$, it means that we have

$$E_r(\lambda^{(r)}\psi) = \lambda^{(r)} E_r \psi \qquad \text{so that} \qquad \langle \lambda^{(r)} E_r \varphi, \psi \rangle = \langle \varphi, \overline{\lambda^{(r)}} E_r \psi \rangle. \tag{3.46}$$

Therefore, since $\lambda^{(r)}(x)$ is real valued, the operator (3.45) is self-adjoint. This modification has the effect of breaking each eigenvalue $\lambda_r \in \mathrm{Sp}(-\Delta)$ with eigenspace $\mathcal{L}_r$ into a collection of eigenvalues

$$\mathrm{range}(\lambda^{(r)}) = \{\lambda_Q : Q \in \Pi_r\} \subseteq \mathrm{Sp}(-\widetilde{\Delta}) \tag{3.47}$$

with eigenspaces properly contained in $\mathcal{L}_r$. In particular, if we allow λ_Q to vary in such a way that the mapping $Q \mapsto \lambda_Q$ is one-to-one, each $\mathcal{L}_Q$ for $Q \in \mathcal{V}_1$ will itself be an eigenspace of $\widetilde{\Delta}$ and the multiplicity of each eigenvalue λ_Q will be exactly $\dim \mathcal{L}_Q = \nu - 1$. In order to rid the spectrum of isolated points, we would like to define $\widetilde{\Delta}$ in such a way that for each $r \geq 1$, the eigenvalues $\{\lambda_Q : Q \in \Pi_r\}$ form a dense subset of the inteveral of length $2\sigma\lambda_r$ centered around λ_r where $\sigma \in (0,1)$ is a coupling constant (measure of disorder). This way, $\mathrm{Sp}(-\Delta)$ is contained in $\mathrm{Sp}(-\widetilde{\Delta})$ but each isolated eigenvalue $\lambda_r \in \mathrm{Sp}(-\Delta)$ is replaced by an interval

$$[\lambda_r(1-\sigma), \lambda_r(1+\sigma)] \subseteq \mathrm{Sp}(-\widetilde{\Delta}) \tag{3.48}$$

The condition $\sigma < 1$ ensures that we do not gain any negative spectrum. Furthermore, as $\sigma \to 0$, $\mathrm{Sp}(-\widetilde{\Delta})$ shrinks to $\mathrm{Sp}(-\Delta)$ and we obtain Δ as a special case of $\widetilde{\Delta}$.

Proposition 3.9. *Suppose $\widetilde{\Delta}$ is defined by* (3.45) *where the functions $\lambda^{(r)} \in \mathcal{M}_r$ are such that the mapping $Q \mapsto \lambda_Q$ from $\mathcal{V}_1$ to $\mathrm{Sp}(-\widetilde{\Delta})$ is one-to-one and for each rank r, the range of $\lambda^{(r)}$ is a dense subset of the interval* (3.48) *where $0 < \sigma < 1$ and $\lambda_r = \mathbf{P}(\rho \geq r)$. Then*

- *$\widetilde{\Delta} \leq 0$ is self-adjoint with $\|\widetilde{\Delta}\| = 1 + \sigma$,*

- $\mathrm{Sp}(-\widetilde{\Delta})$ *consists of zero together with the union of intervals in* (3.48) *for* $r \geq 1$,

- *the eigenspaces for* $-\widetilde{\Delta}$ *consist of* $\mathcal{L}_Q$ *for* $Q \in \mathcal{V}_1$ *with each eigenvalue* λ_Q *having multiplicity* $\nu - 1$.

Proof. Since $\{\lambda_Q : Q \in \Pi_r\}$ is dense in $[\lambda_r(1-\sigma), \lambda_r(1+\sigma)]$, we have

$$\mathrm{Sp}(-\widetilde{\Delta}) \supseteq \overline{\bigcup_{r=1}^{\infty} [\lambda_r(1-\sigma), \lambda_r(1+\sigma)]}.$$

If $\lambda \notin \overline{\bigcup_{r=1}^{\infty} \operatorname{supp} \lambda^{(r)}}$, then there exists an $\varepsilon > 0$ such that $|\lambda - \lambda_Q| \geq \varepsilon$ for every $Q \in \mathcal{V}_1$. Using the diagonalization

$$-\widetilde{\Delta} = \sum_{Q \in \mathcal{V}_1} \lambda_Q E_Q,$$

we see that

$$R_\lambda = (\lambda I + \widetilde{\Delta})^{-1} = \sum_{Q \in \mathcal{V}_1} \frac{1}{\lambda - \lambda_Q} E_Q.$$

Then for every $\psi \in \ell^2(X)$, we have

$$\|(\lambda I + \widetilde{\Delta})^{-1}\psi\|^2 = \sum_{Q \in \mathcal{V}_1} \frac{\|E_Q\psi\|^2}{|\lambda - \lambda_Q|^2} \leq \frac{1}{\varepsilon^2} \sum_{Q \in \mathcal{V}_1} \|E_Q\psi\|^2 = \frac{1}{\varepsilon^2}\|\psi\|^2$$

hence $\|R_\lambda\| \leq \varepsilon^{-1} < \infty$ and it follows that $\mathrm{Sp}(-\widetilde{\Delta}) = \overline{\bigcup_{r=1}^{\infty} \operatorname{supp} \lambda^{(r)}}$. ∎

CHAPTER 4: RANDOM HEIRARCHICAL LAPLACIAN

4.1 Definition

To define a *random* hierarchical Laplacian, let $\{\omega_Q : Q \in \mathcal{V}_1\}$ be an independent family of symmetric random variables with continuously differentiable densities supported on the interval $[-1,1]$ where for each $r \geq 1$, the random variables $\{\omega_Q : Q \in \Pi_r\}$ corresponding to cubes of rank r, are identically distributed. Then for any two different cubes Q and Q', we have $\omega_Q \overset{\text{law}}{=} \omega_{Q'}$ when Q and Q' have the same rank but except for in Section 4.4, we allow for the possibility that ω_Q and $\omega_{Q'}$ are distributed differently whenever Q and Q' have different ranks.

For each $r \geq 1$ we define $\omega^{(r)} : X \to [-1,1]$ by $\omega^{(r)}(x) = \omega_{Q^{(r)}(x)}$ and we define a random coefficient function $\xi^{(r)} : X \to [-p_r, p_r]$, where $p_r = \mathbf{P}(\rho = r)$, by

$$\xi^{(r)}(x) = p_r\big(1 + \sigma\omega^{(r)}(x)\big) \tag{4.1}$$

Then $\xi^{(r)}(x)$ and $\xi^{(r)}(y)$ are independent for $d_h(x,y) > r$ but $\xi^{(r)}(x) = \xi^{(r)}(y)$ whenever $d_h(x,y) \leq r$. For each $\psi \in \ell^2(X)$ we define

$$-\Delta_\omega\psi(x) = \sum_{k=1}^{\infty} \xi^{(k)}(x)(I - A_k)\psi(x) = \sum_{r=1}^{\infty} \lambda^{(r)}(x)E_r\psi(x). \tag{4.2}$$

where

$$\lambda^{(r)}(x) = \sum_{k=r}^{\infty} \xi^{(k)}(x) = \lambda_r + \sigma\sum_{k=r}^{\infty} p_k\omega^{(k)}(x). \tag{4.3}$$

Observe that (3.19) implies

$$p^{\alpha}\sum_{k=r}^{\infty} qp^{k-1}\big(1+\sigma\omega^{(k)}(x)\big) \le \lambda^{(r)}(x) \le p^{-\alpha}\sum_{k=r}^{\infty} qp^{k-1}\big(1+\sigma\omega^{(k)}(x)\big) \tag{4.4}$$

where $q = 1 - p$. Let $\lambda_Q = \lambda_i^{(r)}$ denote the single random value assumed by the function $\lambda^{(r)} : X \to \mathbb{R}$ on the cube $Q = Q_i^{(r)}$ of rank r. Because $\{\lambda_Q : Q \in \mathcal{V}_1\}$ is a continuous family of random variables, it means that $\mathbf{P}$-a.s., the mapping $Q \mapsto \lambda_Q$ is one-to-one. Therefore, each $\mathcal{L}_Q$ is an eigenspace for $-\Delta_\omega$ with eigenvalue λ_Q having finite multiplicity $\dim \mathcal{L}_Q = \nu - 1$. We'll prove that $-\Delta_\omega$ satisfies the conditions of Proposition 3.9 (see Proposition 4.1).

Consider the rescaling of $\lambda^{(r)}(x)$ given by

$$\zeta^{(r)}(x) = \frac{\lambda^{(r)}(x) - \lambda_r}{\sigma\lambda_r} = \frac{1}{\lambda_r}\sum_{k=0}^{\infty} p_{r+k}\omega^{(r+k)}(x). \tag{4.5}$$

By (3.19) and (3.31) the k^{th} term of this series is of order p^k and $\mathbf{P}$-a.s., we have

$$\big|\zeta^{(r)}(x)\big| \le \frac{1}{\lambda_r}\sum_{k=r}^{\infty} p_k = 1. \tag{4.6}$$

We denote the n^{th} partial sum by

$$\zeta_n^{(r)}(x) = \frac{1}{\lambda_r}\sum_{k=0}^{n-1} p_{r+k}\omega^{(r+k)}(x). \tag{4.7}$$

Observe that $\zeta_n^{(r)}(x)$ and $\zeta^{(r+n)}(x)$ are independent and we have

$$\zeta^{(r)}(x) = \zeta_n^{(r)}(x) + \tfrac{\lambda_{r+n}}{\lambda_r}\zeta^{(r+n)}(x). \tag{4.8}$$

Proposition 4.1. *For* $r \ge 1$, $\mathbf{P}$*-a.s.,* $\overline{\{\lambda_Q : Q \in \Pi_r\}} = \big[\lambda_r(1-\sigma), \lambda_r(1+\sigma)\big]$, *Proof.* For $(a,b) \subseteq \big[\lambda_r(1-\sigma), \lambda_r(1+\sigma)\big]$, let

$$(z-\varepsilon, z+\varepsilon) = \tfrac{(a,b)-\lambda_r}{\sigma\lambda_r} = \big(\tfrac{a-\lambda_r}{\sigma\lambda_r}, \tfrac{b-\lambda_r}{\sigma\lambda_r}\big).$$

Then $z \in (-1,1)$ and we have

$$\lambda^{(r)}(x) \in (a,b) \quad \text{if and only if} \quad |\zeta^{(r)}(x) - z| < \varepsilon.$$

