LIMIT THEOREMS FOR RANDOM EXPONENTIAL SUMS AND THEIR APPLICATIONS TO INSURANCE AND RANDOM ENERGY MODEL

by

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ABSTRACT

HUSEYIN ERTURK. Limit theorems for random exponential sums and their applications to insurance and random energy model. (Under the direction of DR. STANISLAV MOLCHANOV)

In this dissertation, we are mainly concerned with the sum of random exponentials, $S_N(t) = \sum_{i=1}^{N(t)} e^{tX_i}$. Here, $t, N(t) \to \infty$ in appropriate form and $\{X_i, i \geq 1\}$ are i.i.d. random variables. Our first goal is to find the limiting distributions of $S_N(t)$ for new class of the random variables, $\{X_i, i \geq 1\}$. For some classes, such results were known; normal distribution, Weibull distribution etc.

Secondly, we apply these limit theorems to some insurance models and random energy model (REM) in statistical physics. Specifically for the first case, we give the estimate of the ruin probability in terms of the empirical data. For REM, we present the analysis of the free energy for new class of distributions of the random variables, X_i. In some particular cases, we prove the existence of several critical points for free energy. In some other cases, we prove the absence of phase transitions.

The technical tool of this study includes the classical limit theory for the sum of i.i.d. random variables and different asymptotic methods like Euler-Maclaurin formula and Laplace method.

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CHAPTER 1: INTRODUCTION

1.1 A General Summary

The main object in this paper is the partial sum of exponentials of the form

$$S_N(t) = \sum_{i=1}^{N(t)} e^{tX_i} \tag{1}$$

where the sequence,

$$\{X_1, X_2, ..., X_{N(t)}\}, \tag{2}$$

is composed of i.i.d. random variables. First, we analyze the limiting behavior of this object for different growth rates of N(t) when the sequence (2) is double exponentially distributed (8). In our analysis, we show that the random exponential sum converges to normal distribution or stable distribution under appropriate additive and multiplicative factors of t. After this theoretical analysis, we explore applications of the statistical sum in insurance mathematics and statistical physics.

1.2 Two Particular Applications

The first application of the partial sum of exponentials is from insurance mathematics. Consider a portfolio consisting of N policies with individual risks $\{X_1, ..., X_N\}$ over a given time period and assume that the nonnegative random variables $\{X_1, ..., X_N\}$ are i.i.d. Here the aggregate claim amount can be calculated as $U = \sum_{i=1}^{N} X_i$ and the risk reserve process is given by $R(s) = u + \beta s - U$ where β is the premium rate, s is time and and u is the initial reserve. One problem is to estimate the Lundberg bounds which approximate the tail distribution of U, $\overline{F}_U(x) = P(U > x)$. This requires the solution of the Laplace equation,

$$m(\gamma) = E(\gamma X) = p^{-1} \tag{3}$$

where p is a small constant. We assume that the solution exists and it is called the adjustment coefficient, γ. Also the same equation helps us to approximate the ruin probability $\psi(s) := P\left(\min_{s \geq 0} R(s) < 0\right)$ for appropriate p which is essential for insurance companies [see Rolski et al. (1999), Sect. 4.5.1, p. 125-126 and Sect. 5.4.1, p. 170-171] [11].

In practical applications, γ is estimated using a statistical method and this estimation utilizes empirical Laplace transform. Hence, we replace $m(\gamma)$ (3) with the empirical Laplace transform. Also, we define p on the right hand side of the Laplace equation as a sequence, p_n. When $n \to \infty$, $p_n \to 0$. Then, we obtain the empirical Laplace equation:

$$\bar{m}_U(\gamma_n) := \frac{1}{N(\gamma_n)} \sum_{i=1}^{N(\gamma_n)} e^{\gamma_n U_i} = p_n^{-1} \tag{4}$$

It means that we have a sequence of adjustment coefficients, γ_n, for a sequence of insurance portfolios which give a sequence of Lundberg bounds to esimate ruin probabilities from below and above. Our interest is to analyze the asymptotic behaviour of γ_n when n is large. We make use of the exponential sum to develop this estimation procedure. The estimation of γ has been studied in the paper by Sándor Csörgö and Jef L. Teugels [6] where classical central theorem has been used. Our approach is slightly different in the sense that we can control the growth rate of number of individual risks.

Another application of this study is REM. REM was first introduced by Derrida [2]. Eisele [7] demonstrated the phase transitions (non-analiticity) of free energy in the class of Weibull type distributions. We will show similar results for Weibull distribution, relatively heavy tailed distribution and relatively light tailed double exponential distribution using order statistics. Also, we will show that there are several critical points for mixed Weibull distributions.

REM [2] introduced in Derrida's paper describes the system of size n with 2^n energy levels where $E_i = \sqrt{n}X_i$ and $\{\, X_i,\ i = 1, 2, ..., N \,\}$ are i.i.d. random variables following N(0,1) distribution. Thermodynamics of the system is quantified by the statistical sum i.e. so called partition function. This partition function in Derrida's model has the following form

$$Z_n(\beta) = \sum_{i=1}^{N} e^{\beta A(n) X_i} \tag{5}$$

where $A(n) = \sqrt{n}$ and $\beta > 0$ is the inverse temperature. We use the same statistical sum with different selection of $A(n)$. Derrida defines free energy by the following formula:

$$\chi(\beta) = \lim_{n\to\infty} \frac{log Z_n(\beta)}{n}$$

According to Derrida's results, free energy is quantified as

$$\chi(\beta) = \begin{cases} \beta^2/2 + \beta_c^2/2, & \text{if } 0 < \beta \leq \beta_c\ , \\ \\ \beta\beta_c, & \text{if } \beta \geq \beta_c \end{cases}$$

where $\beta_c = \sqrt{2log2}$. It is important to note that $\chi(\beta)$ and $\chi'(\beta)$ are continuous but $\chi''(\beta)$ has a jump. This is so called third order phase transition. $\chi(\beta)$ is convex and continuous. The phase transition introduces the presence of two analytic branches in free energy. One branch corresponds to the high temperature i.e. $\beta = \dfrac{1}{kT} < \beta_{critical}$. The second branch corresponds to the low temperature i.e. $\beta \geq \beta_{critical}$.

Derrida's paper was extended in several directions. In Eisele's paper [2], the results of Derrida [2] were proven for Weibull type distributions. Later on, Olivieri and Picco [3] and also L. A. Pastur [4] rigorously derived the limits as well. The mathematical justification of this result as well as the theory of limit theorems for the sum $Z_n(\beta)$ was analyzed in detail in the mathematical paper by A. Bovier, I. Kurkova and M. Löwe [1]. In the paper by G. Ben Arous, L. V. Bogachev and S. A. Molchanov [5], the results were extended to the Weibull/Frechet-type tails. It contains the complete theory of of the limiting distributions for the sum of the random exponentials in the case

$$Z_n(\beta) = \sum_{i=1}^{N} e^{tX_i}$$

$$P\{X_i > a\} = \exp\left\{-\frac{a^{\varrho}L(a)}{\varrho}\right\}$$

where $\varrho \geq 1$ and $L(a)$ is slowly varying function with additional regulatory properties (See in [5]).

The technical tools in A. Bovier, I. Kurkova and M. Löwe [1] and G. Ben Arous, L. V. Bogachev and S. A. Molchanov [5] are traditional Bahr-Esseen inequality and Lyapunov fraction that are used for the proof of LLN and CLT. Also, standard methods for the stable distributions are utilized.

In this paper, we use the methods developed in G. Ben Arous, L. V. Bogachev and S. A. Molchanov [5] for the computation of free energy. In addition to this methodology, we develop the new approach based on the properties of the variational series of exponential random variables, see in Feller Volume 2 (1971) [10]. This approach covers REM outside the Weibull type tails and Bahr-Esseen inequality. We analyze four types of distributions for REM: Weibull, mixed Weibull, light tailed and heavy tailed distribution.

CHAPTER 2: STATEMENT OF THE VARIABLES AND DISTRIBUTIONS

In this chapter, we state the variables and distributions that are used throughout the whole dissertation. All the sequence of random variables in this study are assumed to be i.i.d.

Weibull distribution is the most commonly used distribution. Weibull random variable, X, folllows the law:

$$1. \quad P(X > x) = \begin{cases} \exp\left\{-\dfrac{x^{\varrho}}{\varrho}\right\}, & \text{if x} \geq 0 \\ 1, & \text{o.w.} \end{cases} \tag{6}$$

where $1 < \varrho < \infty$. Also, we make use of the mixed Weibull distribution:

$$2. \quad X = \begin{cases} X_1, & \text{with prob. p and } P(X_1 > x) = \exp\left\{-\dfrac{x^{\varrho}}{\varrho}\right\} \quad (7a) \\ L(n) + X_2, & \text{with prob. 1-p and } P(X_2 > x) = \exp\left\{-\dfrac{x^{\varrho}}{\varrho}\right\} \quad (7b) \end{cases}$$

where n is a large number and $1 < \varrho < \infty$. In the next chapter, we work on double exponential distribution which has lighter tails than Weibull distribution. The paper by G. Ben Arous, L. V. Bogachev and S. A. Molchanov [5] analyzes the limiting distributions of the random exponential sum (1) when X_i's in the statistical sum are Weibull type random variables. We extend this to the double exponential random variable which has the distribution function:

$$3. \quad P(X > x) = \begin{cases} \exp\{1 - e^{x}\}, & \text{if } x \geq 0 \quad (8a) \\ 1, & \text{o.w.} \quad (8b) \end{cases}$$

In addition to the above distributions, we have relatively heavy tailed distribution. Corresponding heavy tailed random variable is defined as a function of standard exponential random variables. Heaviness of the tail behavior is relative to Weibull distribution. Standard exponential distribution and relatively heavy tailed random variable are expressed as

$$4. \quad P(Y > x) = \begin{cases} \exp\{-x\}, & \text{if x} \geq 0 \\ 1, & \text{o.w.} \end{cases} \tag{9}$$

$$5. \quad X = \frac{1+Y}{\ln(1+Y)} \tag{10}$$

respectively.

Assume that w_X stands for ess sup X, and $P(X < w_X) = 1$ which means X is finite with probability 1 and the log tail distribution for above distributions is:

$$h(x) = -\log P(X > x) \tag{11}$$

for $x \in \mathbb{R}$, $h(x)$ is non-negative, non-decreasing and right-continuous. From the above information, we can state that $P(X > x) = e^{-h(x)}$ such that $x < w_X$. If h is regularly varying at infinity with index ϱ, we write $h \in R_\varrho$ where $1 < \varrho < \infty$. It means that for any $\kappa > 0$ we have $h(\kappa x)/h(x) \to \kappa^\varrho$ as $x \to \infty$

We frequently work with Laplace transform and we require that $E\left[e^{tX_i}\right] < \infty$ for finite t. The selected distributions above satisfy this condition and detailed analysis is given in Chapter 6: Appendix. We introduce the cumulant generating function as

$$H(t) = \log E[e^{tX}] \tag{12}$$

where $H(t)$ is well defined, non-decreasing for any $t \geq 0$. $H(t) \to \infty$ as $t \to \infty$. For Weibull distribution,

$$\varrho^{'} = \frac{\varrho}{\varrho - 1} \tag{13}$$

is being used as the exponent of the cumulant generating function with the condition that $1 < \varrho^{'} < \infty$. Note that $1 = \frac{1}{\varrho} + \frac{1}{\varrho^{'}}$. It is important to mention that $h \in R_{\varrho}$ implies $H \in R_{\varrho'}$

As a result of these definitions we express the expected value of the random exponential sum (1) as

$$E[S_N(t)] = \sum_{i=1}^{N} E[e^{tX_i}] = Ne^{H(t)}, \tag{14}$$

For REM, random variables in the statistical sum (5) are expressed as a function of exponential random variables, $Y_1, ..., Y_N$ (9), such that $X_i = f(Y_i)$ (10). This enables us to express the statistical sum in a simplified form and compute free energy using Euler-Maclaurin formula and Laplace Method. The results for free energy depend on the structure of the distribution which is specified by $f(Y_i)$ and the selection of $A(n)$. $A(n)$ is an analytic and increasing function of n. For appropriate selection of $A(n)$, we assume that there exists p-a.s. limit for free energy

$$\chi(\beta) \quad := \quad \lim_{n\to\infty} \frac{\log Z_n(\beta)}{n}$$

Also, there are other variables that we will use for various theorems:

$$B(t) \quad = (\lambda t)^t \tag{15}$$

$$A(t) = \begin{cases} E[S_N(t)], & \text{for } 1 < \lambda < 2 \quad (16a) \\ E[S_N(t)1_{\{Y \leq \tau\}}], & \text{for } \lambda = 1 \quad (16b) \\ 0, & \text{for } 0 < \lambda < 1 \quad (16c) \end{cases}$$

