LIMIT THEOREMS FOR ONE CLASS OF ERGODIC MARKOV CHAINS

by

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ABSTRACT

NEZIHE TURHAN. Limit theorems for one class of ergodic Markov chains.
(Under the direction of DR. STANISLAV MOLCHANOV)

The goal of this dissertation is to develop some classical limit theorems for the additive functionals of the homogeneous Markov chains in the special class of the so-called, Loop Markov Chains. The additive functionals of the Markov chains have the numerous applications; especially in Mathematical Finance, Optimal control, and Random game theory. The first limit theorems for the finite Markov chains were proven by the founder of the theory Andrei Markov, later on by A. Kolmogorov, W. Döblin, J. Doob, W. Feller and many other experts on the topic. However, the situation with infinite Markov chains is more complicated including the case of Loop Markov chains. Therefore, we present a detailed work to prove the limit theorems for the Loop Markov chains in this dissertation.

This dissertation consists of five chapters. Our most significant contribution to the theory is presented in chapters four and five after we familiarize the reader on the essentials of the theory by reviewing the preliminary work given by others in the first three chapters. The structure of this dissertation is organized as follows. In the first chapter, we provide an intuitive background for the theory and introduce the case of Loop Markov chains. We address the difficulties we face during the construction of the limit theorems, especially in the case of the Gaussian limiting law for the Loop Markov chains. Since we construct the theory for both discrete and continuous-time Loop Markov chains, we review the essentials on both cases. Later on in the third chapter we give the specifics on the Döblin method and the Martingale approach to prove the CLT. In the fourth chapter, we introduce three models

of Loop Markov chains, namely, Discrete-time Loop Markov chain with countable phase space, Continuous-time Loop Markov chain with countable phase space and Continuous-time Loop Markov chain with continuous phase space, which are the main objectives of this dissertation. We present the CLT for the first two models and calculate the corresponding limiting variance by using both the Döblin method and the Martingale approach. Moreover, we talk about the Random Number Generators (RNG's) which are appropriate applications of the models constructed on the countable phase. Lastly, in chapter five we analyze and present a complete work on the convergence to the Stable limiting laws on both countable and continuous phase space, in which the latter case enlightens the complexity of the third model.

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CHAPTER 1: INTRODUCTION

Central Limit Theorem (CLT) is one of the most crucial theorems in probability theory and statistical mathematics. This limit theorem was proven for the finite Markov chains for the first time by Andrei Markov, who was a student of Pafnuty Lvovich Chebyshev. As a student of Chebyshev, Markov was undoubtedly influenced by his mentor's work. Markov expanded Chebyshev's initial study of sequences of independent random variables by including certain types of dependent random variables. Two decades later, Markov completed the proof of the central limit theorem by the method of moments which was firstly generalized and presented by Chebyshev. For further reading about Andrei Markov's life and his accomplishments as a mathematician, we refer the reader to the paper by Basharin, Langville, and Naumov [2] .

Ever since the initial work by Chebyshev and Markov, many mathematicians and statisticians have been studying the theory of limit theorems, i.e., CLT, on a variety of different classes of Markov chains enormously. For instance, limit theorems for functionals on countable ergodic Markov chains were approached by Kolmogorov and Döblin [9] first time in the late 1930s, and later on, kept being worked on by many others throughout the years. Maxwell and Woodroofe [17] derived the central limit theorems and invariance principles for additive functionals of a stationary ergodic Markov chain with zero mean and finite second moment.

In [8] Derriennic and Lin considered an additive functional of an ergodic Markov chain on a general state space, and then proved a CLT and an invariance principle with respect to the law of the chain started at a point.

In general, we cannot expect to have analogies with the theory of summation of independent random variables, which are the classical objects of the probability theory, in the case of the additive functionals of the Markov chains. Markov himself proved the first limit theorems for the finite Markov chains. However, the situation with infinite Markov chains, including our case, is more complicated. In this case, the least we will need is the ergodicity of the Markov chain, i.e., positive recurrence. On the other hand, this assumption only gives the Law of Large Numbers (LLN), not a more advanced Central Limit Theorem (CLT). In this study, we will prove the convergence of the distribution of the additive functionals to the Gaussian limiting law after appropriate normalization by using two methods, namely Döblin method, and Martingale approach. Nevertheless, there are still open problems even in this case. Let's formulate some of them which will be the subject of this dissertation as follows:

1. When the limiting distribution is non-degenerated, the situation is well understood for the chains with an excellent mixing. However, in many applications, we have to work with the Markov chains with a well developed the deterministic component.

2. Typical examples of "almost deterministic" chains give the Loop Markov chains proposed by Kai Lai Chung [6] , however, in an entirely different setting.