Choose an integer $n > 2\alpha + 1$. Then by (4.8) and (3.31) we have

$$\begin{aligned}
|\zeta^{(r)}(x) - z| &\le \tfrac{\lambda_{n+r}}{\lambda_r}\big|\zeta^{(r+n)}(x) - z\big| + \big|\zeta_n^{(r)}(x) - \big(1 - \tfrac{\lambda_{n+r}}{\lambda_r}\big)z\big| \\
&\le p^{n-2\alpha}\varepsilon + \sum_{k=0}^{n-1} \tfrac{p_{r+k}}{\lambda_r}|\omega^{(r+k)}(x) - z| \\
&< p\varepsilon + \sum_{k=0}^{n-1} qp^{k-2\alpha}|\omega^{(r+k)}(x) - z|
\end{aligned}$$

where $q = 1 - p$. For each cube $Q_i^{(n+r)} \in \Pi_{n+r}$ let

$$\eta_i^{(n+r)} = \prod_{Q \in \mathcal{S}_i^{(n+r)}} \mathbf{1}_{(z-qp^{2\alpha}\varepsilon, z+qp^{2\alpha}\varepsilon)}(\omega_Q).$$

Then $\big\{\eta_i^{(n+r)}\big\}_{i=0}^{\infty}$ is an i.i.d. sequence of Bernoulli random variables with

$$\mathbf{P}\big(\eta_i^{(n+r)} = 1\big) = \mathbf{P}\big(|\omega_Q - z| < qp^{2\alpha}\varepsilon\big)^{\left|\mathcal{S}_i^{(n+r)}\right|} \in (0,1]$$

Then $\mathbf{P}$-a.s., there exists a cube $Q_i^{(n+r)} \in \Pi_{n+r}$ with $\eta_i^{(n+r)} = 1$. Then for every $x \in Q_i^{(n+r)}$,

$$|\omega^{(r+k)}(x) - z| < qp^{2\alpha}\varepsilon \quad \text{for} \quad 0 \le k \le r+n$$

hence

$$\big|\zeta_n^{(r)}(x) - \big(1 - \tfrac{\lambda_{n+r}}{\lambda_r}\big)z\big| \le \sum_{k=0}^{n-1} qp^{k-2\alpha}|\omega^{(r+k)}(x) - z| < q\varepsilon$$

so that $|\zeta^{(r)}(x) - z| < \varepsilon$. Then $\mathbf{P}$-a.s., we have $\lambda^{(r)}(x) \in (a,b)$ so that

$$(a,b) \cap \{\lambda_Q : Q \in \Pi_r\} \neq \varnothing.$$

Now by considering $(a,b) \subseteq \big[\lambda_r(1-\sigma), \lambda_r(1+\sigma)\big]$ with rational endpoints it follows that $\mathbf{P}$-a.s., $\{\lambda_Q : Q \in \Pi_r\}$ is dense in $\big[\lambda_r(1-\sigma), \lambda_r(1+\sigma)\big]$. ■

The eigenvalues are of course dependent but in a sense which we will make precise, $\lambda^{(m)}(x)$ and $\lambda^{(n)}(y)$ become nearly uncorrelated whenever the graph distance (2.6) between $Q^{(m)}(x)$ and $Q^{(n)}(y)$ is large.

Lemma 4.2. *There exists a constant $c > 0$ such that*

$$\big|\mathbf{E}f\big(\zeta_n^{(r)} + h\big) - \mathbf{E}f\big(\zeta_n^{(r)}\big)\big| \leq c|h| \int_{-1}^{1} |f(z)|\, dz \tag{4.9}$$

and

$$\big|\mathbf{E}f\big(\zeta_n^{(r)} + h\big) - \mathbf{E}f\big(\zeta^{(r)}\big)\big| \leq c(|h| + p^{n-2\alpha})\, \|f\|_1 \tag{4.10}$$

uniformly for $f \in L^1\big([-1,1], dx\big)$, $r \geq 1$, and $1 \leq n \leq \infty$ where $\zeta_\infty^{(r)} = \zeta^{(r)}$.

Proof. Let $g_k(z)$ be the density for $\frac{p_{r+k}}{\lambda_r}\omega^{(r+k)}$ and let $f_n^{(r)}(z)$ the density for $\zeta_n^{(r)}$. Then

$$\hat{f}_n^{(r)}(t) = \mathbf{E}e^{it\zeta_n^{(r)}} = \prod_{k=0}^{n-1} \hat{g}_k(t)$$

where $\hat{g}_k(t) = \mathbf{E}e^{it\frac{p_{r+k}}{\lambda_r}\omega^{(r+k)}}$. For each $k \geq 0$, because the density for $\omega^{(r+k)}$ is continuously differentiable on $[-1,1]$, it follows that $g_k(z)$ is continuously differentiable on $\big[-\frac{p_{r+k}}{\lambda_r}, \frac{p_{r+k}}{\lambda_r}\big]$. This implies g_k' is bounded and that $\hat{g}_k(t)$ is $o(t^{-1})$ as $t \to \infty$ — in particular, $t\hat{g}_k(t)$ is bounded on $\mathbb{R}$. Then there exists a constant $c > \max\{\|g_0'\|_\infty, \|g_0'\|_\infty\}$

such that

$$|\hat{g}_0(t)\hat{g}_1(t)\hat{g}_2(t)| \le c|t|^{-3} \quad \text{for all } t \in \mathbb{R}.$$

Since $f_1^{(r)} = g_0$, by the Mean Value Theorem, for all λ, μ, we have

$$\big|f_1^{(r)}(\lambda) - f_1^{(r)}(\mu)\big| \le c|\lambda - \mu|.$$

Similarly, since $f_2^{(r)} = g_0 * g_1$, we have

$$\big|f_2^{(r)}(\lambda) - f_2^{(r)}(\mu)\big| \le \int g_0(z)|g_1(\lambda - z) - g_1(\mu - z)|\, dz \le c|\lambda - \mu|.$$

For $3 \le n \le \infty$ we have

$$\big|f_n^{(r)}(\lambda) - f_n^{(r)}(\mu)\big| = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}(e^{-it\lambda} - e^{-it\mu})\hat{f}_n^{(r)}(t)\, dt\right| \le \frac{|\lambda - \mu|}{\pi}\int_0^{\infty}\big|t\hat{f}_n^{(r)}(t)\big|\, dt.$$

But since $n \ge 3$,

$$\big|t\hat{f}_n^{(r)}(t)\big| = \left|t\hat{g}_0(t)\hat{g}_1(t)\hat{g}_2(t)\prod_{k=2}^{n-1}\hat{g}_k(t)\right| \le c \cdot \min\left\{1, |t|^{-2}\right\}.$$

Therefore, we have

$$\frac{1}{\pi}\int_0^{\infty}\big|t\hat{f}_n^{(r)}(t)\big|\, dt \le \frac{c}{\pi}\int_0^1 dt + \frac{c}{\pi}\int_1^{\infty} t^{-2}\, dt = \frac{2c}{\pi} \le c$$

hence

$$\big|f_n^{(r)}(\lambda) - f_n^{(r)}(\mu)\big| \le c|\lambda - \mu| \quad \text{for all} \quad 1 \le n \le \infty. \tag{4.11}$$

Next, for any $f \in L^1\big([-1,1], dx\big)$, since

$$\mathbf{E}f\big(\zeta_n^{(r)} + h\big) = \int f(z)f_n^{(r)}(z - h)\, dz,$$

we have

$$\left|\mathbf{E}f\big(\zeta_n^{(r)}+h\big)-\mathbf{E}f\big(\zeta_n^{(r)}\big)\right|\le\int\left|f(z)\right|\left|f_n^{(r)}(z-h)-f_n^{(r)}(z)\right|dz\le c|h|\int_{-1}^{1}|f(z)|\,dz.$$

so that (4.9) is proven. Finally, since $\zeta_n^{(r)}$ and $\zeta^{(r+n)}$ are independent, by (4.8),

$$\mathbf{E}\Big(f\big(\zeta^{(r)}\big)\,\Big|\,\zeta^{(r+n)}\Big)=\varphi\big(\zeta^{(r+n)}\big)\quad\text{where}\quad\varphi(z)=\mathbf{E}f\big(\zeta_n^{(r)}+\tfrac{\lambda_{r+n}}{\lambda_r}z\big).$$

Then, by (4.9), (3.31) and (4.6), we have

$$\left|\,\mathbf{E}f\big(\zeta_n^{(r)}\big)-\varphi\big(\zeta^{(r+n)}\big)\right|\le c\left|\tfrac{\lambda_{r+n}}{\lambda_r}\zeta^{(r+n)}\right|\|f\|_1\le cp^{n-2\alpha}\,\|f\|_1\,,$$

$\mathbf{P}$-a.s., so that

$$\left|\,\mathbf{E}f\big(\zeta_n^{(r)}+h\big)-\varphi\big(\zeta^{(r+n)}\big)\right|\le\left|\,\mathbf{E}f\big(\zeta_n^{(r)}+h\big)-\mathbf{E}f\big(\zeta_n^{(r)}\big)\right|+cp^{n-2\alpha}\,\|f\|_1\,. \qquad (4.12)$$

Taking expectations in (4.12) and again applying (4.9), we obtain

$$\left|\,\mathbf{E}f\big(\zeta_n^{(r)}+h\big)-\mathbf{E}f\big(\zeta^{(r)}\big)\right|\le c|h|\,\|f\|_1+cp^{n-2\alpha}\,\|f\|_1=c(|h|+p^{n-2\alpha})\,\|f\|_1$$

so that (4.10) is proven. ■

If $f(\lambda)$ is an integrable function supported on $\mathrm{Sp}(-\Delta_\omega)$, for each $r\ge 0$, we put

$$f_r(z)=f\big(\lambda_r(1+\sigma z)\big)\mathbf{1}_{[-1,1]}(z).$$

Then the average value of $|f(\lambda)|$ on $\operatorname{supp}\lambda^{(r)}$ is given by

$$\frac{1}{2\sigma\lambda_r}\int_{\operatorname{supp}\lambda^{(r)}}|f(\lambda)|\,d\lambda=\tfrac{1}{2}\,\|f_r\|_1\,.$$