CHAPTER 3: LIMIT THEOREMS FOR WEIBULL AND DOUBLE EXPONENTIAL DISTRIBUTION

This section is devoted to the convergence of the random exponential sum (1) when X_i's (2) have Weibull (6) or double exponential distribution (8). Similar analysis has been done for Weibull distribution in in the paper of Gerard Ben Arous, Leonid V. Bogachev, Stanislav A. Molchanov [5]. We extend this to double exponential distribution. We look for the range of the exponential rate, λ, on $N(t)$ that gives the necessary and sufficient conditions for the existence of law of large numbers (LLN), central limit theorem (CLT) and convergence to the stable distribution. Before starting our theorems we specify the growth rate of $N(t)$. In this chapter, Case 1 refers to the Weibull distribution (6) and Case 2 refers to the double exponential distribution (8). When X_i's have Weibull distribution,

$$N(t) = e^{\lambda H(t)} \tag{17}$$

is being used as the growth rate. $H(t)$ is the cumulant generating function introduced in (12). The asymptotic of $H(t)$, $H_0(t)$, can be found in Appendix 6.3. When X_i's have double exponential distribution (8), the growth rate is

$$N(t) = e^{\lambda t} \tag{18}$$

We first prove lemma 1 that helps us in the proof of LLN and CLT. In later sections, we prove LLN, CLT and convergence to the stable distribution. Similar study for Weibull distribution has been done in the paper by G. Ben Arous, L. V. Bogachev and S. A. Molchanov [5]. We extend it to double exponential distribution.

3.1 Main Lemma

Lemma 1. *Consider the function*

$$v_\lambda(x) \;:=\; \lambda(x-1) - (x^{\varrho'} - x) \quad x \geq 1$$

if $\lambda > \lambda_b$ *(*$\lambda_b = \lambda_1 = \varrho' - 1$ *for Case 1) then there exists* $x_0 > 1$ *such that* $v_\lambda(x) > 0$ *for all* $x \in (1, x_0)$*.*

Proof. Note that $v_\lambda(1) = 0$ and $v'_\lambda(x) = \lambda - (\varrho' x^{\varrho' - 1} - 1)$ so $v'_\lambda(1) = \lambda - (\varrho' - 1) = \lambda - \lambda_b > 0$ where $\lambda_b = \lambda_1$ for Case 1. Based on Taylor's formula, $v_\lambda(x) > 0$ for all $x > 1$ sufficiently close to 1. □

3.2 Main Theorems

Theorem 2. *Law of large numbers (LLN) for different growth rates of* $N(t)$*,*

$$\frac{S_N(t)}{E[S_N(t)]} \xrightarrow{p} 1. \tag{19}$$

1. *Assume that* X_i*'s (2) in the statistical sum (5) have Weibull distribution (6). If* $\lambda > \varrho' - 1 = \lambda_1$ *(17), LLN holds.*

2. *Assume that* X_i*'s (2) in the statistical sum (5) have double exponential distribution (8). If* $\lambda > 1$ *(18), LLN holds.*

Proof. Set

$$S_N^*(t) = \frac{S_N(t)}{E[S_N(t)]} = \frac{1}{N}\sum_{i=1}^{N} e^{tx_i - H(t)}$$

It is sufficient to show that $\lim_{t\to\infty} E\left|S_N^*(t) - 1\right|^r = 0$ for some $r > 1$.

$$E|S_N^*(t) - 1|^r = E\left|\frac{\sum_{i=1}^N e^{tx_i - H(t)}}{N} - 1\right|^r$$
$$= E\left|\frac{\sum_{i=1}^N e^{tx_i - H(t)} - 1}{N}\right|^r = N^{-r} E\left|\sum_{i=1}^N e^{tx_i - H(t)} - 1\right|^r$$

Using Bahr Essen inequality and $(x+1)^r \le 2^{r-1}(x^r + 1)$ where $(x > 0, r \ge 1)$,

$$N^{-r} E\left|\sum_{i=1}^N e^{tx_i - H(t)} - 1\right|^r \le 2N^{-r}\sum_{i=1}^N E\left|e^{tx_i - H(t)} - 1\right|^r$$
$$\le 2N^{1-r} E\left|e^{tx_i - H(t)} + 1\right|^r \le 2N^{1-r}2^{r-1} E\left|e^{rtx_i - rH(t)} + 1\right|$$
$$= 2^r N^{1-r} e^{H(rt) - rH(t)} + 2^r N^{1-r} \tag{20}$$

Case 1: Since $H \in R_{\varrho'}$ [Refer to Appendix 6.2 for details]

$$\liminf_{n\to\infty}\left[\frac{(r-1)\log(N)}{H(t)} - \frac{H(rt)}{H(t)} + r\right] = \lambda(r-1) - \left(r^{\varrho'} - r\right) = v_\lambda(r)$$

By Lemma 1, we can choose $r > 1$ such that $v_\lambda(r) > 0$ when $\lambda > \lambda_1 = \frac{\varrho'}{\varrho} = \varrho' - 1$ and this implies that right hand side converges to 0.

Case 2: For Double Exponential distribution, we use the Bahr-Essen inequality (20),

$$E\left|S_N^*(t) - 1\right|^r < 2^r N^{1-r} e^{H(rt) - rH(t)} + 2^r N^{1-r}.$$

Cumulant generating function $H(t)$ of double exponential distribution has asymptotic equivalent

$$H(t) = t\ln(t) - t + \frac{\ln t}{2} + \underline{\varrho}(1).$$

[Refer to Appendix 6.3 for details] Only the first two terms play a role in the proof of LLN. Using substitution $r = 1 + \epsilon$ as $\epsilon \to 0^+$, we must have

$$(r-1)\log(N) - H(rt) + rH(t) \cong \epsilon \log(N) - (1+\epsilon)\epsilon t + \epsilon/2(\ln t - 1) > 0$$

for the existence of LLN which implies that

$$\frac{\log N}{t} = \lambda > 1$$

□

Theorem 3. *CLT for different growth rates of $N(t)$*

$$\frac{S_N(t) - E[S_N(t)]}{Var[S_N(t)]^{1/2}} \xrightarrow{d} N(0,1), \tag{21}$$

1. *Assume that X_i's (2) in the statistical sum (5) have Weibull distribution (6). If $\lambda > 2^{\varrho'} \frac{\varrho'}{\varrho} = \lambda_2$ (17), CLT holds.*

2. *Assume that X_i's (2) in the statistical sum (5) have double exponential distribution (8). If $\lambda > 2$ (18), CLT holds.*

Proof. Suppose that $e^{tX_1}, e^{tX_2}, \ldots$ is a sequence of independent random variables, each with finite expected value and variance. We know from Lemma 4.1 in [5] that $Var(e^{tX_i}) \cong e^{H(2t)}$ for Weibull distribution. This asymptotic also holds for double exponential distribution which can be proven using the same steps of Lemma 4.1 in [5]. Define

$$s_n^2 = \sum_{i=1}^{N(t)} Var(e^{tX_i}) \cong N(t)e^{H(2t)}$$

If for some $\delta > 0$, the Lyapunov's condition

$$\lim_{t\to\infty} = \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{N(t)} E\left[\left|e^{tX_i} - E\left(e^{tX_i}\right)\right|^{2+\delta}\right] = 0$$

is saitsfied then $\dfrac{S_N(t) - E[S_N(t)]}{Var[S_N(t)]^{1/2}}$ converges to standard normal distribution. Using the Lyapunov's condition and the inequality, $(x+1)^r \leq 2^{r-1}(x^r+1)$ where $(x > 0, r \geq 1)$, we obtain

$$\begin{aligned}
&\frac{1}{s_n^{2+\delta}} \sum_{i=1}^{N(t)} E[|e^{tX_i} - E(e^{tX_i})|^{2+\delta}] \\
&\cong N(t)^{-\delta/2} \exp\{H(t)(2+\delta)\} \exp\{-H(2t)(1+\delta/2)\} E\left[\left|\frac{e^{tX_i}}{e^{H(t)}} - 1\right|^{2+\delta}\right] \\
&\leq \exp\{-\ln(N(t))\delta/2 + H(t)(2+\delta) - H(2t)(1+\delta/2)\}\, E\left[\left(\frac{e^{tX_i}}{e^{H(t)}} + 1\right)^{2+\delta}\right] \\
&\leq 2^{r-1} \exp\{-\ln(N(t))\delta/2 + H(t)(2+\delta) - H(2t)(1+\delta/2)\} \left[E\left[\frac{e^{t(2+\delta)X_i}}{e^{H(t)(2+\delta)}}\right] + 1\right] \\
&= 2^{r-1} \exp\{-\ln(N(t))\delta/2 - H(2t)(1+\delta/2) + H(t(2+\delta))\}\,(1+o(1)) \qquad (22)
\end{aligned}$$

Case 1: Since $H \in R_{\varrho'}$ and using the substitution $r = 1 + \delta/2$,

$$\begin{aligned}
&\liminf_{t\to\infty} \left[\frac{\ln(N(t))\delta/2}{H(t)} + \frac{H(2t)(1+\delta/2)}{H(t)} - \frac{H(t(2+\delta))}{H(t)}\right] \\
&= 2^{\varrho'} \left[\frac{\lambda}{2^{\varrho'}}(r-1) - (r^{\varrho'} - r)\right] = 2^{\varrho'} v_{\lambda/2^{\varrho'}}(r)
\end{aligned}$$

By Lemma 1, we can choose $r > 1$ such that $v_{\lambda/2^{\varrho'}}(r) > 0$ when $\lambda/2^{\varrho'} > \lambda_1 = \dfrac{\varrho'}{\varrho}$ and this implies that CLT holds if $\lambda > \lambda_2 = 2^{\varrho'}\dfrac{\varrho'}{\varrho}$.

Case 2: We make use of the inequality that we obtained in (22)

$$\begin{aligned}
&\frac{1}{s_n^{2+\delta}} \sum_{i=1}^{N(t)} E\left[\left|e^{tX_i} - E\left(e^{tX_i}\right)\right|^{2+\delta}\right] \\
&\leq 2^{r-1} \exp\{-\ln(N(t))\delta/2 - H(2t)(1+\delta/2) + H(t(2+\delta))\}\,(1+o(1))
\end{aligned}$$

where $H(t)$ for double exponential distribution has asymptotic equivalent $H_0(t) = t\ln(t) - t$. Then, the requirement for CLT is the following condition

$$\liminf_{\substack{t\to\infty \\ \delta\to 0^+}} \left[\ln(N(t))\delta/2 + H(2t)(1+\delta/2) - H(t(2+\delta))\right] > 0$$

This inequality implies that we have CLT if

$$\liminf_{t\to\infty} \frac{\ln(N)}{t} = \lambda > \liminf_{\delta\to 0^+} \frac{\ln(1+\delta/2)}{\delta/2}(2+\delta) = 2$$

□

Theorem 4. *Conditions for Convergence to an Infinitely Divisible Distribution*

We use the theorem about the weak convergence of sums of independent random variables from Gerard Ben Arous, Leonid V. Bogachev, Stanislav A. Molchanov [5] which is also given in a similar form in the book of Petrov [8]. Suppose that

$$Y_i(t) = \frac{e^{tX_i}}{B(t)} \tag{23}$$

is a sequence of independent identically distributed random variables where $B(t)$ is a multiplicative factor. Additionally, we define $A(t)$ as an additive factor. Both $A(t)$ and $B(t)$ are increasing function of t such that $A(t), B(t) \to \infty$ as $t \to \infty$. According to classical theorems on weak convergence of sums of independent random variables, in order that

$$S_N^*(t) = \sum_{i=1}^{N(t)} Y_i(t) - \frac{A(t)}{B(t)} \tag{24}$$

converges to an infinitely divisible law with characteristic function

$$\phi(u) = \exp\left\{iau - \frac{\sigma^2 u^2}{2} + \int_{|x|>0} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) dL(x)\right\}, \tag{25}$$

it is necessary and sufficient that the following conditions hold:

1. *At all continuity points, $L(x)$ satisfies*

$$L(x) = \begin{cases} \lim_{t\to\infty} NP\{Y \leq x\} & \text{for } x < 0 \\ -\lim_{t\to\infty} NP\{Y > x\} & \text{for } x > 0. \end{cases} \tag{26}$$

2. *σ^2 satisfies*

$$\sigma^2 = \lim_{\tau\to 0^+} \limsup_{t\to\infty} NVar[Y1_{\{Y\leq\tau\}}] = \lim_{\tau\to 0^+} \liminf_{t\to\infty} NVar[Y1_{\{Y\leq\tau\}}] \tag{27}$$

3. *For each $\tau > 0$ the following identity is satisfied.*

$$\lim_{t\to\infty} \left\{ NE[Y1_{\{Y\leq\tau\}}] - \frac{A(t)}{B(t)} \right\} = a + \int_0^\tau \frac{x^3}{1+x^2}\, dL(x) - \int_\tau^\infty \frac{x}{1+x^2}\, dL(x) \tag{28}$$

where a is a constant depending on the distribution function.