Such chains can be used for the construction of a new class of the Random Number Generators (RNG's), by continuing the random switching between known algorithms. Limit theorems we will state and prove in this dissertation give the information about the quality of such RNG's.

3. We will essentially develop the construction introduced by Chung for the Loop Markov chains with continuous-time or continuous phase space. For these models, we will prove the CLT on the convergence to the Gaussian limiting law, and then we will find the conditions on non-degeneracy including the new case of infinite limiting variance.

4. We will use Döblin method and Martingale approach to prove the limit theorems for the Loop Markov chains as they were used for the general Markov chains. We will justify in detail that Döblin method is stronger than the Martingale approach even though its application is technically more challenging.

5. The limit theorems on the convergence to the Stable, or infinite-divisible distributions are practically not known outside the Döblin condition. We will prove several results on the convergence to the Stable multidimensional distribution for the Loop Markov chains.

As future work, we will publish the results given by 1-4 and 5 as two different scientific papers.

CHAPTER 2: PRELIMINARIES ON THE MARKOV CHAINS

In this section, for the reader's convenience, we shall mention some basic definitions on both discrete and continuous-time Markov chains. First, we will formulate and discuss the most general results for the discrete-time Markov chains on countable phase space. Consider the discrete-time Markov chain $\{x(t), \ t = 0,1,2,\ldots\}$ on the countable phase space $X = \{0,1,2,\ldots,n,\ldots\}$. The matrix $\mathbb{P} = [p(x,y), \ x,y \in X]$ where

$$p(x,y) = \Pr\{x(t+1) = y \mid x(t) = x\}$$
$$\text{with } p(x,y) \geq 0 \text{ and } \sum_{y \in X} p(x,y) = 1,$$

is called the stochastic transition matrix for the chain $x(t)$. Therefore, probability of the transition from state x to y in exactly t steps, $p^{(t)}(x,y)$, is the element of the stochastic transition matrix after t steps, $\mathbb{P}^t = \left[p^{(t)}(x,y)\right]$, $t = 0,1,2,\ldots$ A chain is called *irreducible (connected)* and *aperiodic* if for any $x,y \in X$ there exists N such that $p^{(t)}(x,y) > 0$ for all $t > N$. For our future purposes, we will assume that the chain $x(t)$ is irreducible and aperiodic. A chain $x(t)$ is called *recurrent*, that is, $\Pr\{x(t) = x \text{ for some } t \geq 1 \mid x(0) = x\} = 1$ for any $x \in X$, if

$$\sum_{t}^{\infty} p^{(t)}(x,x) = \infty,$$

and *transient*, that is, $\Pr\{x(t) = x \text{ for some } t \geq 1 \mid x(0) = x\} < 1$ for any $x \in X$, if

$$\sum_{t}^{\infty} p^{(t)}(x, x) < \infty.$$

Note that for an irreducible Markov chain, either all states are recurrent, or all states are transient. If all states are recurrent, then we say that the Markov chain is recurrent; transient otherwise.

Now let $f(x) : X \to \mathbb{R}$ be a real-valued function, and then consider the additive functional of the discrete-time Markov chain $x(t)$ given by

$$S(T) = \sum_{t=0}^{T-1} f(x(t)). \tag{2.1}$$

The sum in Equation (2.1) denotes the price one has to pay for the observation of the chain in the given time interval $[0, T]$. Now assume that μ is the limiting distribution of $S(T)$ after appropriate normalization. Distribution $\mu(\cdot)$ can depend on the initial distribution of the Markov chain $x(t)$,that is, $\nu(x) = \Pr(x(0) = x)$. To prove the limit theorems as $T \to \infty$, such that they are independent of the initial distribution, and preserve the basic properties of limit theorems for sums of independent and identically distributed (i.i.d.) random variables such as the *infinite divisibility of the limiting distribution*, we need strong additional assumption on the chain $x(t)$, namely, *positive recurrence*. We will define positive recurrence as follows. For some recurrent state $x^* \in X$, let τ_{x^*} be the moment of first return from

x^* to x^*, i.e.,

$$\tau_{x^*} = \min\left(t \geq 1 : x(t) = x^* \mid x(0) = x^*\right).$$

State x^* is called positive recurrent if the expected amount of time to return to state x^* given that the chain started in state x^* has finite first moment, i.e.,

$$\mathbb{E}_{x^*}\left[\tau_{x^*}\right] < \infty.$$

Because the chain $x(t)$ is irreducible, we say all states in the chain are positively recurrent, i.e.,

$$\mathbb{E}_x\left[\tau_x\right] < \infty$$

for any $x \in X$.