Proposition 4.3. *There exists a constant $c>0$ such that for any two integrable* *functions* $f(\lambda)$ *and* $g(\lambda)$ *supported on* $\mathrm{Sp}(-\Delta_\omega)$*, we have*

$$\left|\mathbf{Cov}\Big(f\big(\lambda_i^{(m)}\big), g\big(\lambda_k^{(r)}\big)\Big)\right| \le cp^{r-m}\left\|f_m\right\|_1 \mathbf{E}|g(\lambda^{(r)})| \tag{4.13}$$

whenever $Q_i^{(m)} \subsetneq Q_k^{(r)}$ *and*

$$\left|\mathbf{Cov}\Big(f\big(\lambda_i^{(m)}\big), g\big(\lambda_j^{(n)}\big)\Big)\right| \le cp^r\Big(\tfrac{\|f_m\|_1 \mathbf{E}|g(\lambda^{(n)})|}{p^m} + \tfrac{\|g_n\|_1 \mathbf{E}|f(\lambda^{(m)})|}{p^n} + \tfrac{p^r\|f_m\|_1\|g_n\|_1}{p^{m+n}}\Big) \tag{4.14}$$

for $1 \le m \le n < r = d_h(Q_i^{(m)}, Q_j^{(n)})$*.*

Proof. Let $x \in Q_i^{(m)}$ and write $\lambda^{(m)} = \lambda_i^{(m)} = \lambda^{(m)}(x)$ and $\lambda^{(r)} = \lambda_k^{(r)} = \lambda^{(r)}(x)$. Then because $\zeta_{r-m}^{(m)}$ and $\zeta^{(r)}$ are independent and we have $\zeta^{(m)} = \zeta_{r-m}^{(m)} + \frac{\lambda_r}{\lambda_m}\zeta^{(r)}$, it means that

$$\mathbf{E}\Big(f\big(\lambda^{(m)}\big)g\big(\lambda^{(r)}\big) \;\Big|\; \zeta^{(r)}\Big) = \mathbf{E}\Big(f_m\big(\zeta^{(m)}\big) \;\Big|\; \zeta^{(r)}\Big)g\big(\lambda^{(r)}\big) = \varphi\big(\zeta^{(r)}\big)g\big(\lambda^{(r)}\big)$$

where $\varphi(z) = \mathbf{E}f_m\big(\zeta_{r-m}^{(m)} + \frac{\lambda_r}{\lambda_m}z\big)$. By (4.10), (3.31) and (4.6), we have

$$\left|\varphi(z) - \mathbf{E}f_m\big(\zeta^{(m)}\big)\right| \le c\big(\tfrac{\lambda_r}{\lambda_m}|z| + p^{r-m-2\alpha}\big)\left\|f_m\right\|_1 \le \tfrac{2c}{p^{2\alpha}}p^{r-m}\left\|f_m\right\|_1 \tag{4.15}$$

so that

$$\left|\mathbf{E}\Big(f\big(\lambda^{(m)}\big)g\big(\lambda^{(r)}\big) \;\Big|\; \zeta^{(r)}\Big) - g\big(\lambda^{(r)}\big)\,\mathbf{E}f_m\big(\zeta^{(m)}\big)\right| \le \tfrac{c}{p^{2\alpha}}p^{r-m}\left\|f_m\right\|_1\left|g\big(\lambda^{(r)}\big)\right|. \tag{4.16}$$

Taking expectations in (4.16), we obtain (4.13).

Now to prove (4.14), let $x \in Q_i^{(m)}$, $y \in Q_j^{(n)}$, and write

$$\lambda^{(m)} = \lambda^{(m)}(x), \quad \lambda^{(n)} = \lambda^{(n)}(y), \quad \text{and} \quad \lambda^{(r)} = \lambda^{(r)}(x) = \lambda^{(r)}(y).$$

Because $\zeta^{(m)}_{r-m}(x)$, $\zeta^{(n)}_{r-n}(y)$, and $\zeta^{(r)}$ are independent, we have

$$\mathbf{E}\Big(f\big(\lambda^{(m)}\big)g\big(\lambda^{(n)}\big)\ \Big|\ \zeta^{(r)}\Big) = \mathbf{E}\Big(f_m\big(\zeta^{(m)}(x)\big)g_n\big(\zeta^{(n)}(y)\big)\ \Big|\ \zeta^{(r)}\Big) = \varphi\big(\zeta^{(r)}\big)\gamma\big(\zeta^{(r)}\big)$$

where

$$\varphi(z) = \mathbf{E}f_m\big(\zeta^{(m)}_{r-m} + \tfrac{\lambda_r}{\lambda_m}z\big) \quad \text{and} \quad \gamma(z) = \mathbf{E}g_n\big(\zeta^{(n)}_{r-n} + \tfrac{\lambda_r}{\lambda_n}z\big).$$

Let

$$s = \mathbf{E}f\big(\lambda^{(m)}\big) = \mathbf{E}f_m\big(\zeta^{(m)}\big) \quad \text{and} \quad t = \mathbf{E}g\big(\lambda^{(n)}\big) = \mathbf{E}g_n\big(\zeta^{(n)}\big).$$

Then just like in (4.15) we have

$$|\varphi(z) - s| \le \tfrac{2c}{p^{2\alpha}}p^{r-m}\,\|f_m\|_1 \qquad \text{and} \qquad |\gamma(z) - t| \le \tfrac{2c}{p^{2\alpha}}p^{r-n}\,\|g_n\|_1$$

so that

$$\begin{aligned} |\varphi\big(\zeta^{(r)}\big)\gamma\big(\zeta^{(r)}\big) - st| &\le |\varphi\big(\zeta^{(r)}\big) - s||t| + |\gamma\big(\zeta^{(r)}\big) - t||s| + |\varphi\big(\zeta^{(r)}\big) - s||\gamma\big(\zeta^{(r)}\big) - t| \\ &\le c_0 p^r\left(\tfrac{\|f_m\|_1\,\mathbf{E}|g(\lambda^{(n)})|}{p^m} + \tfrac{\|g_n\|_1\,\mathbf{E}|f(\lambda^{(m)})|}{p^n} + \tfrac{p^r\|f_m\|_1\|g_n\|_1}{p^{m+n}}\right) \end{aligned} \tag{4.17}$$

where $c_0 = \max\{2cp^{-2\alpha}, (2cp^{-2\alpha})^2\}$. Since

$$\mathbf{Cov}\Big(f\big(\lambda^{(m)}_i\big), g\big(\lambda^{(n)}_j\big)\Big) = \mathbf{E}\big(\varphi\big(\zeta^{(r)}\big)\gamma\big(\zeta^{(r)}\big) - st\big)\,,$$

taking expectations in (4.17), we obtain (4.14). ■

Proposition 4.4. *For $1 \le m < r$ and for any two measurable sets A and B, we have*

$$\Big|\mathbf{Cov}\Big(\mathbf{1}_A\big(\lambda^{(m)}\big), \mathbf{1}_B\big(\lambda^{(r)}\big)\Big)\Big| \le \min\Big\{4cp^{r-m},\ \tfrac{2cp^{r+1}}{\sigma p^{2m}}|A|\,\mathbf{P}\big(\lambda^{(r)} \in B\big)\Big\} \tag{4.18}$$

where $|A|$ denotes the Lebesgue measure of A.

Proposition 4.5. *For $1 \leq m \leq n < r = d_h(x,y)$, the following estimates are valid for any two measurable sets A and B.*

$$\left|\mathbf{Cov}\Big(\mathbf{1}_A\big(\lambda^{(m)}(x)\big), \mathbf{1}_B\big(\lambda^{(n)}(y)\big)\Big)\right| \leq 8cp^{r-n} \tag{4.19}$$

and

$$\left|\mathbf{Cov}\Big(\mathbf{1}_A\big(\lambda^{(m)}(x)\big), \mathbf{1}_B\big(\lambda^{(n)}(y)\big)\Big)\right| \leq \tfrac{4c^2p^r}{\sigma^2}\Big(\tfrac{|A|P(\lambda^{(n)}\in B)}{p^{2m}} + \tfrac{|B|P(\lambda^{(m)}\in A)}{p^{2n}} + \tfrac{|A||B|}{p^{2(m+n)}}\Big). \tag{4.20}$$

Corollary 4.6. *For any two measurable sets A and B we have*

$$\left|\mathbf{Cov}\Big(\mathbf{1}_A\big(\lambda_i^{(m)}\big), \mathbf{1}_B\big(\lambda_k^{(r)}\big)\Big)\right| \leq 4cp^{r-m} \tag{4.21}$$

whenever $Q_i^{(m)} \subsetneq Q_k^{(r)}$ and

$$\left|\mathbf{Cov}\Big(\mathbf{1}_A\big(\lambda_i^{(m)}\big), \mathbf{1}_B\big(\lambda_j^{(n)}\big)\Big)\right| \leq 8cp^{r-n} \tag{4.22}$$

for $1 \leq m \leq n < r = d_h(Q_i^{(m)}, Q_j^{(n)})$.

4.2 Density of States Measure

For $r \leq L$, let $\mathcal{S}_L$ be the set of all non-degenerate sub-cubes of $Q_0^{(L)}$, i.e.,

$$\mathcal{S}_L = \bigcup_{r=1}^{L} \mathcal{S}_L^{(r)} \quad \text{where} \quad \mathcal{S}_L^{(r)} = \big\{Q \in \Pi_r : Q \subseteq Q_0^{(L)}\big\}.$$

For $1 \leq r \leq L$, there are ν^{L-r} cubes of rank r contained in $Q_0^{(L)}$ hence

$$\big|\mathcal{S}_L\big| = \sum_{r=1}^{L} \big|\mathcal{S}_L^{(r)}\big| = \sum_{r=1}^{L} \nu^{L-r} = \frac{\nu^L - 1}{\nu - 1}.$$

Then, counting multiplicities, the spectral problem

$$-\Delta_\omega \psi = \lambda\psi, \quad \psi \equiv 0 \ \text{ on } \ X \diagdown Q_0^{(L)} \tag{4.23}$$

has $\nu^L - 1$ eigenvalues $\{\lambda_Q : Q \in \mathcal{S}_L\}$ (recall that each eigenvalue has multiplicity $\dim \mathcal{L}_Q = \nu - 1$).

Let $\mathcal{N}_L(A)$ be the random number of eigenvalues for (4.23) which belong to the measurable set $A \subseteq \mathrm{Sp}(-\Delta_\omega)$, i.e.,

$$\mathcal{N}_L(A) = \sum_{Q\in\mathcal{S}_L} \mathbf{1}_A(\lambda_Q) = \sum_{r=1}^{L} \mathcal{N}_L^{(r)}(A) \tag{4.24}$$

where

$$\mathcal{N}_L^{(r)}(A) = \sum_{Q\in\mathcal{S}_L^{(r)}} \mathbf{1}_A(\lambda_Q) = \sum_{i<\nu^{L-r}} \mathbf{1}_A(\lambda_i^{(r)}). \tag{4.25}$$

We see that

$$\mathbf{E}\big[\mathcal{N}_L^{(r)}(A)\big] = \sum_{Q\in\mathcal{S}_L^{(r)}} \mathbf{P}(\lambda_Q \in A) = \nu^{L-r}\,\mathbf{P}(\lambda^{(r)} \in A)$$

and

$$\mathbf{E}\big[\mathcal{N}_L(A)\big] = \sum_{r=1}^{L} \nu^{L-r}\,\mathbf{P}(\lambda^{(r)} \in A).$$

Lemma 4.7. *For any two measurable sets $A, B \subseteq \mathrm{Sp}(-\Delta_\omega)$ we have*

$$\big|\,\mathbf{Cov}\big(\mathcal{N}_L(A), \mathcal{N}_L(B)\big)\big| \lesssim \nu^L\big(L^2 + (\nu p)^L\big) \tag{4.26}$$

and for $1 \le k \le r \le L$ we have

$$\big|\,\mathbf{Cov}\big(\mathcal{N}_L^{(k)}(A), \mathcal{N}_L^{(r)}(B)\big)\big| \lesssim \nu^L\big(L + (\nu p)^L\big) \tag{4.27}$$

where $x \lesssim y$ means $x = O(y)$ as $L \to \infty$.