Theorem 5. *Suppose that X_i's in (23) are i.i.d. double exponentially distributed random variables (8). Also, suppose that $N(t)$ is defined as in (18) and λ satisfies the inequality, $0 < \lambda < 2$. Then*

$$\frac{S_N(t) - A(t)}{B(t)} \xrightarrow{d} F_\lambda \tag{29}$$

for large t where $A(t)$ and $B(t)$ are given in (16) and (15) respectively. F_λ is an infinitely divisible distribution with the characteristic function,

$$\phi_\lambda(u) = \exp\left\{ iau + \lambda \int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{dx}{x^{\lambda+1}} \right\}, \tag{30}$$

where a is given by

$$a = \begin{cases} \dfrac{\lambda e\pi}{2\cos\dfrac{\lambda\pi}{2}} & \textit{for } \lambda \neq 1 \\ 0 & \textit{for } \lambda = 1 \end{cases} \tag{31}$$

Proof. To prove this theorem, we need to show that the 3 conditions in Theorem (4) are satisfied.

1. For selected $B(t) = (\lambda t)^t$ (15), the function $L(x)$ (26) is given by

$$L(x) = \begin{cases} \lim_{t\to\infty} NP\{Y \leq x\} = 0 & \text{for } x < 0 \\ -\lim_{t\to\infty} NP\{Y > x\} = -x^{-\lambda} & \text{for } x > 0. \end{cases} \tag{32}$$

   where Y is given in (23). Because $Y \geq 0$, $L(x) = 0$ holds in the case $x < 0$. Assume that $x > 0$. By using (15), (18) and (23) we obtain

$$\begin{aligned} & NP\{Y(t) > x)\} = e^{\lambda t} P\left\{\frac{e^{tX}}{B(t)} > x\right\} = e^{\lambda t} P\left\{X > \frac{\ln x + \ln B(t)}{t}\right\} \\ & \cong \exp\left\{1 + \lambda t - \exp\left(\frac{\ln x + \ln B(t)}{t}\right)\right\} \cong \exp\left\{1 + \lambda t - \left(1 + \frac{\ln x}{t}\right)\lambda t\right\} \\ & = -x^{-\lambda} e \end{aligned}$$

   for large t which shows that (32) holds.

2. We claim that (27) holds and $\sigma^2 = 0$ for all $\lambda \in (0, 2)$. Since

$$0 \leq Var\left[Y 1_{\{Y\leq\tau\}}\right] \leq E\left[Y^2 1_{\{Y\leq\tau\}}\right],$$

   we just need to prove that

$$\sigma^2 = \lim_{\tau\to 0^+} \lim_{t\to\infty} NE\left[Y^2 1_{\{Y\leq\tau\}}\right] = 0 \tag{33}$$

We introduce a common variable which will be used throughout this theorem,

$$\eta(t,\tau) = \frac{\ln B(t) + \ln\tau}{t} \tag{34}$$

Using (15), (18) and (34) for any $\tau > 0$,

$$NE\left[Y^2 1_{\{Y\le\tau\}}\right] \cong N(t)E\left[\frac{e^{2tX}}{B^2(t)}1_{\{X\le\eta(t,\tau)\}}\right]$$
$$\cong \frac{N(t)e}{B^2(t)}\int_0^{+\infty}\exp\left\{2tx + x - e^x\right\}1_{\{x\le\eta(t,\tau)\}}\,dx$$

We use the substitution $x = y + \ln(2t+1)$, (18) and Appendix 6.3.2, 6.3.3 which gives

$$C(t)\int_{-\ln(2t+1)}^{+\infty}\exp\left\{(2t+1)(y-e^y)\right\}1_{\{y\le\eta(t,\tau)-\ln(2t+1)\}}\,dy$$
$$= C(t)\int_{-\ln(2t+1)}^{K}\exp\left\{(2t+1)(y-e^y)\right\}\,dy$$
$$= C(t)\exp\left\{(2t+1)(K-e^K)\right\}\frac{1}{(2t+1)\mid g'(K)\mid} \tag{35}$$

where

$$C(t) = \frac{N(t)e}{B^2(t)}\exp\{(2t+1)\ln(2t+1)\}$$
$$K = \ln(\lambda/2) + \frac{\ln\tau}{t} - \ln\left(1+\frac{1}{2t}\right) \tag{36}$$

Substitution of K (36) into (35) gives us [Refer to Appendix 6.3.4, for details]

$$NE\left[Y^2 1_{\{Y\le\tau\}}\right] \cong \frac{\lambda\tau^{2-\lambda}e}{|g'(K)|}$$

where $|g'(K)| \cong 1-\lambda/2 < 0$ when t is large. Then

$$\sigma^2 = \lim_{\tau\to0^+}\lim_{t\to\infty}NE[Y^2 1_{\{Y\le\tau\}}] = \lim_{\tau\to0^+}\frac{\lambda\tau^{2-\lambda}e}{|g'(K)|} = \lim_{\tau\to0^+}\frac{\lambda\tau^{2-\lambda}e}{1-\lambda/2} = 0$$

3. When $\lambda \in (0,2)$, for each $\tau > 0$ the limit,

$$D_\lambda(\tau) = \lim_{t\to\infty} \left\{ NE[Y1_{\{Y\le\tau\}}] - \frac{A(t)}{B(t)} \right\},$$

exists where A(t) and B(t) are given in (16), (15) respectively. Then $D_\lambda(\tau)$ can be expressed

$$D_\lambda(\tau) = \begin{cases} \dfrac{\lambda e}{1-\lambda}\tau^{1-\lambda} & \text{for } \lambda \neq 1 \\ e\ln\tau & \text{for } \lambda = 1. \end{cases} \tag{37}$$

3.a) Assume that $\lambda \in (0,1)$. Then $A(t) = 0$ (16c). Using the substitution $x = y + \ln(t+1)$, (34) and Appendix 6.3.2

$$\begin{aligned} NE\left[Y1_{\{Y\le\tau\}}\right] &\cong \frac{N(t)e}{B(t)} \int_0^{+\infty} \exp\left\{tx + x - e^X\right\} 1_{\{x\le\eta(t,\tau)\}}\,dx \\ &= D(t)\int_{-\ln(t+1)}^{+\infty} \exp\{(t+1)(y-e^y)\}\, 1_{\{y\le\eta(t,\tau)-\ln(t+1)\}}\,dy \\ &= D(t)\int_{-\ln(t+1)}^{K} \exp\{(t+1)(y-e^y)\}\,dy \\ &= D(t)\exp\left\{(t+1)(K-e^K)\right\}\frac{1}{(t+1)\mid g'(K)\mid} \end{aligned} \tag{38}$$

where

$$D(t) = \frac{N(t)e}{B(t)}\exp\{(t+1)\ln(t+1)\} \tag{39}$$

$$K = \ln(\lambda) + \frac{\ln\tau}{t} - \ln\left(1+\frac{1}{t}\right) \tag{40}$$

Substitution of K (40) into (38) gives us [Refer to Appendix 6.3.4, 6.3.5 for details]

$$NE\left[Y1_{\{Y\le\tau\}}\right] \cong \frac{\lambda\tau^{1-\lambda}e}{|g'(K)|} \cong \frac{\lambda\tau^{1-\lambda}e}{1-\lambda}$$

when t is large.

3.b) Assume that $\lambda \in (1,2)$. Also, $A(t) = E[S_N(t)]$ (16a). Using the substitution

$x = y + \ln(t+1)$, (39) and Appendix 6.3.3

$$\begin{aligned}
& NE[Y1_{\{Y\leq\tau\}}] - \frac{A(t)}{B(t)} \\
& \cong \frac{N(t)e}{B(t)} \int_0^{+\infty} \exp\left\{tx + x - e^X\right\} 1_{\{x>\eta(t,\tau)\}}\, dx \\
& = D(t) \int_{-\ln(t+1)}^{+\infty} \exp\left\{(t+1)(y - e^y)\right\} 1_{\{y>\eta(t,\tau)-\ln(t+1)\}}\, dy \\
& = D(t) \int_K^{\infty} \exp\left\{(t+1)(y - e^y)\right\}\, dy \\
& = D(t) \exp\left\{(t+1)(K - e^K)\right\} \frac{1}{(t+1) \mid g'(K) \mid} \qquad (41)
\end{aligned}$$

where

$$K = \ln(\lambda) + \frac{\ln\tau}{t} - \ln\left(1 + \frac{1}{t}\right) > 0 \qquad (42)$$

for large t. Substitution of K (42) into (41) gives us [Refer to Appendix 6.3.4 and 6.3.6 for details]

$$NE\left[Y1_{\{Y\leq\tau\}}\right] - \frac{A(t)}{B(t)} \cong \frac{\lambda\tau^{1-\lambda}e}{|g'(K)|} \cong \frac{\lambda\tau^{1-\lambda}e}{1-\lambda}$$

when t is large.

3.c) Assume that $\lambda = 1$ and $\tau > 1$ for definiteness. Also, $A(t) = E[S_N(t)1_{\{Y\leq 1\}}]$ (16b). Using N(t) (18) and B(t) (15),

$$\begin{aligned}
& NE[Y1_{\{Y\leq\tau\}}] - \frac{A(t)}{B(t)} \\
& \cong \frac{N(t)e}{B(t)} \int_0^{+\infty} \exp\left\{tx + x - e^X\right\} [1_{\{x\leq\eta(t,\tau)\}} - 1_{\{x\leq\ln B(t)/t\}}]\, dx \\
& = \frac{N(t)e}{B(t)} \int_{\ln B(t)/t}^{\eta(t,\tau)} \exp\left\{tx + x - e^X\right\}\, dx \\
& \cong \frac{N(t)e}{B(t)} \frac{\ln\tau}{t} \exp\left\{tK + K - e^K\right\} \qquad (43)
\end{aligned}$$

where

$$K = \ln t + \frac{\ln \tau}{t} \tag{44}$$

for large t. Substitution of K (44) into (43) gives us

$$NE\left[Y1_{\{Y\leq\tau\}}\right] - \frac{A(t)}{B(t)} \cong e \ln \tau$$

when t is large.

3.d) The parameter a defined in (31) satisfies the identity (28) with $L(x)$ specified by (32),

$$\begin{aligned} D_\lambda(\tau) \quad &= \lim_{t\to\infty} \left\{ NE[Y1_{\{Y\leq\tau\}}] - \frac{A(t)}{B(t)} \right\} \\ &\cong a + \int_0^\tau \frac{ex^{2-\lambda}}{1+x^2}\, dx - \int_\tau^\infty \frac{ex^{-\lambda}}{1+x^2}\, dx \end{aligned} \tag{45}$$

where $D_\lambda(\tau)$ is given by (37).

Assume that $\lambda \in (0,1)$. It is known that

$$\int_0^\tau \frac{x^{2-\lambda}}{1+x^2}\, d(x) = \frac{\tau^{1-\lambda}}{1-\lambda} - \int_0^\tau \frac{x^{-\lambda}}{1+x^2}\, dx \tag{46}$$

Using (32) and (31), equation (45) turns out to be

$$\frac{\pi}{2cos\left(\frac{\lambda\pi}{2}\right)} = \int_0^\infty \frac{x^{-\lambda}}{1+x^2}\, dx$$

which is true from Gradshteyn and Ryzhik [9].