Now assume that the discrete time Markov chain $\{x(t), t = 0, 1, 2, \ldots\}$ is positively recurrent, that is, any state $x \in X$ is positively recurrent. Then the limit

$$\pi(y) = \lim_{t\to\infty} p^{(t)}(x, y) = \lim_{t\to\infty} \Pr\{x(t) = y \mid x(0) = x, \ \forall x \in X\} \tag{2.2}$$

exists for all $y \in X$, which is called the unique stationary distribution of the chain $x(t)$. The stationary distribution is independent of the initial state $x(0)$, and given by

$$\pi(y) = \frac{1}{\mathbb{E}_y\left[\tau_y\right]} > 0, \text{ for all states } y \in X.$$

Next, we will give the definition of the Döblin condition which relates to the existence and uniqueness of the stationary distribution provided by (2.2).

Definition 2.1 *There exists an integer $k_0 \geq 1$ such that for some $\varepsilon > 0$,*

$$\begin{gathered}\mathbb{P}^{(k_0)} \geq \epsilon \mathbf{\Pi} \\ \text{i.e., } p^{(k_0)}(x,y) \geq \varepsilon \pi_y \end{gathered} \tag{2.3}$$

for $\forall x, y \in X$. Here $\mathbf{\Pi}$ is a rank-one stochastic matrix, i.e.,

$$\mathbf{\Pi} = \begin{bmatrix} \pi_1 & \pi_2 & \cdots \\ \pi_1 & \pi_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

The following lemma shows that, if we have the Döblin condition then there exists a unique stationary probability measure to which the Markov chain converges at a geometric rate from any starting point.

Lemma 2.2 *Under condition* (2.3), $\exists \gamma > 0$ *such that*

$$\left| p^{(t)}(x,y) - \pi(y) \right| \leq 2e^{-\gamma t} \tag{2.4}$$

for $\forall x, y \in X$.

A Markov chain $x(t)$ that is irreducible, aperiodic and positively recurrent is called *ergodic*, see $[6,10]$. For an ergodic Markov chain, the unique stationary distribution $\pi(\cdot)$ is the solution of the equation

$$\begin{gathered}\pi = \pi \mathbb{P} \\ \text{s.t. } \forall x \in X,\ \pi(x) > 0 \text{ and } \sum_{x \in X} \pi(x) = 1. \end{gathered} \tag{2.5}$$

If the initial distribution of the chain $x(t)$ equals to the stationary distribution $\pi(\cdot)$, then the chain $\{x(t)\}$ will be the stationary ergodic process.

Next, we will state some facts about the countable continuous-time Markov chains. One should know that most properties of continuous-time Markov chains are related to the results of discrete-time Markov chains, the Poisson process, and the exponential distribution. Therefore, to understand the continuous-time case we need to explain the connection between these concepts. One can consider the continuous-time Markov chain as a discrete-time chain with tailored transition times. That means the transition times; subsequently, the number of occurrences are both random, unlike the discrete case. In continuous-time Markov chains, the amount of time chain $x(t)$ spends in state x before making a transition is exponentially distributed with rate $-\lambda_{x\to x} = \sum\limits_{y\neq x} \lambda_{x\to y}$, and thus, mean $1/\lambda_{x\to x}$.

Now consider the continuous-time Markov chain $\{x(t) : t \geq 0\}$ where time t is understood to be any nonnegative real number. Similar to the discrete case, we can assume the phase space is countable, i.e., $X = \{0, 1, 2, \ldots, n, \ldots\}$, and the chain $x(t)$ is ergodic. Before we proceed, we will give the definition of the matrix exponent. Let $\lambda_{x\to y}$ be the rate of transition from state x to y, and $\mathbb{A} = \left[\lambda_{x\to y},\ x, y \in X\right]$ be the generator matrix for the chain. Then by definition, for any real number $t \geq 0$ we have

$$\exp(t\,\mathbb{A}) = e^{t\,\mathbb{A}} = \sum_{n=0}^{\infty} \frac{t^n\,\mathbb{A}^n}{n!} \text{ and } \mathbb{A}^0 = \mathbb{I}, \tag{2.6}$$

where $\mathbb{A}^n = \left[\lambda^{(n)}_{x\to y}\right]$ is the n^{th} power of the matrix $\mathbb{A}$. Because $\lambda^{(n)}_{x\to y}$ can be estimated as $\left|\lambda^{(n)}_{x\to y}\right| \leq \| \mathbb{A}\|^{n}$, we have

$$\left|[\exp(t\,\mathbb{A})]_{x,y}\right| = \left|\sum_{n=0}^{\infty} \frac{t^n \lambda^{(n)}_{x\to y}}{n!}\right| \leq \sum_{n=0}^{\infty} \frac{t^n \| \mathbb{A}\|^{n}}{n!} = \exp(t \| \mathbb{A}\|).$$