Proof. For $i, j < \nu^{L-k}$, considering the isometry $\varphi : X \to X$ which swaps the cubes $Q_i^{(k)}$ and $Q_j^{(k)}$, we see that $\big(\lambda_i^{(k)}, \mathcal{N}_L^{(r)}(B)\big) \overset{\text{law}}{=} \big(\lambda_j^{(k)}, \mathcal{N}_L^{(r)}(B)\big)$. This means we have

$$\mathbf{Cov}\big(\mathcal{N}_L^{(k)}(A), \mathcal{N}_L^{(r)}(B)\big) = \sum_{i<\nu^{L-k}} \mathbf{Cov}\big(\mathbf{1}_A(\lambda_i^{(k)}), \mathcal{N}_L^{(r)}\big) = \nu^{L-k}\, \mathbf{Cov}\big(\mathbf{1}_A(\lambda_0^{(k)}), \mathcal{N}_L^{(r)}\big). \tag{4.28}$$

But

$$\mathbf{Cov}\big(\mathbf{1}_A(\lambda_0^{(k)}), \mathcal{N}_L^{(r)}\big) = \mathbf{Cov}\big(\mathbf{1}_A(\lambda_0^{(k)}), \mathbf{1}_B(\lambda_0^{(r)})\big) + \sum_{n=1}^{L-r} \sum_{i=\nu^{n-1}}^{\nu^n-1} \mathbf{Cov}\big(\mathbf{1}_A(\lambda_0^{(k)}), \mathbf{1}_B(\lambda_i^{(r)})\big).$$

Since $Q_0^{(k)} \subseteq Q_0^{(r)}$, we have $\big|\,\mathbf{Cov}\big(\mathbf{1}_A(\lambda_0^{(k)}), \mathbf{1}_B(\lambda_0^{(r)})\big)\big| \lesssim p^{r-k}$. For $\nu^{n-1} \le i < \nu^n$ we have $d_h\big(Q_0^{(k)}, Q_i^{(r)}\big) = r + n$ so that $\big|\,\mathbf{Cov}\big(\mathbf{1}_A(\lambda_0^{(k)}), \mathbf{1}_B(\lambda_i^{(r)})\big)\big| \lesssim p^{(r+n)-r} = p^n$ hence

$$\left|\sum_{i=\nu^{n-1}}^{\nu^n-1} \mathbf{Cov}\big(\mathbf{1}_A(\lambda_0^{(k)}), \mathbf{1}_B(\lambda_i^{(r)})\big)\right| \lesssim (\nu^n - \nu^{n-1})p^n \lesssim (\nu p)^n. \tag{4.29}$$

It means that

$$\big|\,\mathbf{Cov}\big(\mathcal{N}_L^{(k)}, \mathcal{N}_L^{(r)}\big)\big| \lesssim \nu^{L-k}\left(p^{r-k} + \sum_{n=1}^{L-r}(\nu p)^n\right) = \nu^L\left(\frac{p^r}{(\nu p)^k} + \frac{1}{\nu^k}\sum_{n=1}^{L-r}(\nu p)^n\right). \tag{4.30}$$

Now, because we have

$$\mathbf{Cov}\big(\mathcal{N}_L(A), \mathcal{N}_L(B)\big) = \sum_{r=1}^{L}\sum_{k=1}^{r} 2\,\mathbf{Cov}\big(\mathcal{N}_L^{(k)}(A), \mathcal{N}_L^{(r)}(B)\big) \tag{4.31}$$

it follows from (4.30) that

$$\begin{aligned}\big|\,\mathbf{Cov}\big(\mathcal{N}_L(A), \mathcal{N}_L(B)\big)\big| &\lesssim \nu^L \sum_{r=1}^{L}\left[\sum_{k=1}^{r}\frac{p^r}{(\nu p)^k} + \left(\sum_{k=1}^{r}\frac{1}{\nu^k}\right)\left(\sum_{n=1}^{L-r}(\nu p)^n\right)\right] \\ &\lesssim \nu^L \sum_{r=1}^{L}\left[\sum_{k=1}^{r}\frac{p^r}{(\nu p)^k} + \sum_{n=1}^{L-r}(\nu p)^n\right]\end{aligned} \tag{4.32}$$

For $\nu p \neq 1$ we have

$$\sum_{k=1}^{r} \frac{p^r}{(\nu p)^k} + \sum_{n=1}^{L-r} (\nu p)^n = \tfrac{\nu p}{|\nu p - 1|} \Big(\tfrac{1}{\nu p} |p^r - \nu^{-r}| + \big|(\nu p)^{L-r} - 1\big| \Big) \lesssim L + (\nu p)^{L-r} \quad (4.33)$$

and for $\nu p = 1$ we have

$$\sum_{k=1}^{r} \frac{p^r}{(\nu p)^k} + \sum_{n=1}^{L-r} (\nu p)^n = r p^r + L - r \leq L + (\nu p)^{L-r} \quad (4.34)$$

so that

$$\big| \mathbf{Cov}\big(\mathcal{N}_L(A), \mathcal{N}_L(B)\big) \big| \lesssim \nu^L \sum_{r=1}^{L} \big(L + (\nu p)^{L-r} \big) = \nu^L \left(L^2 + \sum_{r=0}^{L-1} (\nu p)^r \right) \lesssim \nu^L \big(L^2 + (\nu p)^L \big).$$

From (4.30) we also find that

$$\big| \mathbf{Cov}\big(\mathcal{N}_L^{(k)}, \mathcal{N}_L^{(r)}\big) \big| \lesssim \nu^L \left(1 + \sum_{n=1}^{L-r} (\nu p)^n \right) \lesssim \nu^L \big(L + (\nu p)^L \big) \quad (4.35)$$

which completes the proof. ■

The empirical measures for $\{\lambda_Q : Q \in \mathcal{S}_L\}$ and $\{\lambda_Q : Q \in \mathcal{S}_L^{(r)}\}$ are given by

$$N_L(A) = \frac{\mathcal{N}_L(A)}{|\mathcal{S}_L|} = \frac{(\nu - 1)\mathcal{N}_L(A)}{(1 - \nu^{-L})\nu^L} \qquad \text{and} \qquad N_L^{(r)}(A) = \frac{\mathcal{N}_L^{(r)}(A)}{\big|\mathcal{S}_L^{(r)}\big|} = \frac{\mathcal{N}_L^{(r)}(A)}{\nu^{L-r}}.$$

Observe that

$$\mathbf{E}\big[N_L^{(r)}(A)\big] = \frac{1}{\big|\mathcal{S}_L^{(r)}\big|} \sum_{Q \in \mathcal{S}_L^{(r)}} \mathbf{P}(\lambda_Q \in A) = \mathbf{P}(\lambda^{(r)} \in A).$$

Therefore, since

$$\mathbf{Var}\big[N_L^{(r)}(A)\big] \lesssim \frac{\nu^L \big(L + (\nu p)^L \big)}{\big|\mathcal{S}_L^{(r)}\big|^2} \lesssim \frac{L}{\nu^L} + p^L \to 0 \quad \text{as} \quad L \to \infty,$$

we see that $\mathbf{P}$-a.s., $N_L^{(r)}(A) \to \mathbf{P}(\lambda^{(r)} \in A)$ as $L \to \infty$. We also have

$$\mathbf{E}\big[N_L(A)\big] = \frac{1}{|\mathcal{S}_L|} \sum_{r=1}^{L} \big|\mathcal{S}_L^{(r)}\big| \, \mathbf{E}\big[N_L^{(r)}(A)\big] = \frac{\nu - 1}{1 - \nu^{-L}} \sum_{r=1}^{L} \frac{\mathbf{P}(\lambda^{(r)} \in A)}{\nu^r}.$$

Let

$$N(A) = \lim_{L\to\infty} \mathbf{E}\big[N_L(A)\big] = \sum_{r=1}^{\infty} \frac{\nu - 1}{\nu^r} \, \mathbf{P}(\lambda^{(r)} \in A). \tag{4.36}$$

Then because

$$\mathbf{Var}\big[N_L(A)\big] \lesssim \frac{\nu^L\big(L^2 + (\nu p)^L\big)}{|\mathcal{S}_L|^2} \lesssim \frac{L^2}{\nu^L} + p^L \to 0 \quad \text{as} \quad L \to \infty,$$

we see that $\mathbf{P}$-a.s., $N_L(A) \to N(A)$ as $L \to \infty$.

Proposition 4.8. *For each measurable set* $A \subseteq \mathrm{Sp}(-\Delta_\omega)$, $\lim_{L\to\infty} \mathbf{Var}\big[N_L(A)\big] = 0$. *Therefore, with probability one,* $\lim_{L\to\infty} N_L(A) = N(A)$.

4.3 Density of states

It is clear from (4.36) that the measure $N(d\lambda)$, which depends on the parameter $0 < \sigma < 1$, is supported on $\mathrm{Sp}(-\Delta_\omega)$ and has a continuous distribution function and density given by

$$N(0, \lambda] = \sum_{r=1}^{\infty} \frac{\nu - 1}{\nu^r} \, \mathbf{P}\big(\lambda^{(r)} \le \lambda\big) \tag{4.37}$$

and

$$n(\lambda) = \tfrac{d}{d\lambda} N(0, \lambda] = \sum_{r=1}^{\infty} \frac{\nu - 1}{\nu^r} g^{(r)}(\lambda) \tag{4.38}$$

where

$$g^{(r)}(\lambda) = \tfrac{d}{d\lambda} \, \mathbf{P}(\lambda^{(r)} \le \lambda) \tag{4.39}$$

is the density for $\lambda^{(r)}$. Since **P**-a.s., $\lambda^{(r)}$ lies between $(1 \pm \sigma)\lambda_r$, if we let $\beta = \log_p \frac{1-\sigma}{1+\sigma}$, then we may write

$$\operatorname{supp} \lambda^{(r)} = (1+\sigma)[p^\beta \lambda_r, \lambda_r]$$

and we have

$$\mathrm{Sp}(-\Delta_\omega) = (1+\sigma) \bigcup_{r=1}^{\infty} [p^\beta \lambda_r, \lambda_r]. \tag{4.40}$$

Observe that (3.31) implies that for all $r \geq 1$,

$$p^{1+2\alpha} \leq \tfrac{\lambda_{r+1}}{\lambda_r} \leq p^{1-2\alpha}. \tag{4.41}$$

The expression (4.40) shows that $\mathrm{Sp}(-\Delta_\omega)$ is connected, i.e., $\mathrm{Sp}(-\Delta_\omega) = [0, 1+\sigma]$, if and only if $p^\beta \leq \inf_{r \geq 1} \frac{\lambda_{r+1}}{\lambda_r}$ which by (4.41) is the case whenever

$$\tfrac{1-p^{1+2\alpha}}{1+p^{1+2\alpha}} \leq \sigma < 1.$$

On the other hand, whenever

$$0 < \sigma < \tfrac{1-p^{1-2\alpha}}{1+p^{1-2\alpha}},$$

the union in (4.40) is disjoint and it is "physically impossible" that two eigenvalues of different rank assume the same value.