When $\lambda \in (1,2)$, it is known that

$$\int_\tau^\infty \frac{x^\lambda}{1+x^2}\, d(x) = \frac{\tau^{1-\lambda}}{\lambda-1} - \int_\tau^\infty \frac{x^{2-\lambda}}{1+x^2}\, dx \tag{47}$$

Using (32) and (31), equation (45) turns out to be

$$\frac{\pi}{2cos\left(\frac{\lambda\pi}{2}\right)} + \int_0^\infty \frac{x^{2-\lambda}}{1+x^2}\,dx = 0,$$

which is true again from Gradshteyn and Ryzhik [9].

For $\lambda = 1$, equation (45) has the form

$$\ln\tau = \int_0^\tau \frac{x}{1+x^2}\,dx + \int_\tau^\infty \frac{1}{(1+x^2)x}\,dx$$

The integral on the right can be computed using calculus as

$$\frac{1}{2}\ln\left(1+x^2\right)\Big|_0^\tau + \frac{1}{2}\ln\left(\frac{x^2}{1+x^2}\right)\Big|_\tau^\infty = \ln\tau$$

This completes the proof. □

Theorem 6. *The charachteristic function ϕ_λ determined by Theorem 5 corresponds to a stable probability law with exponent $\lambda \in (0,2)$ and skewness parameter $\beta = 1$ and can be represented in canonical form by*

$$\phi_\lambda(u) = \begin{cases} \exp\left\{-\Gamma(1-\lambda) \mid u \mid^\lambda \exp\left(-\frac{i\pi\lambda}{2}\, sgn(u)\right)\right\} & \textit{for } \lambda \in (0,1) \\ \exp\left\{\frac{\Gamma(2-\lambda)}{\lambda-1} \mid u \mid^\lambda \exp\left(-\frac{i\pi\lambda}{2}\, sgn(u)\right)\right\} & \textit{for } \lambda \in (1,2) \\ \exp\left\{iu(1-\gamma) - \frac{\pi}{2} \mid u \mid \left(1 + i\, sgn(u)\, \frac{2}{\pi} \ln \mid u \mid\right)\right\} & \textit{for } \lambda = 1 \end{cases} \tag{48}$$

where $\Gamma(s) = \int_\tau^\infty x^{s-1}e^{-x}\,dx$ is the gamma function, $sgn(u) : u/ \mid u \mid$ for $u \neq 0$ and $sgn(u) := 0$, and $\gamma = 0.5772...$ is the Euler constant. The proof of this theorem can be found in the paper of Gerard Ben Arous, Leonid V. Bogachev, Stanislav A. Molchanov [5].

CHAPTER 4: APPLICATION: STATISTICAL ESTIMATION OF THE LUNDBERG ROOT USING EMPIRICAL LAPLACE TRANSFORM

Many applications in insurance mathematics are related to compound distributions and their corresponding ruin probabilities. The ruin probability of an insurance portfolio is one of the major concerns of an insurance company and it depends on the tail behavior of the insurance portfolio. Hence, it is important to understand how the tails behave. In practice, it is difficult to quantify the tail probability exactly so we estimate the upper and lower bounds. The main technical tool for this estimation procedure is CLT.

Consider a portfolio consisting of infinitely many policies with individual risks $\{X_1, X_2, ...\}$ over a given time period. Assume that the non-negative random variables $\{X_1, X_2, ...\}$ are i.i.d. Weibull type random variables (6) with distribution function F_X. First, we investigate the asymptotic behaviour of the tail probability $\overline{F}_U(x) = P(U > x)$ of the compound $U = \sum_{i=1}^{N} X_i$ when $N \to \infty$. Here, N has geometric distribution with parameter $p \in (0,1)$. We are able to determine this by finding the upper and lower Lundberg bounds but this procedure requires the existence of a solution to the following Lundberg equation,

$$L(\gamma) = \int_0^\infty e^{\gamma x}\, dF(x) = \frac{1}{p}. \tag{49}$$

The solution of this equation, γ, is so called the adjustment coefficient in risk theory.

The question is how to estimate the unknown solution of this equation. In practical applications, we do not know the form of $L(t)$ precisely. Only a sample version of $L(t)$, $L_N(t)$, is known for an insurance company. This is defined as the empirical Laplace transform,

$$L_N(\gamma) = \frac{1}{N}\sum_{j=1}^{N}(e^{\gamma X_j}). \tag{50}$$

Hence, an estimation procedure was developed in the paper of Sandor Csorgo and Jef L. Teugels [6] where the empirical Laplace transform of $L(t)$ has been used. We extend this procedure by customizing the growth rate of the number of individual risks i.e. the number of claims. The adjustment coefficient which is the solution of the estimation procedure gives us the Lundberg bounds which also gives the tail probability. When p in the Lundberg equation is replaced by a different constant, runin probability can be obtained. Details can be found in the book of T. Rolski, H. Schmidli, V. Schmidt and J. Teugels [11] (P. 125-131 and P. 170-171).

4.1 Geometric Compounds

Consider the case where N has geometric distribution with parameter $p \in (0, 1)$. Then, the compound geometric distribution F_U is given by

$$F_U(x) = \sum_{i=0}^{\infty}(1-p)p^i\, F_X^{*i}(x) \tag{51}$$

Writing the first summand in (51) separately, we get

$$F_U = (1-p)\delta_0 + pF_X * F_U \tag{52}$$

where

$$\delta_0 = \delta_0(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ \\ 0, & \text{if } x < 0 \end{cases}$$

This is so called the defective renewal equation or transient renewal equation. Replacing the distribution F_U on the right hand side of (52) by the term $(1-p)\,\delta_0 + p\,F_X * F_U$ and iterating this procedure, we get

$$F_U(x) = \lim_{n\to\infty} F_n(x) \tag{53}$$

for $x \geq 0$ where F_n is defined as

$$F_n = (1-p)\delta_0 + p\,F_X * F_{n-1} \tag{54}$$

for all $n \geq 1$ and F_0 is an arbitrary initial distribution on $\mathbb{R}_+$. Additionally, assume that

$$L(\gamma) = \int_0^\infty e^{\gamma x}\,dF_X(x) = \frac{1}{p} \tag{55}$$

has a solution where p is the parameter of the geometric distribution and $F_X(x)$ is the distribution function of Weibull distributed individual risks, $\{X_1, X_2, ...\}$. γ is the adjustment coefficient here. Let $x_0 = \sup\{x : F_U(x) < 1\}$. Then the following theorem gives us the Lundberg bounds based on the existence of the adjustment coefficient.

Theorem 7. *If X is a geometric compound with characteristics (p, F_X) such that*

(55) admits a positive solution ,γ, then

$$a_- e^{-\gamma x} \leq \overline{F}_U(x) \leq a_+ e^{-\gamma x}$$

where $x \geq 0$ *and*

$$a_- = \inf_{x \in [0, x_0)} \frac{e^{\gamma x} \overline{F}_X(x)}{\int_x^\infty e^{\gamma y}\, dF_X(y)}$$
$$a_+ = \sup_{x \in [0, x_0)} \frac{e^{\gamma x} \overline{F}_X(x)}{\int_x^\infty e^{\gamma y}\, dF_X(y)} \tag{56}$$

Proof. To find the upper bound in (56) we aim to find an initial distribution F_0 such that the corresponding distribution F_1 defined in (54) for $n = 1$ satisfies

$$F_1(x) \geq F_0(x) \tag{57}$$

for $x \geq 0$. Then $F_X(x) * F_1(x) \geq F_X(x) * F_0(x)$ for $x \geq 0$ and by induction, $F_{n+1}(x) \geq F_n(x)$ for all $x \geq 0$ and $n \in \mathbb{N}$. This means that

$$\overline{F}_U(x) \leq \overline{F}_0(x) \tag{58}$$

for $x \geq 0$. Let $F_0(x) = 1 - a\, e^{-\gamma x} = (1-a)\delta_0(x) + a\, G(x)$ where $a \in (0, 1]$ is some constant and $G(x) = 1 - e^{-\gamma x}$. Inserting this into (54) we obtain

$$\begin{aligned} F_1(x) \quad &= 1 - p + p \left((1-a) F_X(x) + a \int_0^x G(x-y)\, dF_X(y) \right) \\ &= 1 - p + p \left(F_X(x) - a \int_0^x e^{-\gamma(x-y)}\, dF_X(y) \right) \end{aligned}$$

for all $x \geq 0$. Since we want to arrive at (57) we look for a such that

$$1 - p + p\left(F_X(x) - a\int_0^x e^{-\gamma(x-y)}\,dF_X(y)\right) \geq 1 - ae^{\gamma x} \tag{59}$$

This inequality can be simplified to

$$a\left(1 - p\int_0^x e^{\gamma y}\,dF_X(y)\right) \geq pe^{\gamma x}\,\overline{F}_X(x)$$

which is trivial for $x \geq x_0$ using

$$1 = p\int_0^\infty e^{\gamma y}\,dF_X(y) = p\int_0^x e^{\gamma y}\,dF_X(y) + p\int_x^\infty e^{\gamma y}\,dF_X(y)$$

Then (59) is equivalent to

$$ap\int_x^\infty e^{\gamma y}\,dF_X(y) \geq pe^{\gamma x}\overline{F}_X(x) \tag{60}$$

Setting $a_+ = \sup\limits_{x\in[0,x_0)} \dfrac{e^{\gamma x}\overline{F}_X(x)}{\int_x^\infty e^{\gamma y}\,dF_X(y)}$, we get (57) and consequently (58). Upper bound follows and lower bound can be driven similarly. □

This section is mainly taken from the book of T. Rolski, H. Schmidli, V. Schmidt and J. Teugels [11].

4.2 Estimation of the Adjustment Coefficient

When we found the bounds to the tail probability in the previous section, we assumed the existence of a solution to the Lundberg equation (55). As stated earlier, we estimate this adjustment coefficient, γ, using empirical Laplace transform.

We made use of Laplace transform for Lundberg equation (55). We always assume that Laplace transform $L(\gamma)$ exists in an open neighborhood of the origin, $I = (-\infty, \sigma)$ where σ is the abscissa of convergence of $L(\gamma)$. $L(\gamma)$ is arbitrarily many times differentiable in I. Also, $L(\gamma)$ is an increasing convex function on I and it has non-negative random variables.

Let's assume that we have a sequence of insurance portfolios $\{0, 1, 2, ..., n, ...\}$. n'th portfolio has N_n individual risks and N_n is geometrically distributed with parameter $p_n \in (0, 1)$. Individual risks $\{X_1, X_2, ..., X_{N_n}\}$ follow Weibull law with parameter ϱ and distribution function F_X (6). They are i.i.d. random variables. Assume that p_n is a decreasing function of n and $p_n \to 0$ as $n \to \infty$.

Also, assume that there exists a solution to the following Lundberg equation for each n.

$$L(\gamma) = \int_0^{\infty} e^{\gamma x}\, dF(x) = \frac{1}{p_n}. \tag{61}$$

We define each solution as t_n and $t_n \to \infty$ when $n \to \infty$. t_n is a large number even for small n because p_n is small for every n. Note that t_n is the real Lunberg root that we estimate.

We only have a sample as available information which means we do not have the precise form of $L(\gamma)$. Hence, we replace the Laplace transform with the empirical Laplace transform and assume that there exist a solution to the following empirical Lundberg equation for each n,

$$L_{N_n}(\gamma) = \frac{1}{N_n} \sum_{j=1}^{N_n} (e^{\gamma X_j}) = \frac{1}{p_n} \tag{62}$$

N_n in (62) is defined as in (17):

$$N_n(\gamma) = e^{\lambda H(\gamma)}. \tag{63}$$

where $H(\gamma) = \frac{\gamma^{\varrho'}}{\varrho'}$ for Weibull distribution when γ is large and λ is a constant. We define these solutions as a sequence of adjustment coefficients, τ_n.

Based on the above definitions, we have the following array scheme which contains i.i.d. individual risks.

$$X_{11}, X_{12}, ..., X_{1N_1}$$

$$X_{21}, X_{22}, ..., X_{2N_2}$$

$$............................$$

$$X_{n1}, X_{n2}, ..., X_{nN_n}$$

$$............................$$

Each line refers to a portfolio and has its own associated adjustment coefficient depending on n, τ_n. On the other hand, there is a sequence of real solutions, t_n to the Lundberg Equations,

$$L(t_n) = E(e^{t_n X_j}) = \int_0^\infty e^{t_n x}\, dF_X(x) = \frac{1}{p_n} \tag{64}$$

Combining the above variables and equations, we solve

$$L_{N_n}(\tau_n) = \frac{1}{N_n(\tau_n)} \sum_{j=1}^{N_n(\tau_n)} (e^{\tau_n X_j}) = \int_0^\infty e^{\tau_n x}\, dF_{N_n}(x) = \frac{1}{p_n} \tag{65}$$

where

$$F_{N_n}(x) \quad = \quad \frac{1}{N_n} \#\{1 \leq j \leq N_n : X_j \leq x\}$$

is the empirical distribution function of the sample. $L_{N_n}(\tau_n)$ is a random analytic function for all values of τ_n. We obtain sample based estimator of the adjustment coefficient using empirical Laplace transform.

We introduce some useful functions to express limits:

$$W_{N_n}(\tau_n) = \sum_{j=1}^{N_n(\tau_n)} \{e^{\tau_n X_{nj}} - E(e^{\tau_n X_j})\} \tag{66}$$

Also, one term Taylor expansion gives us the identity

$$\exp(\tau_n X_j) = \exp(t_n X_j) + (\tau_n - t_n) X_j \exp(\tau_n(j) X_j) \tag{67}$$

where $\tau_n(j)$ satisfies the inequalities $min(\tau_n, t_n) \leq \tau_n(j) \leq \max(\tau_n, t_n)$. Additionally, We use the following abbreviations

$$S_{N_n}(\tau_n) = \sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n X_j} \tag{68}$$

$$S_{N_n}(\tau_n, t_n) = \sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n(j) X_j}$$

where $\tau_n(j)$ is determined by by the above equations.

We estimate t_n by solving the equation $L_{N_n}(\tau_n) = 1/p_n$ and $L(t_n) = 1/p_n$ when p_n is small.

$$\begin{aligned} 0 &= N_n(\tau_n)\,(L_{N_n}(\tau_n) - L(t_n)) \\ &= \sum_{j=1}^{N_n(\tau_n)} (e^{\tau_n X_j} - E(e^{t_n X_j})) \end{aligned}$$

Applying taylor series approximation from (67) gives us

$$\sum_{j=1}^{N_n(\tau_n)} (e^{\tau_n X_j} - E(e^{t_n X_j})) = \sum_{j=1}^{N_n(\tau_n)} (e^{t_n X_j} - E(e^{t_n X_j})) + (\tau_n - t_n) \sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n(j) X_j} = 0$$

Rearrangement of terms leads to

$$t_n - \tau_n = \frac{\sum_{j=1}^{N_n(\tau_n)} (e^{t_n X_j} - E(e^{t_n X_j}))}{\sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n(j) X_j}} \cong \frac{W_{N_n}(t_n)}{S_{N_n}(\tau_n, t_n)} \tag{69}$$

Assume that central limit theorem holds for the random exponential sums in (66) and (68). It means that the growth rate satisfies the inequality $\lambda > \lambda_2 = 2^{\varrho'} \frac{\varrho'}{\varrho}$ using Theorem 3 about CLT. Then, law of large number already holds, Theorem 2. Using these theorems, we can state that the following limit exists:

$$S_{N_n}(\tau_n, t_n) = \sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n(j) X_j} \xrightarrow{p} E[S_{N_n}(\tau_n, t_n)]$$

where

$$E[S_{N_n}(\tau_n, t_n)] \cong N_n(\tau_n) E\left[X_j e^{\tau_n(j) X_j}\right] \cong N_n(\tau_n) \tau_n^{\varrho' - 1} e^{H(\tau_n)}$$

In theorems 2 and 3, we did not have X_j as a multiplier but this multiplier is asymptotically small and do not change the boundaries of the limits. The last approximation, $N_n(\tau_n) E\left[X_j e^{\tau_n(j) X_j}\right] \cong N_n(\tau_n) \tau_n^{\varrho' - 1} e^{H(\tau_n)}$, comes from a similar integration for Weibull which is shown in Appendix 6. The central limit theorem 3 provides us

$$W_{N_n}(\tau_n) = \sum_{j=1}^{N_n(\tau_n)} \{e^{t_n X_j} - E(e^{t_n X_j})\} \xrightarrow{d} N(0,1) \left(Var \sum_{j=1}^{N_n(t_n)} e^{t_n X_j} \right)^{1/2} \tag{70}$$

where

$$Var \sum_{j=1}^{N_n(t_n)} e^{t_n X_j} = \sum_{j=1}^{N_n(t_n)} Var(e^{t_n X_i}) \cong N_n(t_n) e^{H(2t_n)}$$

As a result of these limits, we get an asymptotic confidence interval for t_n

$$\lim_{n\to\infty} P\left\{\tau_n - z_{\alpha/2}\, \tau_n^{\varrho'-1}\, J(\tau_n) \le t_n \le \tau_n + z_{\alpha/2}\, \tau_n^{\varrho'-1}\, J(\tau_n)\right\}$$
$$= 1 - \alpha \tag{71}$$

where $\phi(z_{\alpha/2}) = 1 - \alpha/2$ for $0 < \alpha < 1$ and

$$J(\tau_n) = \exp\{H(\tau_n) - H(2\tau_n)\}$$

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CHAPTER 5: APPLICATION: REM MODEL

Free energy was driven using concepts of convergence in probability in papers by Eisele's [7] and A. Bovier, I. Kurkova and M. Löwe [1]. This computation required long derivations though. Hence, we develop a different approach using order statistics, Euler-Maclaurin series and Laplace method which simplifies the process. In the first part, we introduce variables for our computations. Then, we drive free energy for Weibull distribution using limiting distributions similar to the paper G. Ben Arous, L. V. Bogachev and S. A. Molchanov [5]. Then, we develop the new approach using order statistics. Free energy is calculated for Weibull, relatively heavy tailed (10) and relatively light tailed (8) distributions using this method. Once the statistical sum is represented in terms of exponential random variables, driving free energy is quite straight forward.

5.1 Variable Definitions

Assume that $\{ X_i,\ i = 1, 2, ..., N \}$ are i.i.d. random variables. We already defined free energy in Chapter 1 as

$$\chi(\beta) \qquad = \lim_{n\to\infty} \frac{logZ_n(\beta)}{n} \tag{72}$$

where

$$Z_n(\beta) = \sum_{i=1}^{N} e^{\beta A(n) X_i} \tag{73}$$

is the statistical sum or partition function. β is strictly positive. For simplicity, we assume that

$$N = [e^n] \tag{74}$$

$$\ln N = n + \underline{\underline{O}}\left(e^{-n}\right)$$

$A(n)$ in the statistical sum is selected in such a way that free energy converges. For different distributions, we will select the proper growth factor for $A(n)$.

5.2 Free Energy Using Limit Theorems for Weibull Distribution

Assume that $\{\, X_i,\ i = 1, 2, ..., N \,\}$ are i.i.d. random variables with Weibull distribution and we select

$$A(n) = n^{1/\varrho'} \tag{75}$$

as the proper growth factor where ϱ' is introduced in Chapter 2 for Weibull distribution.

The cumulant generating function for Weibull is

$$H(t) = logE[e^{tX}] \cong \frac{t^{\varrho'}}{\varrho'} \tag{76}$$

for large t. $H(t)$ is well defined, non-decreasing and $H(t)/t \to \infty$ as $t \to \infty$. Also $A(n)$ is an increasing funtion of n. As a result of these definitions, we express the expected value of the statistical sum for large n as

$$E[Z_n(\beta)] = \sum_{i=1}^{[e^n]} E\left[e^{\beta A(n) X_i}\right] \cong [e^n] exp\left\{\frac{\beta^{\varrho'} n}{\varrho'}\right\} \tag{77}$$

5.2.1 Main Theorems

Theorem 8. *(Law of Large Numbers for the statistical sum)*

Let $\ln E[e^{\beta A(n) X_i}] = H(\beta A(n))$. *For sufficiently small* ϵ, *if*

$$n\epsilon - H((1+\epsilon)\beta A(n)) + (1+\epsilon)H(\beta A(n))] > 0$$

then we have

$$\frac{Z_n(\beta)}{E[Z_n(\beta)]} \xrightarrow{p} 1. \tag{78}$$

for large n.

Proof. Set t= $\beta A(n)$. Also define

$$Z_n^*(\beta) = \frac{Z_n(\beta)}{E[Z_n(\beta)]} = \frac{1}{N}\sum_{i=1}^{N} e^{tx_i - H(t)} \tag{79}$$

We have to prove that $Z_n^*(\beta) \xrightarrow{p} 0$ as $n \to \infty$. It is sufficient to show that

$$\lim_{n\to\infty} E|Z_n^*(\beta) - 1|^r = 0$$

for some $r > 1$. Using Bahr Essen inequality and

$$(x+1)^r \leq 2^{r-1}(x^r + 1)$$

where $(x > 0, r \geq 1)$, we obtained in the proof of Law of Large numbers, Theorem 2

$$E\,|Z_n^*(\beta) - 1|^r \leq 2^r N^{1-r} e^{H(rt) - rH(t)} + 2^r N^{1-r} \tag{80}$$

For the existence of the limit we must have $\lim_{n\to\infty} E|Z_n^*(\beta) - 1|^r = 0$. Substituting $t = \beta A(n)$, $r = 1 + \epsilon$, $N = [e^n]$ (74) to the right hand side of the inequality (80), we obtain the condition

$$\liminf_{\substack{t\to\infty \\ \epsilon\to 0^+}} \left[n\epsilon - H((1+\epsilon)\beta A(n)) + (1+\epsilon)H(\beta A(n))\right] \quad > \quad 0$$

for the existence of LLN. □

Theorem 9. *X_i are independent and identically distributed Weibull type random variables and assume that we have the following conditions:*

$$M_{1,N}(n) = \max\left(e^{\beta A(n) X_i}\right) \tag{81}$$

for $i = 1, ..., N = [e]^n$ and $\beta A(n) = \beta n^{1/\varrho'}$ Then,

$$P\left(\frac{M_{1,N}(n)}{B(n)} < x\right) \quad \to \quad K(x) = e^{-x^{-\alpha}}$$
$$\log M_{1,N}(n) \quad \cong \quad \ln B(n)$$

for large n where $\ln B(n) = \beta\varrho^{1/\varrho}n$.

Proof. Let's call $\beta A(n) = A$ shortly which implies that $n = \dfrac{A^{\varrho'}}{\beta^{\varrho'}}$. Then,

$$\begin{aligned} P\left(\frac{M_{1,N}(n)}{B(n)} < x\right) &= \left[P\left(\frac{e^{AX_i}}{B(n)} < x\right)\right]^N \\ &= \left[P\left(X_i < \frac{\ln(B(n)x)}{A}\right)\right]^N \\ &= \left[1 - exp\left\{-\frac{A^{-\varrho}\ln^{\varrho}(B(n)x)}{\varrho}\right\}\right]^N \\ &\cong exp\left\{-exp\left\{\ln N - \frac{A^{-\varrho}\ln^{\varrho}(B(n)x)}{\varrho}\right\}\right\} \end{aligned}$$

Then, the asymptotic of the exponent can be computed using the binomial formula

$$\begin{aligned} \ln N - \frac{A^{-\varrho}\ln^{\varrho}(B(n)x)}{\varrho} &= n - n^{1-\varrho}\beta^{-\varrho}\ln^{\varrho} B(n)\left(1 + \varrho\frac{\ln x}{\ln B(n)}\right)/\varrho \\ &= n - n^{1-\varrho}\beta^{-\varrho}\ln^{\varrho} B(n)/\varrho + n^{1-\varrho}\beta^{-\varrho}\ln^{\varrho-1} B(n)\ln x \end{aligned}$$

Plugging into this equation $\ln B(n) = \beta\varrho^{1/\varrho}n$, we obtain

$$P\left(\frac{M_{1,N}(n)}{B(n)} < x\right) \to -\frac{\varrho^{1/\varrho^{'}}}{\beta}\ln x$$