We would like to estimate the number $|I(\lambda)|$ where

$$I(\lambda) = \{r \geq 1 : g_r(\lambda) > 0\}$$

is the set of all ranks for which it is physically possible that some eigenvalue $\lambda_i^{(r)}$ assumes the value $\lambda \in \mathrm{Sp}(-\Delta_\omega)$. Then

$$r \in I(\lambda) \quad \Leftrightarrow \quad (1-\sigma)\lambda_r < \lambda < (1+\sigma)\lambda_r \quad \Leftrightarrow \quad \tfrac{\lambda}{1+\sigma} < \lambda_r < \tfrac{\lambda}{1-\sigma} \tag{4.42}$$

which implies the sum for $n(\lambda)$ is actually finite (see Lemma 5.5 below). If we write

$$I(\lambda) = \{m+1, m+2, \ldots, M\}$$

then (4.38) becomes

$$n(\lambda) = \sum_{r=m+1}^{M} \frac{\nu - 1}{\nu^r} g^{(r)}(\lambda) \tag{4.43}$$

and we have

$$\lambda_{M+1} \leq \tfrac{\lambda}{1+\sigma} < \lambda_M < \cdots < \lambda_{m+2} < \lambda_{m+1} < \tfrac{\lambda}{1-\sigma} \leq \lambda_m$$

so that by (3.31),

$$p^{\alpha+1} \tfrac{\lambda}{1+\sigma} < p^M \leq p^{-\alpha} \tfrac{\lambda}{1+\sigma} \qquad \text{and} \qquad p^{\alpha+1} \tfrac{\lambda}{1+\sigma} \leq p^{m+\beta} < p^{-\alpha} \tfrac{\lambda}{1+\sigma}. \tag{4.44}$$

hence

$$-\alpha \leq M - \log_p \tfrac{\lambda}{1+\sigma} < \alpha + 1 \qquad \text{and} \qquad -\alpha < m + \beta - \log_p \tfrac{\lambda}{1+\sigma} \leq \alpha + 1.$$

Since $|I(\lambda)| = M - m$, it follows that

$$\beta - (2\alpha+1) \leq |I(\lambda)| < \beta + (2\alpha + 1). \tag{4.45}$$

This means for each $\lambda \in \mathrm{Sp}(-\Delta_\omega)$, there are approximately $\beta \pm (2\alpha + 1)$ values of r where $g_r(\lambda) > 0$ and for each of these,

$$-\alpha - \beta < r - \log_p \tfrac{\lambda}{1+\sigma} < \alpha + 1. \tag{4.46}$$

4.4 Lifshitz Exponent

In the case where $\omega^{(r)} \overset{\text{law}}{=} \omega^{(s)}$ even for $r \neq s$, (4.4) implies that for $L = 1, 2, 3, \ldots$,

$$\mathbf{P}(\lambda^{(r)} \leq p^{2\alpha}\lambda) \leq \mathbf{P}(\lambda^{(r+L)} \leq p^{L}\lambda) \leq \mathbf{P}(\lambda^{(r)} \leq p^{-2\alpha}\lambda) \tag{4.47}$$

and from (4.46), it follows that $I(p^L\lambda) = L + I(\lambda)$. Therefore, since

$$\frac{N(0, p^L\lambda]}{\nu - 1} = \sum_{r=L+m+1}^{\infty} \frac{\mathbf{P}(\lambda^{(r)} \leq p^L\lambda)}{\nu^r} = \sum_{r=m+1}^{\infty} \frac{\mathbf{P}(\lambda^{(r+L)} \leq p^L\lambda)}{\nu^{r+L}}, \tag{4.48}$$

we have

$$\nu^{-L}N(0, p^{2\alpha}\lambda] \leq N(0, p^L\lambda] \leq \nu^{-L}N(0, p^{-2\alpha}\lambda]. \tag{4.49}$$

The inequality (4.49) allows us to compute the Lifshitz exponent.

Proposition 4.9. *Provided* $\omega^{(r)} \overset{\text{law}}{=} \omega^{(s)}$ *for all* r, s, $\lim_{\lambda \searrow 0} \frac{\log N(0,\lambda]}{\log \lambda} = \frac{s_h}{2}$.

Proof. Write $\lambda = p^{L+x}$ where $L = \lfloor \log_p \lambda \rfloor$ and $x = \{\log_p \lambda\}$. Then we have

$$\frac{-L\log\nu + \log N(0, p^{x-2\alpha}]}{(L+x)\log p} \leq \frac{\log N(0,\lambda]}{\log\lambda} \leq \frac{-L\log\nu + \log N(0, p^{x+2\alpha}]}{(L+x)\log p}$$

where both the right and left-hand sides tend to $-\frac{\log\nu}{\log p} = \frac{s_h}{2}$ as $L \to \infty$. ∎

CHAPTER 5: POISSON STATISTICS

5.1 Preliminaries

In the spirit of [7, 6], we will study the distribution eigenvalues for $-\Delta_\omega$ in the near vicinity of a given point $\lambda \in \mathrm{Sp}(-\Delta_\omega)$. The set of eigenvalues for the spectral problem (4.23) is a point process in $\mathbb{R}$ (see [2, 3, 6]). After applying a scaling transformation $x \mapsto |\mathcal{S}_L|(x-\lambda)$ to these eigenvalues, i.e., we are really considering the spectrum of the operator

$$H_L^\lambda = -|\mathcal{S}_L|(\lambda + \Delta_\omega)\mathbf{1}_{Q_0^L} \ ,$$

we will prove that the set

$$\mathrm{Sp}(H_L^\lambda) = \{|\mathcal{S}_L|(\lambda_Q - \lambda) : Q \in \mathcal{S}_L\} \tag{5.1}$$

of rescaled eigenvalues converges, as $L \to \infty$, to a Poissson point process. Let

$$\mu_L^\lambda(A) = \big|A \cap \mathrm{Sp}(H_L^\lambda)\big| \tag{5.2}$$

be the number of rescaled eigenvalues which belong to a bounded measurable set $A \subseteq \mathbb{R}$. Alternately, we may write

$$\mu_L^\lambda(A) = \sum_{Q \in \mathcal{S}_L} \mathbf{1}_A\big(|\mathcal{S}_L|(\lambda_Q - \lambda)\big) = \sum_{Q \in \mathcal{S}_L} \mathbf{1}_{A_L^\lambda}(\lambda_Q) = \mathcal{N}_L(A_L^\lambda) \tag{5.3}$$

where

$$A_L^\lambda = \lambda + \tfrac{1}{|\mathcal{S}_L|}A = \big\{\lambda + \tfrac{x}{|\mathcal{S}_L|} : x \in A\big\}.$$

In view of Proposition 4.8 but ignoring the fact that there is a double limit involved, we should expect that as $L \to \infty$,

$$\mathcal{N}_L(A_L^\lambda) \approx |\mathcal{S}_L| N(A_L^\lambda) = |\mathcal{S}_L| \int_{A_L^\lambda} n(x)\,dx = \int_A n(\lambda + \tfrac{x}{|\mathcal{S}_L|})\,dx \approx n(\lambda)|A|.$$

We want to prove that μ_L^λ converges weakly as $L \to \infty$ to an integer-valued random measure μ^λ which possesses the property that for any collection $A_1, \ldots, A_n$ of pairwise disjoint measurable sets, $\mu^\lambda(A_1), \ldots, \mu^\lambda(A_n)$ is a collection of independent Poissonian distributed random variables with

$$\mathbf{E}\big[\mu^\lambda(A)\big] = n(\lambda)|A|$$

where $|A|$ is the Lebesgue measure of A. We must prove that

$$\lim_{L\to\infty} \mathbf{E} z^{\mathcal{N}_L(A_L^\lambda)} = e^{n(\lambda)|A|(z-1)} \text{ for all } z \in \mathbb{C} \text{ with } |z| = 1.$$

Observe that for a continuous function $f(x)$,

$$\int_{A_L^\lambda} f(x)\,dx = \frac{1}{|\mathcal{S}_L|} \int_A f(\lambda + \tfrac{x}{|\mathcal{S}_L|})\,dx \approx \frac{|A| f(\lambda)}{|\mathcal{S}_L|} \approx \frac{|A|(\nu - 1) f(\lambda)}{\nu^L}. \tag{5.4}$$

Because $|\mathcal{S}_L| = \frac{\nu^L - 1}{\nu - 1} \geq \nu^{L-1}$, we obtain the following estimate.

Lemma 5.1. *If $f(x)$ is bounded then*

$$\left| \int_{A_L^\lambda} f(x)\,dx \right| \leq \nu^{1-L} |A| \|f\|_\infty$$

Using the fact that $I(\lambda)$ is finite, the next lemma will allow us to keep the error in the approximation (5.4) of order $O(\nu^{-2L})$.

Lemma 5.2. *Define an operator T by*

$$Tf(\lambda) = \int_{A_L^\lambda} f(x)\,dx - \frac{|A|(\nu-1)}{\nu^L} f(\lambda).$$

Whenever $f(x)$ is continuously differentiable,

$$\|Tf\|_\infty \le \nu^{2-2L}\left(|A| + \textstyle\int_A |x|dx\right)\left(\|f\|_\infty + \|f'\|_\infty\right). \tag{5.5}$$

Proof. We have

$$\begin{aligned}
\left|\int_{A_L^\lambda} f(x)\,dx - \frac{(\nu-1)|A|f(\lambda)}{\nu^L}\right| &\le \frac{1}{|\mathbb{S}_L|}\int_A \left|f(\lambda + \tfrac{x}{|\mathbb{S}_L|}) - (1-\nu^{-L})f(\lambda)\right|dx \\
&\le \frac{|A||f(\lambda)|}{\nu^L|\mathbb{S}_L|} + \frac{1}{|\mathbb{S}_L|}\int_A \left|f(\lambda + \tfrac{x}{|\mathbb{S}_L|}) - f(\lambda)\right|dx \\
&\le \frac{|A|\|f\|_\infty}{\nu^L|\mathbb{S}_L|} + \frac{\|f'\|_\infty \int_A |x|dx}{|\mathbb{S}_L|^2} \\
&\le \frac{|A|\|f\|_\infty + \|f'\|_\infty \int_A |x|dx}{\nu^{2L-2}}
\end{aligned}$$

which implies (5.5). ■

Lemma 5.3. *Let $z_{n,k}$ and $w_{n,k}$ be two triangular arrays of complex numbers. If there exists a constant $c > 0$ such that $|z_{n,k}| \le \frac{c}{n}$, $|w_{n,k}| \le \frac{c}{n}$, and $|z_{n,k} - w_{n,k}| \le \frac{c}{n^2}$ for all n, k with $1 \le k \le n$, then*

$$\left|\prod_{k=1}^{n}(1 + z_{n,k}) - \exp\left(\sum_{k=1}^{n} w_{n,k}\right)\right| \le \frac{C}{n} \tag{5.6}$$

for every $n \ge 1$ where $C = c(1 + ce^c)e^{c(2+ce^c)}$.