Then, we can state that $logM_{1,N}(n) \cong \ln B(n)$ for large n. $\square$

5.2.2 Computation of Random Energy

Lemma 10. *Assume that we have a sequence of i.i.d. Weibull type random variables* $X_1, ..., X_N$ *(6). When we select* $\beta A(n) = \beta n^{1/\varrho^{'}}$*, the statistical sum satisfies LLN for* $0 < \beta < \varrho^{1/\varrho^{'}} = \beta_{critical}$*. Also, free energy can be quantified by the following formula in this interval*

$$\chi(\beta) \quad := \quad 1 + \frac{\beta^{\varrho^{'}}}{\varrho^{'}}$$

Proof. In the appendix, we have proven that, The moment generating function satisfies $H(\beta A(n)) = H(\beta n^{1/\varrho^{'}}) = nf(\beta) + o(n)$ for large n. Using the equivalent of $H(t)$, we obtain that

$$H(\beta n^{1/\varrho^{'}}) \cong \frac{(\beta n^{1/\varrho^{'}})^{\varrho^{'}}}{\varrho^{'}} = \frac{\beta^{\varrho^{'}} n}{\varrho^{'}} \tag{82}$$

Using Theorem 8, we must have the following condition for LLN for small ϵ:

$$n\epsilon - H((1+\epsilon)\beta A(n)) + (1+\epsilon)H(\beta A(n)) > 0$$

Using binomial formula

$$\begin{aligned} n\epsilon - H((1+\epsilon)\beta A(n)) + (1+\epsilon)H(\beta A(n)) &= n\epsilon - (1+\epsilon)^{\varrho^{'}}\frac{\beta^{\varrho^{'}} n}{\varrho^{'}} + (1+\epsilon)\frac{\beta^{\varrho^{'}} n}{\varrho^{'}} \\ \cong n\epsilon - \left(1+\epsilon\varrho^{'}\right)\frac{\beta^{\varrho^{'}} n}{\varrho^{'}} + (1+\epsilon)\frac{\beta^{\varrho^{'}} n}{\varrho^{'}} &= n\epsilon - \epsilon\left(\varrho^{'}-1\right)\frac{\beta^{\varrho^{'}} n}{\varrho^{'}} \end{aligned} \tag{83}$$

for small ϵ. (83) should be positive for LLN which implies that β must satisfy inequality $0 < \beta < \varrho^{1/\varrho'}$. Also, we formulated statistical sum in (77). When LLN holds,

$$\chi(\beta) = \lim_{n\to\infty} \frac{logZ_n(\beta)}{n} = 1 + f(\beta) = 1 + \frac{\beta^{\varrho'}}{\varrho'}$$

□

Theorem 11. *([3] on Page 48) When LLN is not satisfied which means $\beta \geq \beta_{critical}$,*

$$\frac{\ln M_{1,N}(n)}{\ln Z_N(\beta)} \quad \xrightarrow{p} \quad 1$$

where $n \to \infty$, $M_{1,N}(n) = \max(e^{\beta A(n) X_i}, i = 1, ..., N = [e]^n)$, $\beta A(n)$ is an increasing function of n and X_i are i.i.d. Weibull type random variables.

Proof. The proof of this theorem can be found in paper G. Ben Arous, L. V. Bogachev and S. A. Molchanov [5]. □

Free Energy for Weibull Type Distribution

Using Theorem 9 and Theorem 11, we can state that

$$\chi(\beta) \quad := \quad \lim_{n\to\infty} \frac{\ln Z_n(\beta)}{n} = \lim_{n\to\infty} \frac{\ln M_{1,N}(n)}{n}$$

$$= \quad \lim_{n\to\infty} \frac{\ln B(n)}{n} = \beta \varrho^{1/\varrho} \qquad \text{if } \beta \geq \beta_{critical} = \varrho^{1/\varrho'}$$

Combining this result and Lemma 10, the free energy can be calculated as follows:

$$\chi(\beta) = \begin{cases} 1 + \dfrac{\beta^{\varrho'}}{\varrho'}, & \text{if } \beta < \varrho^{1/\varrho'} = \beta_{critical} \\ \beta \varrho^{1/\varrho}, & \text{if } \beta \geq \beta_{critical} \end{cases}$$

This result is obtained using convergence in probability concepts. In the next section we introduce the method of order statistics.

5.3 Free Energy Using Order Statistics

We compute free energy using order statistics. The central assumption in this section is the random variables in the statistical sum (73) can be expressed as an increasing function of standard exponentially distributed random variables.

5.3.1 Formulation of the Statistical Sum

We introduce exponential random variables that will be rearranged in the statistical sum. Let

$$\{Y_1, Y_2, ..., Y_i, ..., Y_N\} \tag{84}$$

$$P\{Y_i > x\} = \begin{cases} e^{-x}, & \text{if x} \geq 0 \\ 1, & \text{o.w.} \end{cases}$$

such that $X_i = f(Y_i)$ and f is a monotone increasing function of standard exponentially distributed random variables, Y_i. Also, we reorder the sequence in (84) to obtain

$$Y_{(1)} > Y_{(2)} > .... > Y_{(i)} > ... > Y_{(N)} \tag{85}$$

the variational sequence of the sample (84). We make use of a proposition from Feller Volume 2 [see W. Feller (1971), Section 1.6, p. 19] [10] to express each ordered random variable in (85) and we obtain

$$\begin{aligned}
Y_{(1)} &= W_1 + \frac{W_2}{2} + \ldots\ldots\ldots\ldots + \frac{W_N}{N} \\
Y_{(2)} &= \quad\;\; \frac{W_2}{2} + \ldots\ldots\ldots\ldots + \frac{W_N}{N} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
Y_{(i)} &= \qquad\qquad \frac{W_i}{i} + \ldots + \frac{W_N}{N} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
Y_{(N)} &= \qquad\qquad\qquad\quad \frac{W_N}{N}
\end{aligned} \tag{86}$$

where $\{W_1, W_2, ..., W_i, ..., W_N\}$ is another set of i.i.d. standard exponential random variables.

This helps us to drive the partition function in terms of standard exponential random variables:

$$\begin{aligned}
Z_n(\beta) &= \sum_{i=1}^{N} e^{\beta A(n) X_i} = \sum_{i=1}^{N} e^{\beta A(n) f(Y_i)} \\
&= \sum_{i=1}^{N} exp\left\{\beta A(n) f\left(\frac{W_i}{i} + ... + \frac{W_N}{N}\right)\right\}
\end{aligned} \tag{87}$$

To be able to simplify the above expression, $\frac{W_i}{i} + ... + \frac{W_N}{N}$, we prove three propositions to obtain its asymptotic equivalent:

Proposition 12. *Suppose that* $\{Y_1, Y_2, ..., Y_l, ..., Y_N\}$ *are standard exponentially distributed random variables. Then,* $M_{Y_N} = \max(Y_1, Y_2, ..., Y_N) - \ln N$ *converges to the standard Gumbel distribution as* $N \to \infty$*.*

Proof. Let $F(x) = 1 - e^{-x}$ for $x \in [0, \infty)$. When $x \in \mathbb{R}$, the cumulative distribution function of M_{Y_N} can be expressed as

$$\begin{aligned} P\left\{M_{Y_N} \leq x\right\} &= P\left\{\max\{Y_1, Y_2 ..., Y_N\} \leq x + \ln N\right\} \\ &= F^N\left(x + \ln N\right) \\ &= \left\{1 - exp\{-x - \ln N\}\right\}^N \end{aligned}$$

which converges to $exp\{-e^{-x}\}$ as $N \to \infty$. □

Proposition 13. *Let $\{W_1, W_2, ..., W_l, ..., W_N\}$ be a set of i.i.d. standard exponential random variables. Then,*

$$\sum_{l=i}^{\infty} \frac{W_l - 1}{l}$$

follows Gumbel distribution such that

$$P\left\{\sum_{l=i}^{\infty} \frac{W_l - 1}{l} \leq x\right\} = exp\left\{-e^{-x+\gamma}\right\}$$

where γ is the Euler constant.

Proof. Let $M_{Y_N} = \max(Y_1, Y_2, ..., Y_N) - \ln N$ where $\{Y_1, Y_2, ..., Y_l, ..., Y_N\}$ are i.i.d. standard exponentially distributed random variables. Using the representation in (86), we can write for any $x \in \mathbb{R}$ that

$$\begin{aligned} &P\left\{M_{Y_N} \leq x\right\} \\ &= P\left\{\max(Y_1, Y_2, ..., Y_N) - 1 - \frac{1}{2} - ... - \frac{1}{N} \leq x + \ln N - 1 - \frac{1}{2} - ... - \frac{1}{N}\right\} \\ &= P\left\{\sum_{l=i}^{N} \frac{W_l - 1}{l} \leq x + \ln N - \left(1 + \frac{1}{2} + ... + \frac{1}{l} + ... + \frac{1}{N}\right)\right\} \end{aligned}$$

converges to a standard Gumbel distribution as $N \to \infty$ which was proven in Proposition 12. Note that

$$\gamma = \lim_{N\to\infty}\left[\ln N - \left(1 + \frac{1}{2} + ... + \frac{1}{l} + ... + \frac{1}{N}\right)\right]$$

where γ is the Euler constant. Then we can drive the the distribution function as follows

$$P\left\{\sum_{l=i}^{\infty}\frac{W_l - 1}{l} \leq x\right\} = exp\left\{-e^{-x+\gamma}\right\}$$

Note that $E\left[\sum_{l=i}^{\infty}\frac{W_l - 1}{l}\right] = 0$ and $Var\left(\sum_{l=i}^{\infty}\frac{W_l - 1}{l}\right) = \frac{\Pi^2}{6}$ □

Proposition 14. *Let $\{W_1, W_2, ..., W_i, ..., W_N\}$ be a set of i.i.d. standard exponential random variables. Then the summation, $\frac{W_i}{i} + ... + \frac{W_N}{N}$, can be approximated by*

$$\frac{W_i}{i} + ... + \frac{W_N}{N} \cong \ln N - \ln i + \sum_{l=i}^{N}\frac{W_l - 1}{l} = \ln N - \ln i + \underline{\underline{O}}(1) \tag{88}$$

when N is large.

Proof. From Euler-Maclaurin formula we get the following approximation for large N

$$\sum_{l=i}^{N}\frac{1}{l} = \int_i^N \frac{1}{x}dx + \underline{\underline{O}}(1) = \ln N - \ln i + \underline{\underline{O}}\left(\frac{1}{N}\right) + \underline{\underline{O}}\left(\frac{1}{i}\right) \tag{89}$$

Using this result we get

$$\frac{W_i}{i} + ... + \frac{W_N}{N} = \sum_{l=i}^{N}\frac{1}{l} + \sum_{l=i}^{N}\frac{W_l - 1}{l} = \ln N - \ln i + \sum_{l=i}^{N}\frac{W_l - 1}{l} + \bar{\bar{o}}(1) \tag{90}$$

when N and i are large. Also Kolmogorov's two series theorem implies that the series $\sum_{l=i}^{N}\frac{W_l - 1}{l}$ is convergent as $\sum_{l=i}^{N}Var\left(\frac{W_l - 1}{l}\right)$ and $\sum_{l=i}^{N}E\left[\frac{W_l - 1}{l}\right]$ are convergent. Also Proposition 13 states that $\sum_{l=i}^{N}\frac{W_l - 1}{l}$ converges to Gumbel distribution. This proves the approximation (88).

□

By substituting (88) into (87), the statistical sum is expressed as follows:

$$\begin{aligned} Z_n(\beta) &= \sum_{i=1}^{N} exp\left\{\beta A(n) f\left(\frac{W_i}{i} + ... + \frac{W_N}{N}\right)\right\} \\ &= \sum_{i=1}^{N} exp\left\{\beta A(n) f\left(\ln N - \ln i + \underline{\underline{O}}(1)\right)\right\} \end{aligned} \quad (91)$$

5.3.2 Computation of Limits

In this section, we compute free energy for i.i.d. random variables in the statistical sum, (73), which are functions of exponential random variables such that $X_i = f(Y_i)$ (84). We make use of the simplified statistical sum formula (91) and obtain the asymptotic behavior of the free energy. At the very end, we show two phase transitions for a mixed Weibull distribution.