Proof. First observe that for all $z, w \in \mathbb{C}$, we have

$$\left|e^w - (1+z)\right| \le |w-z| + \sum_{n=2}^{\infty}\frac{|w|^n}{n!} \le |w-z| + |w|^2 e^{|w|}$$

so that

$$\left|e^{w}_{n,k} - (1 + z_{n,k})\right| \le \frac{c}{n^2} + \left(\frac{c}{n}\right)^2 e^{c/n} \le \frac{c + c^2 e^c}{n^2}.$$

Therefore, using the inequality

$$\left(1 + \frac{c}{n}\right)^n \le e^c \tag{5.7}$$

and the formula

$$\prod_{k\in S} x_k - \prod_{k\in S} y_k = \sum_{\varnothing\neq T\subseteq S} \left[\left(\prod_{k\notin T} y_k\right)\left(\prod_{k\in T}(x_k - y_k)\right)\right] \tag{5.8}$$

for a difference of products with $S = \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}\left|\prod_{k=1}^{n}(1 + z_{n,k}) - \prod_{k=1}^{n} e^{w_{n,k}}\right| &\le \sum_{\varnothing\neq T\subseteq S}\left(\prod_{k\notin T}|1 + z_{n,k}| \cdot \prod_{k\in T}\left|e^{w_{n,k}} - (1 + z_{n,k})\right|\right) \\ &\le \sum_{\varnothing\neq T\subseteq S}\left(\prod_{k\notin T}\left(1 + \frac{c}{n}\right) \cdot \prod_{k\in T}\frac{c(1 + ce^c)}{n^2}\right) \\ &= \left(1 + \frac{c}{n}\right)^n \sum_{\varnothing\neq T\subseteq S}\left(\frac{c(1 + ce^c)}{n^2 + nc}\right)^{|T|} \\ &\le e^c \sum_{\varnothing\neq T\subseteq S}\left(\frac{c(1 + ce^c)}{n^2}\right)^{|T|} = e^c\left[\left(1 + \frac{c(1 + ce^c)}{n^2}\right)^n - 1\right].\end{aligned}$$

Applying the inequality

$$\left|(1 + z)^n - 1\right| \le n|z|(1 + |z|)^{n-1}, \tag{5.9}$$

we have

$$\begin{aligned}\left|\prod_{k=1}^{n}(1 + z_{n,k}) - \prod_{k=1}^{n} e^{w_{n,k}}\right| &\le e^c \cdot n \cdot \frac{c(1 + ce^c)}{n^2} \cdot \left(1 + \frac{c(1 + ce^c)}{n^2}\right)^{n-1} \\ &\le \frac{c(1 + ce^c)e^c}{n} \cdot e^{c(1+ce^c)/n} \le \frac{C}{n}\end{aligned}$$

which establishes (5.6). ■

Lemma 5.4. *Let* $\{z_Q, w_Q : Q \in \mathcal{S}_L^{(r)}\} \subseteq \mathbb{C}$ *and assume there exists a constant* $c > 0$ *such that* $|z_Q| \leq 1 + \frac{c}{\nu^L}$*,* $|w_Q| \leq \frac{c}{\nu^L}$*, and* $|z_Q - (1 + w_Q)| \leq \frac{c}{\nu^{2L}}$ *for all* $Q \in \mathcal{S}_L^{(r)}$*. Then*

$$\left| \prod_{Q \in \mathcal{S}_L^{(r)}} (z_Q)^\nu - \prod_{Q \in \mathcal{S}_L^{(r)}} (1 + \nu w_Q) \right| \leq \frac{e^{c/\nu^{r-1}}}{\nu^{r-1}} \nu^{2-L}.$$

Proof. First, we have

$$\big|(z_Q)^\nu - (1 + w_Q)^\nu\big| \leq \big|(z_Q) - (1 + w_Q)\big| \sum_{k=0}^{\nu-1} |z_Q|^{\nu-k} |1 + w_Q|^k \leq \frac{\nu c}{\nu^{2L}} \left(1 + \frac{c}{\nu^L}\right)^\nu$$

and

$$\big|(1 + w_Q)^\nu - (1 + \nu w_Q)\big| \leq \nu^2 |w_Q|^2 (1 + |w_Q|)^{\nu-2} \leq \frac{\nu^2 c^2}{\nu^{2L}} \left(1 + \frac{c}{\nu^L}\right)^\nu$$

hence

$$|(z_Q)^\nu - (1 + \nu w_Q)| \leq \frac{\nu c}{\nu^{2L}} \left(1 + \frac{c}{\nu^L}\right)^\nu + \frac{\nu^2 c^2}{\nu^{2L}} \left(1 + \frac{c}{\nu^L}\right)^\nu \leq \frac{\nu^2 (c+1)^2}{\nu^{2L}} \left(1 + \frac{c}{\nu^L}\right)^\nu.$$

Therefore, using the formula (5.8) for a difference of products, we have

$$\begin{aligned} \left| \prod_{Q \in \mathcal{S}_L^{(r)}} (z_Q)^\nu - \prod_{Q \in \mathcal{S}_L^{(r)}} (1 + \nu w_Q) \right| &\leq \sum_{\varnothing \neq \mathcal{T} \subseteq \mathcal{S}_L^{(r)}} \left[\left(\prod_{Q \notin \mathcal{T}} |z_Q|^\nu \right) \left(\prod_{Q \in \mathcal{T}} \big|(z_Q)^\nu - (1 + \nu w_Q)\big| \right) \right] \\ &\leq \sum_{\varnothing \neq \mathcal{T} \subseteq \mathcal{S}_L^{(r)}} \left[\prod_{Q \notin \mathcal{T}} \left(1 + \frac{c}{\nu^L}\right)^\nu \prod_{Q \in \mathcal{T}} \left(\frac{\nu^2 (c+1)^2}{\nu^{2L}} \left(1 + \frac{c}{\nu^L}\right)^\nu \right) \right] \\ &= \left(1 + \frac{c}{\nu^L}\right)^{\nu^{L-r-1}} \left[\left(1 + \frac{\nu^2 (c+1)^2}{\nu^{2L}}\right)^{\nu^{L-r}} - 1 \right]. \end{aligned}$$

Finally, applying the inequalities (5.7) and (5.9), we obtain

$$\begin{aligned} \left| \prod_{Q \in \mathcal{S}_L^{(r)}} (z_Q)^\nu - \prod_{Q \in \mathcal{S}_L^{(r)}} (1 + \nu w_Q) \right| &\leq e^{c/\nu^{r-1}} \nu^{L-r} \frac{1}{\nu^{2L-2}} \left(1 + \frac{\nu^2 (c+1)^2}{\nu^{2L}}\right)^{\nu^{L-r}-1} \\ &\leq \frac{e^{c/\nu^{r-1}}}{\nu^r} e^{\nu^2 (c+1)^2 / \nu^{L+r}} \nu^{2-L} \leq \frac{e^{c/\nu^{r-1}}}{\nu^{r-1}} \nu^{2-L} \end{aligned}$$

as required. ■

Lemma 5.5. *Let M be an integer which exceeds $\max I(\lambda)$. If L is taken so large that $|x| \leq p_M |\mathcal{S}_L|$ for every $x \in A$, then*

$$\bigcup_{x \in A} I(\lambda + \tfrac{x}{|\mathcal{S}_L|}) \subseteq \{1, 2, \ldots, M\} \tag{5.10}$$

hence

$$n(\lambda + \tfrac{x}{|\mathcal{S}_L|}) = \sum_{r=1}^{M} \frac{g^{(r)}(\lambda + \tfrac{x}{|\mathcal{S}_L|})}{\nu^r} \quad \text{for all } x \in A \tag{5.11}$$

and

$$\mathcal{N}_L(A_L^\lambda) = \sum_{r=1}^{M} \mathcal{N}_L^{(r)}(A_L^\lambda), \quad \mathbf{P}\text{-almost surely.} \tag{5.12}$$

Proof. Since M exceeds $\max I(\lambda)$, we have $\lambda \geq (1+\sigma)\lambda_M$. It means for each $r > M$,

$$\lambda + \tfrac{x}{|\mathcal{S}_L|} \geq (1+\sigma)(\lambda_M - p_M) = (1+\sigma)\lambda_{M+1} \geq (1+\sigma)\lambda_r$$

hence $g^{(r)}(\lambda + \tfrac{x}{|\mathcal{S}_L|}) = 0$ which establishes (5.10) and (5.11). This further implies that

$$\mathbf{P}\big(\lambda^{(r)} \in A_L^\lambda\big) = 0 \quad \text{for each } r > M$$

which establishes (5.12). ■

Let $g_n^{(r)}$ be the density for

$$\lambda^{(r)} - \lambda^{(r+n)} = \xi^{(r)} + \xi^{(r+1)} + \cdots + \xi^{(r+n-1)}$$

Then $g_1^{(r)}$ is the density for $\xi^{(r)}$ so we have

$$g_{n+1}^{(r)} = g_n^{(r)} * g_1^{(r+n)} \tag{5.13}$$

and

$$g^{(r)} = g_n^{(r)} * g^{(r+n)} \tag{5.14}$$

where $g^{(r)}$ is the density for $\lambda^{(r)}$. Now put

$$h_n = \sum_{r=1}^{n} \nu^{n-r} g^{(r)}_{n-r+1} = \nu^{n-1} g^{(1)}_n + \nu^{n-2} g^{(2)}_{n-1} + \cdots + \nu g^{(n-1)}_2 + g^{(n)}_1. \tag{5.15}$$

Notice that $h_1 = g^{(1)}_1$ and for $n > 1$, by (5.13), we obtain the recursive formula

$$h_n = \nu h_{n-1} * g^{(n)}_1 + g^{(n)}_1 = (\nu h_{n-1} + \delta) * g^{(n)}_1. \tag{5.16}$$

By (5.14), we obtain $n(\lambda)$ by convolution of $\nu^{-M} h_M(\lambda)$ with $(\nu - 1) g^{(M+1)}(\lambda)$

$$n(\lambda) = \sum_{r=1}^{M} \frac{(\nu-1) g^{(r)}(\lambda)}{\nu^r} = \frac{(\nu-1)(h_M * g^{(M+1)})(\lambda)}{\nu^M} \tag{5.17}$$

where M exceeds $\max I(\lambda)$.

5.2 Proof of Poisson statistics

Let $\mathcal{F}_{\geq r}$ be the σ-algebra generated by all eigenvalues of rank at least r, i.e.,

$$\mathcal{F}_{\geq r} = \sigma\big(\lambda_Q : Q \in \mathcal{V}_r\big)$$

and similarly,

$$\mathcal{F}_{> r} = \sigma\big(\lambda_Q : Q \in \mathcal{V}_{r+1}\big).$$

Proposition 5.6. *Let M and L be as in Lemma 5.5. Then for $2 \leq n \leq M+1$,*

$$\mathbf{E}\big(z^{\mathcal{N}_L(A^\lambda_L)} \,\big|\, \mathcal{F}_{\geq n}\big) = z^{\sum_{r=n}^{M} \mathcal{N}^{(r)}_L(A^\lambda_L)} \prod_{Q \in \mathcal{S}^{(n)}_L} \Big(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L} \nu h_{n-1}(\lambda - \lambda_Q)\Big) + \varepsilon_n \tag{5.18}$$

where

$$\|\varepsilon_n\|_\infty \leq \nu^{2-L} \sum_{k=1}^{n-1} \frac{e^{2\nu^{2+\gamma-k}}}{\nu^k} \tag{5.19}$$

and γ is chosen so large that

$$\nu^{\gamma/4} > \max\left\{|A|, \int_A |x|dx, \|g_1^{(n)}\|_\infty, \|(g_1^{(n)})'\|_\infty, \|h_n\|_\infty\right\}$$

for all $n \le M+1$.

Proof. The proof is by induction. We will first establish (5.18) and (5.19) for $n=2$. Since $\mathcal{N}_L^{(r)}(A_L^\lambda)$ is $\mathcal{F}_{\geq 2}$-measurable for $r \geq 2$, it follows from (5.12) that

$$\mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \mid \mathcal{F}_{\geq 2}\big) = z^{\sum_{r=2}^{M} \mathcal{N}_L^{(r)}(A_L^\lambda)}\, \mathbf{E}\big(z^{\mathcal{N}_L^{(1)}(A_L^\lambda)} \mid \mathcal{F}_{\geq 2}\big). \tag{5.20}$$

Note also that

$$z^{\mathcal{N}_L^{(1)}(A_L^\lambda)} = \prod_{Q \in \mathcal{S}_L^{(1)}} z^{\mathbf{1}_{A_L^\lambda}(\lambda_Q)} = \prod_{Q \in \mathcal{S}_L^{(2)}} \prod_{Q^{(1)} \subseteq Q} z^{\mathbf{1}_{A_L^\lambda}(\xi^{(1)}+\lambda_Q)}.$$

Since $\mathbf{E}\big(z^{\mathcal{N}_L^{(1)}(A_L^\lambda)} \mid \mathcal{F}_{\geq 2}\big)$ depends only on λ_Q for $Q \in \mathcal{S}_L^{(2)}$, since the $\xi^{(1)}$'s are i.i.d. and independent of $\mathcal{F}_{\geq 2}$, and since each cube contains ν cubes of preceeding rank, we have

$$\mathbf{E}\big(z^{\mathcal{N}_L^{(1)}(A_L^\lambda)} \mid \mathcal{F}_{\geq 2}\big) = \psi_2\big(\lambda_Q : Q \in \mathcal{S}_L^{(2)}\big)$$

where for constants $\{\ell_Q : Q \in \mathcal{S}_L^{(2)}\} \subseteq \operatorname{supp}(\lambda^{(2)})$,

$$\psi_2\big(\ell_Q : Q \in \mathcal{S}_L^{(2)}\big) = \mathbf{E} \prod_{Q \in \mathcal{S}_L^{(2)}} \prod_{Q^{(1)} \subseteq Q} z^{\mathbf{1}_{A_L^\lambda}(\xi^{(1)}+\ell_Q)} = \prod_{Q \in \mathcal{S}_L^{(2)}} \Big(\mathbf{E} z^{\mathbf{1}_{A_L^\lambda}(\xi^{(1)}+\ell_Q)}\Big)^\nu.$$

Recalling that $h_1 = g_1^{(1)}$ is the density for $\xi^{(1)}$, we see that

$$\begin{aligned}
\mathbf{E} z^{\mathbf{1}_{A_L^\lambda}(\xi^{(1)}+\ell)} &= 1 + (z-1)\,\mathbf{P}\big(\xi^{(1)} + \ell \in A_L^\lambda\big) \\
&= 1 + (z-1) \int_{A_L^\lambda} h_1(x-\ell)\,dx \\
&= 1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L} h_1(\lambda-\ell) + (z-1) T g_1^{(1)}(\lambda-\ell)
\end{aligned}$$

where by Lemma 5.2, the remainder $Tg_1^{(1)}(\lambda - \ell)$ is of order $O(\nu^{-2L})$

$$\|Tg_1^{(1)}\|_\infty \le \frac{|A|\|g_1^{(1)}\|_\infty + \|(g_1^{(1)})'\|_\infty \int_A |x|dx}{\nu^{2L-2}} \le \frac{\nu^{\gamma/2} + \nu^{\gamma/2}}{\nu^{2L-2}} \le \frac{\nu^{3+\gamma/2}}{\nu^{2L}}.$$

Therefore, since

$$\left|\frac{(z-1)|A|(\nu-1)h_1(\lambda-\ell)}{\nu^L}\right| \le \frac{2|A|(\nu-1)\|h_1\|_\infty}{\nu^L} \le \frac{2\nu^{1+\gamma/2}}{\nu^L},$$

it follows by Lemma 5.4 with $c = 2\nu^{3+\gamma/2}$ that

$$\left|\psi_2\big(\lambda_Q : Q \in \mathcal{S}_L^{(2)}\big) - \prod_{Q\in\mathcal{S}_L^{(2)}} \left(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L}\nu h_1(\lambda-\lambda_Q)\right)\right| \le \frac{e^{2\nu^{2+\gamma/2}}}{\nu}\nu^{2-L}$$

hence taking

$$\begin{aligned}\varepsilon_2 &= \mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \mid \mathcal{F}_{\ge 2}\big) - z^{\sum_{r=2}^{M}\mathcal{N}_L^{(r)}(A_L^\lambda)} \prod_{Q\in\mathcal{S}_L^{(2)}} \left(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L}\nu h_1(\lambda-\lambda_Q)\right) \\ &= \left(\mathbf{E}\big(z^{\mathcal{N}_L^{(1)}(A_L^\lambda)} \mid \mathcal{F}_{\ge 2}\big) - \prod_{Q\in\mathcal{S}_L^{(2)}} \left(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L}\nu h_1(\lambda-\lambda_Q)\right)\right) z^{\sum_{r=2}^{M}\mathcal{N}_L^{(r)}(A_L^\lambda)}\end{aligned}$$

and keeping in mind that $|z| = 1$, we obtain (5.18) and (5.19) for $n = 2$.

Now assume (5.18) and (5.19) have been proven for n. Observe that

$$z^{\mathcal{N}_L^{(n)}(A_L^\lambda)} = \prod_{Q\in\mathcal{S}_L^{(n)}} z^{\mathbf{1}_{A_L^\lambda}(\lambda_Q)} = \prod_{Q\in\mathcal{S}_L^{(n+1)}} \prod_{Q^{(n)}\subseteq Q} z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\lambda_Q)}.$$

Subtracting ε_n and then dividing (5.18) by $z^{\mathcal{N}_L^{(r)}(A_L^\lambda)}$ for $r \ge n+1$, we obtain

$$\begin{aligned}&\left(\mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \,\big|\, \mathcal{F}_{\ge n}\big) - \varepsilon_n\right) z^{-\sum_{r=n+1}^{M}\mathcal{N}_L^{(r)}(A_L^\lambda)} \\ &\qquad = z^{\mathcal{N}_L^{(n)}(A_L^\lambda)} \prod_{Q\in\mathcal{S}_L^{(n)}} \left(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L}\nu h_{n-1}(\lambda-\lambda_Q)\right) \\ &\qquad = \prod_{Q\in\mathcal{S}_L^{(n+1)}} \prod_{Q^{(n)}\subseteq Q} \left[z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\lambda_Q)} \left(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L}\nu h_{n-1}(\lambda-\xi^{(n)}-\lambda_Q)\right)\right].\end{aligned} \tag{5.21}$$

Since $\mathcal{F}_{>n} \subseteq \mathcal{F}_{\geq n}$, we have

$$\mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \bigm| \mathcal{F}_{>n}\big) = \mathbf{E}\big(\,\mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \bigm| \mathcal{F}_{\geq n}\big) \bigm| \mathcal{F}_{>n}\big).$$

Re-conditioning the right-hand side of (5.21) on $\mathcal{F}_{>n}$, keeping in mind that each $z^{\mathcal{N}_L^{(r)}(A_L^\lambda)}$ is $\mathcal{F}_{>n}$-measurable for $r \geq n+1$, we obtain

$$\mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \bigm| \mathcal{F}_{>n}\big) = \psi_{n+1}\big(\lambda_Q : Q \in \mathcal{S}_L^{(n+1)}\big) z^{\sum_{r=n+1}^{M} \mathcal{N}_L^{(r)}(A_L^\lambda)} + \mathbf{E}\big(\varepsilon_n \bigm| \mathcal{F}_{>n}\big)$$

where

$$\begin{aligned}
&\psi_{n+1}\big(\ell_Q : Q \in \mathcal{S}_L^{(n+1)}\big) \\
&\qquad = \mathbf{E} \prod_{Q \in \mathcal{S}_L^{(n+1)}} \prod_{Q^{(n)} \subseteq Q} \Big[z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\ell_Q)} \Big(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L} \nu h_{n-1}(\lambda - \xi^{(n)} - \ell_Q)\Big)\Big] \\
&\qquad = \prod_{Q \in \mathcal{S}_L^{(n+1)}} \left[\mathbf{E}\left(z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\ell_Q)} \Big(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L} \nu h_{n-1}(\lambda - \xi^{(n)} - \ell_Q)\Big)\right)\right]^{\nu}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathbf{E} z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\ell)} &= 1 + (z-1) \int_{A_L^\lambda} g_1^{(n)}(x-\ell)dx \\
&= 1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L} g_1^{(n)}(\lambda - \ell) + (z-1) T g_1^{(n)}(\lambda - \ell)
\end{aligned}$$

while

$$\begin{aligned}
\mathbf{E}\Big(z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\ell)} h_{n-1}(\lambda - \xi^{(n)} - \ell)\Big) &= \int z^{\mathbf{1}_{A_L^\lambda}(x)} g_1^{(n)}(x-\ell) h_{n-1}(\lambda - x) dx \\
&= \left(\int + (z-1) \int_{A_L^\lambda} \right) g_1^{(n)}(x-\ell) h_{n-1}(\lambda - x) dx \\
&= \tfrac{1}{\nu}(h_n - g_1^{(n)})(\lambda - \ell) + (z-1) \int_{A_L^\lambda} g_1^{(n)}(x-\ell) h_{n-1}(\lambda - x) dx
\end{aligned}$$

where we have used (5.16) to arrive at the last equality. Then

$$\begin{aligned}
&\mathbf{E}\Big(z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\ell)}\Big(1+\tfrac{(z-1)|A|(\nu-1)\nu h_{n-1}(\lambda-\xi^{(n)}-\ell)}{\nu^L}\Big)\Big)\\
&=\mathbf{E}z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\ell)}+\tfrac{(z-1)|A|(\nu-1)}{\nu^L}\nu\,\mathbf{E}\Big(z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\ell)}h_{n-1}(\lambda-\xi^{(n)}-\ell)\Big)\\
&=1+\tfrac{(z-1)|A|(\nu-1)h_n(\lambda-\ell)}{\nu^L}+(z-1)\Big[Tg_1^{(n)}(\lambda-\ell)+\tfrac{|A|(\nu-1)}{\nu^{L-1}}\int_{A_L^\lambda}g_1^{(n)}(x-\ell)h_{n-1}(\lambda-x)dx\Big].
\end{aligned}$$