5.3.3 Weibull Type Distribution

Let $X_i = f(Y_i) = Y_i^{1/\varrho} \varrho^{1/\varrho}$ are i.i.d. random variables because Y_i's are i.i.d standard exponential random variables as in (84). It is easily seen that X_i's have Weibull distribution:

$$P\{X_i > a\} = P\left\{Y_i > \frac{a^{\varrho}}{\varrho}\right\} = exp\left\{-\frac{a^{\varrho}}{\varrho}\right\} \quad (92)$$

where $a \geq 0$. Also, we select $A(n) = n^{1/\varrho'}$ in the statistical sum (73). The statistical sum can be expressed as

$$\begin{aligned} Z_n(\beta) &= \sum_{i=1}^{N} exp\{\beta A(n) X_i\} \\ &= \sum_{i=1}^{N} exp\left\{\beta n^{1/\varrho'} \varrho^{1/\varrho} \left(n - \ln i + \underline{\underline{O}}(1)\right)^{1/\varrho}\right\} \end{aligned} \tag{93}$$

by using (74), (88). Also, Euler-MacLaurin series gives us the approximate integral of this series:

$$\begin{aligned} &\sum_{i=1}^{N} exp\left\{\beta n^{1/\varrho'} \varrho^{1/\varrho} (n - \ln i + \underline{\underline{O}}(1))^{1/\varrho}\right\} \\ &= \int_1^N exp\left\{\beta \varrho^{1/\varrho} n^{1/\varrho'} (n - \ln x + \underline{\underline{O}}(1))^{1/\varrho}\right\} dx + \underline{\underline{O}}(1) \end{aligned} \tag{94}$$

Note that for some $c > 0$, we can find bounds on $Z_N(\beta)$ such that

$$\sum_{i=1}^{N_1} exp\left\{\beta n^{1/\varrho'} \varrho^{1/\varrho} \left(n - \ln i - c\right)^{1/\varrho}\right\} < Z_N(\beta) < \sum_{i=1}^{N} exp\left\{\beta n^{1/\varrho'} \varrho^{1/\varrho} \left(n - \ln i + c\right)^{1/\varrho}\right\}$$

where $N_1 = [e^{n-c}]$. The integral in (94) is computed by replacing $\underline{\underline{O}}(1)$ with c in Appendix 6 using Laplace method. This helps us to find the lower and upper bounds of $Z_N(\beta)$. As they only differ by a constant multiplier, these constant multipliers cancel out in the limit so as to give free energy as:

$$\chi(\beta) = \lim_{n\to\infty} \frac{log Z_n(\beta)}{n} = \begin{cases} 1 + \dfrac{\beta^{\varrho'}}{\varrho'}, & \text{if } 0 < \beta < \beta_c = \varrho^{1/\varrho'}, \\ \beta \varrho^{1/\varrho}, & \text{if } \beta \geq \beta_c. \end{cases}$$

Note that $\chi(\beta_c) = \varrho$ and $\chi'(\beta_c) = 1$

5.3.4 Relatively Heavy Tailed Distribution

Let $x_i = f(Y_i) = \dfrac{1+Y_i}{\ln(1+Y_i)}$ where Y_i s are i.i.d random variables with standard exponential distribution (84). Note that these random variables have heavier tails than Weibull distribution.

$$P\{X_i > a\} = P\left\{\frac{1+Y_i}{\ln(1+Y_i)} > a\right\} = exp\left\{-a\ln a - a\ln\ln a + \overline{\overline{O}}(1)\right\} \tag{95}$$

We select $A(n) = \ln n$ in the statistical sum (73). By using (74), (88), the asymptotic of the statistical sum can be expressed as

$$\begin{aligned} Z_n(\beta) &= \sum_{i=1}^{N} exp\{\beta A(n) X_i\} \cong \sum_{i=1}^{N} exp\left\{\beta\, \ln n\, \frac{1+n-\ln i}{\ln(1+n-\ln i)}\right\} \\ &\cong \sum_{i=1}^{N} exp\left\{\beta\left(1+n-\ln i\right)\right\} = e^{\beta n}\sum_{i=1}^{N}\frac{e^{\beta}}{i^{\beta}} \end{aligned} \tag{96}$$

The sequence of the sums, $\sum_{i=1}^{N}\frac{e^{\beta}}{i^{\beta}}$, converges to the finite limit, $\sum_{i=1}^{\infty}\frac{e^{\beta}}{i^{\beta}}$ iff $\beta > 1$. When $\beta < 1$, we use Euler-MacLaurin series to approximate the asymptotic of the the series in terms of an integral. It gives us:

$$\sum_{i=1}^{N}\frac{e^{\beta}}{i^{\beta}} \cong e^{\beta}\int_{1}^{N}\frac{1}{i^{\beta}}\,dx \cong e^{\beta}\,\frac{exp\{n(1-\beta)\}}{1-\beta} \tag{97}$$

When $\beta = 1$, we again use Euler-MacLaurin series to approximate the asymptotic of the the series in terms of an integral. It gives us:

$$\sum_{i=1}^{N}\frac{e^{\beta}}{i} \cong e\int_{1}^{N}\frac{1}{i}\,dx \cong e\,\ln N \tag{98}$$

Then, free energy is given as:

$$\chi(\beta) = \lim_{n\to\infty} \frac{logZ_n(\beta)}{n} = \begin{cases} 1, & \text{if } 0 < \beta \leq \beta_c = 1, \\ \\ \beta, & \text{if } \beta > \beta_c. \end{cases}$$

Also, note that $\chi(\beta_c) = 1$, $\chi'(\beta_c) = 1$ and they are continuous.

5.3.5 Relatively Light Tailed Double Exponential Distribution

Let $X_i = f(Y_i) = \ln Y_i$ where Y_i s are i.i.d random variables with standard exponential distribution (84). Note that these random variables have lighter tails than Weibull distribution.

$$P\{X_i > a\} = P\left\{\ln Y_i > a\right\} = exp\left\{-e^a\right\} \tag{99}$$

for $a \geq 0$. We select $A(n) = \dfrac{n}{\ln n}$ in the statistical sum (73). By using (74), (88), the asymptotic of the statistical sum can be expressed as

$$Z_n(\beta) = \sum_{i=1}^{N} exp\{\beta A(n) X_i\} \cong \sum_{i=1}^{N} exp\left\{\beta \frac{n}{\ln n} \ln(n - \ln i)\right\}$$

We express the upper and lower bounds of $\chi_n(\beta) = \dfrac{logZ_n(\beta)}{n}$ for large n in the following inequality,

$$\frac{logN_1}{n} + \frac{\beta \dfrac{n}{\ln n} \ln(n - logN_1)}{n} < \frac{logZ_n(\beta)}{n} < \frac{logN}{n} + \frac{\beta \dfrac{n}{\ln n} \ln(n)}{n} \tag{100}$$

where $N_1 = [e^{\lambda n}]$ for $\lambda < 1$. Simplification of this inequality gives us

$$\lambda + \beta \frac{n + \ln(1-\lambda)}{n} < \frac{logZ_n(\beta)}{n} < 1 + \beta \tag{101}$$

for large n. Because λ is arbitrarily close to 1, we deduce that

$$\lim_{n\to\infty} \frac{logZ_n(\beta)}{n} = 1 + \beta$$

for any $\beta > 0$.

5.4 Mixed Weibull Type Distribution

We repeat the experiment of selecting mixed Weibull type random variables. In this experiment, we either choose Weibull type random variable with probability p or shifted Weibull type random variable with probability $q = 1 - p$. As a result of this experiment, the random variables in the statistical sum (87) can be expressed as

$$X = \begin{cases} Y_1, & \text{with probability p and } P(Y_1 > x) = exp\left\{-\frac{x^{\varrho}}{\varrho}\right\} \\ an^{1/\varrho} + \sigma Y_2, & \text{with probability q and } P(Y_2 > x) = exp\left\{-\frac{x^{\varrho}}{\varrho}\right\} \end{cases}$$

Also, assume that we repeat this experiment $N = [e^n]$ times. We obtain v_n Weibull and $N - v_n$ shifted Weibull random variables. Such a mixed distribution has the following interpretation. We have N independent and identically distributed random variables, $\{X_1, X_2, ..., X_i, ...X_N\}$, in the statistical sum. The set of indexes is the union of v_n successes, $Y_1^j, j = 1, 2, ..., v_n$, and $N - v_n$ failures, $an^{1/\varrho} + \sigma Y_2^k$, $k = 1, 2, ..., N - v_n$. Here v_n has the binomial distribution $B(N, p)$. Also,

$A(n) = n^{1/\varrho'}$. Then we can express the statistical sum (87) as follows.

$$\begin{aligned} Z_n(\beta) &= \sum_{i=1}^{N} exp\left\{\beta A(n)X_i\right\} \\ &= \sum_{j=1}^{v^n} exp\left\{\beta A(n)Y_1^j\right\} + \sum_{k=1}^{N-v_n} exp\left\{\beta A(n)\left(an^{1/\varrho} + \sigma Y_2^k\right)\right\} \\ &= \sum_{j=1}^{v^n} exp\left\{\beta A(n)Y_1^j\right\} + exp\left\{a\beta n\right\} \sum_{k=1}^{N-v_n} exp\left\{\beta A(n)\sigma Y_2^k\right\} \\ &= Z_n^1(\beta) + Z_n^2(\beta) \end{aligned}$$

It means that the exponent in the sum varies depending on the result of the experiment. If Weibull type sample is selected in a single draw, exponent is $\beta A(n) = \beta n^{1/\varrho'}$. If shifted Weibull type sample is selected in the same single draw, exponent is $\sigma\beta A(n) = \beta n^{1/\varrho'}\sigma$. When σ is different than 1, this gives 2 phase transitions in free energy.

For mixed Weibull case, we have v_n successes as a result of random sampling. We assume that v_n, Y_1^j, Y_2^k are independent. In the previous section 5.3.3, we obtained free energy for Weibull case. We can still make use of this section's results for Y_1^j's. For Y_2^k's, we simply use the same formulation from 5.3.3 by replacing β with $\sigma\beta$. Using the independence of the random variables, it can be stated that

$$\begin{aligned} Z_n^1(\beta) &= \sum_{i=1}^{v_n} e^{\beta n^{1/\varrho'} X_i} \cong \sum_{i=1}^{p[e]^n} e^{\beta n^{1/\varrho'} X_i} \\ &\cong \begin{cases} exp\left\{\left(1 + \dfrac{\beta^{\varrho'}}{\varrho'}\right)n\right\}, & \text{if } \beta < \varrho^{1/\varrho'} = \beta_{critical_1} \\ exp\left\{\beta\varrho^{1/\varrho}n\right\}, & \text{if } \beta \geq \beta_{critical_1} \end{cases} \end{aligned}$$

In the case of shifted samples, we obtain:

$$
\begin{aligned}
Z_n^2(\beta) &= e^{a\beta n} \sum_{i=1}^{[e]^n - v_n} e^{\beta n^{1/\varrho'} \sigma X_i} \cong e^{a\beta n} \sum_{i=1}^{(1-p)[e]^n} e^{\beta\sigma n^{1/\varrho'} X_i} \\
&\cong \begin{cases} exp\left\{ \left(1 + a\beta + \dfrac{(\beta\sigma)^{\varrho'}}{\varrho'} \right) n \right\}, & \text{if } \beta < \dfrac{\varrho^{1/\varrho'}}{\sigma} = \beta_{critical_2} \\ exp\left\{ \left(a\beta + \beta\sigma\varrho^{1/\varrho} \right) n \right\}, & \text{if } \beta \geq \beta_{critical_2} \end{cases}
\end{aligned}
$$

To be able to calculate free energy, $\chi(\beta) = \lim_{n\to\infty} \dfrac{\ln Z_n(\beta)}{n}$, we make use of the following inequality

$$
\begin{aligned}
\max(Z_n^1(\beta), Z_n^2(\beta)) &< Z_n(\beta) < 2\max(Z_n^1(\beta), Z_n^2(\beta)) \\
\ln\max(Z_n^1(\beta), Z_n^2(\beta)) &< \ln Z_n(\beta) < \ln 2 + \ln\max(Z_n^1(\beta), Z_n^2(\beta)) \\
\lim_{n\to\infty} \frac{\ln\max(Z_n^1(\beta), Z_n^2(\beta))}{n} &< \lim_{n\to\infty} \frac{\ln Z_n(\beta)}{n} < \lim_{n\to\infty} \frac{\ln 2 + \ln\max(Z_n^1(\beta), Z_n^2(\beta))}{n}
\end{aligned}
$$

which implies that

$$
\lim_{n\to\infty} \frac{\ln Z_n(\beta)}{n} = \lim_{n\to\infty} \frac{\ln\, \max(Z_n^1(\beta), Z_n^2(\beta))}{n}
$$