The remainder term is $O(\nu^{-2L})$. By Lemmas 5.1 and 5.2 we have

$$\begin{aligned}
&\left|Tg_1^{(n)}(\lambda-\ell)+\frac{|A|(\nu-1)}{\nu^{L-1}}\int_{A_L^\lambda}g_1^{(n)}(x-\ell)h_{n-1}(\lambda-x)dx\right|\\
&\qquad\le\frac{|A|\|g_1^{(n)}\|_\infty+\left(\int_A|x|dx\right)\|(g_1^{(n)})'\|_\infty}{\nu^{2L-2}}+\frac{(\nu-1)|A|^2\|g_1^{(n)}\|_\infty\|h_{n-1}\|_\infty}{\nu^{2L-2}}\\
&\qquad\le\frac{2\nu^{\gamma/2}}{\nu^{2L-2}}+\frac{(\nu-1)\nu^\gamma}{\nu^{2L-2}}\le 2\nu^{3+\gamma-2L}.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&\left|\mathbf{E}\Big(z^{\mathbf{1}_{A_L^\lambda}(\xi^{(n)}+\ell)}\Big(1+\tfrac{(z-1)|A|(\nu-1)}{\nu^L}\nu h_{n-1}(\lambda-\xi^{(n)}-\ell)\Big)\Big)\right|\\
&\qquad\le 1+\tfrac{2|A|(\nu-1)\nu\|h_{n-1}\|_\infty}{\nu^L}\le 1+\tfrac{2(\nu-1)\nu^{1+\gamma/2}}{\nu^L}\le 1+\tfrac{2\nu^{3+\gamma}}{\nu^L}
\end{aligned}$$

and

$$\left|1+\tfrac{(z-1)|A|(\nu-1)}{\nu^L}h_n(\lambda-\ell)\right|\le 1+\tfrac{2|A|(\nu-1)\|h_n\|_\infty}{\nu^L}\le 1+\tfrac{2(\nu-1)\nu^{\gamma/2}}{\nu^L}\le 1+\tfrac{2\nu^{3+\gamma}}{\nu^L}.$$

It follows by Lemma 5.4 with $c=2\nu^{3+\gamma}$ that

$$\left|\psi_{n+1}\big(\lambda_Q:Q\in\mathcal{S}_L^{(n+1)}\big)-\prod_{Q\in\mathcal{S}_L^{(n+1)}}\Big(1+\tfrac{(z-1)|A|(\nu-1)}{\nu^L}\nu h_n(\lambda-\lambda_Q)\Big)\right|\le\frac{e^{2\nu^{3+\gamma-n}}}{\nu^n}\nu^{2-L}.$$

Finally, taking

$$\varepsilon_{n+1} = \mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \;\big|\; \mathcal{F}_{>n}\big) - z^{\sum_{r=n+1}^{M} \mathcal{N}_L^{(r)}(A_L^\lambda)} \prod_{Q\in \mathcal{S}_L^{(n+1)}} \Big(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^L} \nu h_n(\lambda - \lambda_Q)\Big)$$

$$= \Bigg[\psi_{n+1}\big(\lambda_Q : Q \in \mathcal{S}_L^{(n+1)}\big) - \prod_{Q\in \mathcal{S}_L^{(n+1)}} \Big(1 + \tfrac{(z-1)|A|(\nu-1)\nu h_n(\lambda-\lambda_Q)}{\nu^L}\Big)\Bigg] z^{\sum_{r=n+1}^{M} \mathcal{N}_L^{(r)}(A_L^\lambda)} + \mathbf{E}\big(\varepsilon_n \,\big|\, \mathcal{F}_{>n}\big)$$

and keeping in mind that $|z| = 1$, we obtain (5.18) and (5.19) for $n + 1$. ∎

Corollary 5.7. *Let M, L and γ be as in Lemma 5.6. Then*

$$\mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \,\big|\, \mathcal{F}_{>M}\big) = \prod_{Q\in \mathcal{S}_L^{(M+1)}} \Big(1 + \tfrac{(z-1)|A|(\nu-1)}{\nu^{L-1}} h_M(\lambda - \lambda_Q)\Big) + O(\nu^{-L}). \tag{5.22}$$

Proof.

$$\|\varepsilon_{M+1}\|_\infty \le \nu^{2-L} \sum_{k=1}^{M} \frac{e^{2\nu^{2+\gamma-k}}}{\nu^k} \le \frac{2\nu^2 e^{2\nu^{1+\gamma}}}{\nu^L}. \tag{5.23}$$

∎

Theorem 5.8. $\lim_{L\to\infty} \mathbf{E} z^{\mathcal{N}_L(A_L^\lambda)} = e^{(z-1)n(\lambda)|A|}$.

Proof. Applying Lemma 5.3 to (5.22) we see that

$$\mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \,\big|\, \mathcal{F}_{>M}\big) = \exp\Bigg(\sum_{Q\in \mathcal{S}_L^{(M+1)}} \frac{(z-1)|A|(\nu-1)h_M(\lambda-\lambda_Q)}{\nu^{L-1}}\Bigg) + O(\nu^{-L}).$$

as $L \to \infty$. We may rewrite the sum inside the exponent as an integral with respect to the empirical measure $N_L^{(M+1)}(dx)$ of eigenvalues for (4.23) of rank $M + 1$. We have

$$\sum_{Q\in \mathcal{S}_L^{(M+1)}} \tfrac{(\nu-1)h_M(\lambda-\lambda_Q)}{\nu^{L-1}} = \tfrac{1}{\left|\mathcal{S}_L^{(M+1)}\right|} \sum_{Q\in \mathcal{S}_L^{(M+1)}} \tfrac{(\nu-1)h_M(\lambda-\lambda_Q)}{\nu^M} = \int \tfrac{(\nu-1)h_M(\lambda-x)}{\nu^M} N_L^{(M+1)}(dx)$$

so that, as $L \to \infty$,

$$\mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \,\big|\, \mathcal{F}_{>M}\big) = \exp\!\left(\frac{(z-1)|A|(\nu-1)}{\nu^M}\int h_M(\lambda - x) N_L^{(M+1)}(dx)\right) + O(\nu^{-L}).$$

But $N_L^{(M+1)}(dx)$ converges weakly as $L \to \infty$ to $N^{(M+1)}(dx) = g^{(M+1)}(x)dx$. Therefore, applying (5.17)

$$n(\lambda) = \frac{(\nu-1)(h_M * g^{(M+1)})(\lambda)}{\nu^M} = \frac{\nu-1}{\nu^M}\int h_M(\lambda - x) g^{(M+1)}(x)\,dx,$$

we see that

$$\lim_{L\to\infty} \mathbf{E}\big(z^{\mathcal{N}_L(A_L^\lambda)} \,\big|\, \mathcal{F}_{>M}\big) = e^{(z-1)n(\lambda)|A|}.$$

Taking expectations, we obtain our result. ■

5.3 Statistics for eigenvalues of rank r

Theorem 5.9. $\lim_{L\to\infty} \mathbf{E} z^{\mathcal{N}_L^{(r)}(A_L^\lambda)} = e^{(z-1)|A|(\nu-1)g^{(r)}(\lambda)/\nu^r}$

Proof. We may directly condition on $\lambda^{(L)}$. Observe that the random variables

$$\lambda_Q - \lambda^{(L)} = \lambda^{(r)} + \lambda^{(r+1)} + \cdots + \lambda^{(L-1)}, \ \text{ for } \ Q \in \mathcal{S}_L^{(r)},$$

are i.i.d. with density $g_{L-r}^{(r)}$. Then

$$\mathbf{E}\big(z^{\mathcal{N}_L^{(r)}(A_L^\lambda)} \,\big|\, \lambda^{(L)}\big) = \psi(\lambda^{(L)})$$

where for $\ell \in \text{supp}(\lambda^{(L)})$,

$$\begin{aligned}
\psi(\ell) &= \mathbf{E} \prod_{Q \in \mathcal{S}_L^{(r)}} z^{\mathbf{1}_{A_L^\lambda}(\lambda_Q - \lambda^{(L)} + \ell)} = \left(z^{\mathbf{1}_{A_L^\lambda}(\lambda_Q - \lambda^{(L)} + \ell)} \right)^{\nu^{L-r}} \\
&= \left(1 + (z-1) \int_{A_L^\lambda} g_{L-r}^{(r)}(x - \ell) dx \right)^{\nu^{L-r}} \\
&= \left(1 + \frac{(z-1)|A|(\nu-1)}{\nu^L} g_{L-r}^{(r)}(\lambda - \ell) + (z-1) T g_{L-r}^{(r)}(\lambda - \ell) \right)^{\nu^{L-r}}.
\end{aligned}$$

In the proof of Lemma 4.2, it was shown by considering the characteristic functions of $g_n^{(r)}$ that there exists a constant c for which

$$\left| g_n^{(r)}(x) - g_n^{(r)}(y) \right| \leq \frac{c|x-y|}{(\sigma\lambda_r)^2}$$

uniformly for all $r \geq 1$ and $1 \leq n \leq \infty$. From this it follows that $\|g_n^{(r)}\|_\infty \leq \frac{2c}{\sigma\lambda_r}$ for all n hence $(z-1)Tg_{L-r}^{(r)}(\lambda - \lambda^{(L)})$ is $O(\nu^{-2L})$ as $L \to \infty$. Then, applying Lemma 5.3, we have

$$\mathbf{E}\big(z^{\mathcal{N}_L^{(r)}(A_L^\lambda)} \,\big|\, \lambda^{(L)} \big) = \exp\left(\frac{(z-1)|A|(\nu-1)}{\nu^r} g_{L-r}^{(r)}(\lambda - \lambda^{(L)}) \right) + O(\nu^{-L}) \ \text{ as } L \to \infty$$

hence

$$\mathbf{E} z^{\mathcal{N}_L^{(r)}(A_L^\lambda)} = \mathbf{E} e^{(z-1)|A|(\nu-1) g_{L-r}^{(r)}(\lambda - \lambda^{(L)})/\nu^r} + O(\nu^{-L}) \ \text{ as } L \to \infty.$$

Again appealing to Lemma 4.2, one can prove that $g_{L-r}^{(r)}(\lambda - \lambda^{(L)}) \to g^{(r)}(\lambda)$ in some sense and thereby obtain the result. ■

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