For large n, when $\sigma > 1$

$$
\begin{aligned}
\chi(\beta) &= \lim_{n\to\infty} \frac{\ln Z_n(\beta)}{n} \\
&\cong \begin{cases} 1 + \max\left(\dfrac{\beta^{\varrho'}}{\varrho'}, a\beta + \dfrac{(\beta\sigma)^{\varrho'}}{\varrho'} \right), & \text{if } \beta < \dfrac{\varrho^{1/\varrho'}}{\sigma} = \beta_{critical_2} \\ \max(1 + \dfrac{\beta^{\varrho'}}{\varrho'}, a\beta + \beta\sigma\varrho^{1/\varrho}), & \text{if } \beta_{critical_2} \leq \beta < \varrho^{1/\varrho'} = \beta_{critical_1} \\ \beta\max(\varrho^{1/\varrho}, a + \sigma\varrho^{1/\varrho}), & \text{if } \beta_{critical_1} \leq \beta \end{cases}
\end{aligned}
$$

For large n, when $\sigma < 1$

$$
\begin{aligned}
\chi(\beta) &= \lim_{n\to\infty} \frac{\ln Z_n(\beta)}{n} \\
&\cong \begin{cases}
1 + \max\left(\dfrac{\beta^{\varrho'}}{\varrho'}\,,\, a\beta + \dfrac{(\beta\sigma)^{\varrho'}}{\varrho'}\right), & \text{if } \beta < \beta_{critical_1} \\
\max\left(\varrho^{1/\varrho}, 1 + a\beta + \dfrac{(\beta\sigma)^{\varrho'}}{\varrho'}\right), & \text{if } \beta_{critical_1} \le \beta < \beta_{critical_2} \\
\beta \max\left(\varrho^{1/\varrho}, a + \sigma\varrho^{1/\varrho}\right), & \text{if } \beta_{critical_2} \le \beta
\end{cases}
\end{aligned}
$$

CHAPTER 6: APPENDIX

Appendix A: Asymptotic Behavior of Weibull Integral for REM

6.1 Integral for Weibull

We claim that the follwing integral's asymptotic equivalent for large n is as follows

$$\begin{aligned}
\ln I(\beta) &= \ln \int_1^N \exp\left\{\beta \varrho^{1/\varrho} n^{1/\varrho'} (n - \ln x + c)^{1/\varrho}\right\} dx \\
&= \begin{cases} n\left(1 + \dfrac{\beta \varrho'}{\varrho'}\right) + \bar{\bar{o}}(n), & \text{if } 0 < \beta < \beta_c = \varrho^{1\varrho'}, \\[2ex] n\beta\varrho^{1/\varrho} + \bar{\bar{o}}(1), & \text{if } \beta \geq \beta_c. \end{cases}
\end{aligned} \tag{102}$$

where N is defined in (74).

Proof. Let $y = \ln x - c$ and $y = nz$. These substitutions provide us

$$\begin{aligned}
&\int_1^N \exp\left\{\beta \varrho^{1/\varrho} n^{1/\varrho'} (n - \ln x + c)^{1/\varrho}\right\} dx \\
&= ne^c \int_0^1 \exp\left\{n\left(z + \beta\varrho^{1/\varrho}(1-z)^{1/\varrho}\right)\right\} dx
\end{aligned}$$

where we define $g(z) = z + \beta\varrho^{1/\varrho}(1-z)^{1/\varrho}$ and it follows that

$$g'(z) = 1 - \frac{\beta}{\varrho^{1/\varrho'}(1-z)^{1/\varrho'}}$$

Then the conditions below are satisfied:

If $\beta < \varrho^{1/\varrho'} = \beta_c$, then $g(z)$ has a maximum at $z_l = 1 - \dfrac{\beta\varrho'}{\varrho}$.

If $\beta \geq \varrho^{1/\varrho'}$ then $g(z)$ has maximum at $z_l = 0$ as our integration region is restricted to $(0,1)$. Then, the asymptotic of the integral can be driven using Laplace Method. It is expressed as:

$$\ln I(\beta) = \begin{cases} n\left(1 + \dfrac{\beta \varrho^{'}}{\varrho^{'}}\right) + \overline{\overline{o}}(n), & \text{if } 0 < \beta < \beta_c = \varrho^{1/\varrho^{'}}, \\[2ex] n\beta\varrho^{1/\varrho} + \overline{\overline{o}}(n), & \text{if } \beta \geq \beta_c. \end{cases}$$

□

Appendix B: Application of Laplace Method to Weibull and Double Exponential Distribution

6.2 Weibull Distribution

Suppose that X has Weibull distribution (6). We are interested in finding the asymptotic equivalent of the cumulant generating function (12). Using the substitution $x = t^{\varrho' - 1} y$, we obtain

$$
\begin{aligned}
E[e^{tX}] &= \int_0^{+\infty} e^{tX} f_X(x)\, d(x) \\
&= t^{\varrho'} \int_0^{+\infty} y^{\varrho - 1} \exp\left\{ t^{\varrho'} \left(y - \frac{y^{\varrho}}{\varrho} \right) \right\} d(y)
\end{aligned}
$$

Also, $g(y) = y - \dfrac{y^{\varrho}}{\varrho}$ has a maximum at $y = 1$ and the following conditions are satisfied:

$$
\begin{aligned}
g(1) &= 1 - \frac{1}{\varrho} \\
g'(1) &= 0 \\
g''(1) &= -\varrho + 1 < 0
\end{aligned}
$$

Then we can find the asymptotic expression using Laplace method:

$$
H(t) = \log E[e^{tX}] = \frac{t^{\varrho'}}{\varrho'} + \frac{\varrho'}{2} \log(t) + \frac{1}{2} \log(\frac{2\pi}{\varrho - 1}) + o(1)
$$

as $t \to \infty$. In the limit we can see that $H_0(t) = t^{\varrho'}/\varrho'$.

6.3 Double Exponential Distribution

1. We calculate cumulant generating function (12) for large t when X has double exponential distribution (8). The density of this distribution is expressed as

$$
f_X(x) = \exp\{1 + x - e^x\}
$$

for $x > 0$. Use the substitution $x = y + \ln(t+1)$ to obtain

$$
\begin{aligned}
E[e^{tX}] &= \int_0^{+\infty} e^{tX} f_X(x)\, d(x) \\
&= \exp\{1 + \ln(t+1) + t\ln(t+1)\} \int_{-\ln(t+1)}^{+\infty} \exp\{(t+1)(y - e^y)\}\, dy
\end{aligned}
$$

Also, $g(y) := y - e^y$, has a maximum at $y = 0$ from $g^{'}(y) = 1 - e^y = 0$. Then we can apply Laplace method to obtain:

$$H(t) \quad = \quad \log E[e^{tX}] \quad \cong \quad t\ln(t) - t + \frac{\ln t}{2} + \textit{smaller terms} \quad \textit{as } t \to \infty$$

2. We try to evaluate integrals of type

$$\int_{-M_1(t)}^{+\infty} \exp\{M_2(t)(y - e^y)\}\, 1_{\{y \leq K\}}\, dy$$

using Laplace transform where $K < 0$ is the maximum point of the region of integration and $M_1(t), M_2(t) \to \infty$ as $t \to \infty$. For large t this integral is equivalent to

$$\int_{-M_1(t)}^{K} \exp\{M_2(t)(y - e^y)\}\, dy$$

Because $g(y) = y - e^y$ has a maximum at 0, $g'(y) > 0$ for negative y and the maximum point is outside the interval of integration, the major contribution to the integral comes from the neighborhood of the boundary point K. Then, the Laplace method gives us

$$\int_{-M_1(t)}^{K} \exp\{M_2(t)(y - e^y)\}\, dy \cong \exp\left\{M_2(t)(K - e^K)\right\} \frac{1}{M_2(t) \mid g'(K) \mid}$$

3. We evaluate integrals of type

$$\int_0^{+\infty} \exp\{M_2(t)(y - e^y)\} 1_{\{y>K\}}\, dy$$

using Laplace transform where $K > 0$ is the maximum point of the region of integration and $M_2(t) \to \infty$ as $t \to \infty$. For large t this integral is equivalent to

$$\int_K^{\infty} \exp\{M_2(t)(y - e^y)\}\ dy$$

Because $g(y) = y - e^y$ has a maximum at 0, $g'(y) < 0$ for positive y and the maximum point is outside the interval of integration, the major contribution to the integral comes from the neighborhood of the boundary point K. Then, the Laplace method gives us

$$\int_K^{\infty} \exp\{M_2(t)(y - e^y)\}\ dy \cong \exp\left\{M_2(t)(K - e^K)\right\} \frac{1}{M_2(t) \mid g'(K) \mid}$$

4. We drive the asymptotic equivalent of $K = \dfrac{\ln B(t) + \ln\tau}{t} - \ln(at + 1)$ for large t. By (15),

$$\begin{aligned} K &= \frac{\ln B(t) + \ln\tau}{t} - \ln(at+1) = \ln\lambda + \ln t + \frac{\ln\tau}{t} - \ln(at+1) \\ &\cong \ln(\lambda/a) + \frac{\ln\tau}{t} - \ln\left(1 + \frac{1}{at}\right) \end{aligned}$$

5. We want to simplify $NE[Y^a 1_{\{Y\leq\tau\}}]$ using the result, (35),

$$\begin{aligned} &NE[Y^a 1_{\{Y\leq\tau\}}] \\ &= \frac{N(t)}{B^a(t)} \exp\{(at+1)\ln(at+1)\} \exp\left\{(at+1)(K - e^K)\right\} \frac{e}{(at+1) \mid g'(K) \mid} \end{aligned}$$

Assume that $\lambda/a < 1$. Using (15), (18), and $K = \ln(\lambda/a) + \dfrac{\ln\tau}{t} - \ln\left(1 + \dfrac{1}{at}\right)$, we obtain

$$NE[Y^a 1_{\{Y\le\tau\}}] = e^{\lambda t}\frac{(at+1)^{at+1}}{(\lambda t)^{at}}\exp\left\{(at+1)(K-e^K)\right\}\frac{e}{(at+1)\mid g'(K)\mid}$$
$$= aet\left(\frac{a^a e^\lambda}{\lambda^a}\right)^t \exp\left\{(at+1)(K-e^K)\right\}\frac{e}{(at+1)\mid g'(K)\mid}$$
$$= aet\left(\frac{a^a e^\lambda}{\lambda^a}\right)^t \left(\frac{\lambda}{a}\right)^{at+1}\tau^{a-\lambda}e^{-1-t\lambda}\frac{e}{(at+1)|g'(K)|}$$
$$= \frac{\lambda\tau^{a-\lambda}e}{|g'(K)|}$$

where $g(y) = y - \exp\{y\}$ and $|g'(K)| \cong 1 - \lambda/a$

6. We want to simplify $NE[Y1_{\{Y\le\tau\}}]$ using the result (41),

$$NE[Y1_{\{Y\le\tau\}}] - \frac{A(t)}{B(t)}$$
$$= \frac{N(t)}{B(t)}\exp\{(t+1)\ln(t+1)\}\exp\left\{(t+1)(K-e^K)\right\}\frac{e}{(t+1)\mid g'(K)\mid}$$

where A(t) is given in (16a) for $0 < \lambda < 1$. Using (15), (18), and

$$K = \ln\lambda + \frac{\ln\tau}{t} - \ln\left(1+\frac{1}{t}\right)$$

we obtain

$$NE[Y1_{\{Y>\tau\}}] = e^{\lambda t}\frac{(t+1)^{t+1}}{(\lambda t)^t}\exp\left\{(t+1)(K-e^K)\right\}\frac{e}{(t+1)\mid g'(K)\mid}$$
$$= et\left(\frac{e^\lambda}{\lambda}\right)^t \exp\left\{(t+1)(K-e^K)\right\}\frac{e}{(t+1)\mid g'(K)\mid}$$
$$= et\left(\frac{e^\lambda}{\lambda}\right)^t \lambda^{t+1}\tau^{1-\lambda}e^{-1-t\lambda}\frac{e}{(t+1)|g'(K)|}$$
$$= \frac{\lambda\tau^{1-\lambda}e}{|g'(K)|}$$

where $g(y) = y - \exp\{y\}$ and $|g'(K)| \cong 1 - \lambda$

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