Thus, the series $\sum\limits_{n=0}^{\infty} \frac{t^n \lambda^{(n)}_{x\to y}}{n!}$ is analytic with infinite radius of convergence which means it is differentiable. By taking derivative of the expression in (2.6) with respect to t we have

$$\frac{de^{t\,\mathbb{A}}}{dt} = \sum_{n=0}^{\infty} \frac{nt^{n-1}\,\mathbb{A}^{n}}{n!} = \sum_{n=1}^{\infty} \frac{t^{n-1}\,\mathbb{A}^{n-1}}{(n-1)!}\,\mathbb{A} = e^{t\,\mathbb{A}}\,\mathbb{A} = \mathbb{A}e^{t\,\mathbb{A}},$$

i.e., the matrix function $e^{t\,\mathbb{A}} = [p\,(t, x, y)]$ is the unique solution of the system

$$\begin{cases} \dfrac{de^{t\,\mathbb{A}}}{dt} = \mathbb{A}e^{t\,\mathbb{A}} = e^{t\,\mathbb{A}}\mathbb{A} \\ p\,(0, x, y) = \delta_y\,(x) \\ p\,(t, x, y) > 0,\ \forall x, y \in \mathrm{X} \text{ and } t \geq 0. \end{cases}$$

We say matrix $\mathbb{P}(t) = e^{t\,\mathbb{A}}$ is stochastic for any $t \geq 0$ iff

$$\begin{aligned} &i)\ \lambda_{x\to y} \geq 0,\ \ x \neq y \\ &ii) -\ \lambda_{x\to x} \leq 0 \\ &ii)\ \sum_{y\neq x} \lambda_{x\to y} = 0,\ \ \forall x, y \in \mathrm{X}. \end{aligned}$$

Hence matrix $\mathbb{P}(t) = e^{t\,\mathbb{A}} = [p\,(t, x, y)]$ is called the stochastic transition matrix of the chain.

Next, define the moment of first return from state x to x as

$$\tau_x = \min\left\{t > 0,\ x(t) = x \ \mid x(0) = x\right\},$$

for any $x \in X$. Then we have

$$\mathbb{P}_x\left\{\tau_x > t\right\} = e^{t\lambda_{x\to x}} \ \text{ and } \ \mathbb{E}_x\left[\tau_x\right] = \frac{1}{\left|\lambda_{x\to x}\right|},$$

which underlines the fact that, time spent in state x is exponentially distributed with parameter $\lambda_{x\to x}$, i.e., lifetime of state $x \sim Exp\left(\lambda_{x\to x}\right)$. Similar to the discrete case, for a continuous-time ergodic Markov chain the limiting distribution $\pi(\cdot)$ is given by

$$\pi\left(y\right) = \lim_{t\to\infty} p\left(t, x, y\right) \tag{2.7}$$

which exists for all $y \in X$. This limit value in equation (2.7) is independent of the initial state x, and it is called the stationary distribution of the ergodic Markov chain.

Lemma 2.3 *Under condition* (2.3) , $\exists\gamma > 0$ *such that*

$$\left|p\left(t, x, y\right) - \pi\left(y\right)\right| \leq ce^{-\gamma t} \tag{2.8}$$

for $\forall x, y \in X$.

We conclude from Lemma 2.3 that, for a continuous-time ergodic Markov chain,

$\pi(y)$ is the unique solution of the equation

$$\begin{aligned} &\pi\,\mathbb{A} = 0 \\ \text{s.t.}\ \ &\pi\,\mathbb{1} = 0\ \text{ and } \sum_{x\in X} \pi(x) = 1. \end{aligned} \tag{2.9}$$

For further reading, we refer the reader to the books by P. Billingsley [3], K. L. Chung [6], and W. Feller [10, 11].

CHAPTER 3: METHODS TO PROVE THE LIMIT THEOREMS

In this chapter we will introduce two methods, namely Döblin method and Martingale approach, to prove the Central Limit Theorem for the Loop Markov chains. We will demonstrate both methods on the discrete case only. For further reading and the proofs in the continuous case, we refer the reader to Chung [5, 6], Bhattacharya [1] and Holzmann [13] for Döblin method and Martingale approach, respectively.

Assume that the countable phase space X is provided with the unique stationary distribution $\pi(\cdot)$. On this phase space, we define two functional spaces

$$\begin{aligned} \ell^1(X,\pi) &= \left\{ f(x) : X \to \mathbb{R} \mid \sum_{x\in X} \pi(x)\,|f(x)| = \|f\|_1 < \infty \right\} \\ \ell^2(X,\pi) &= \left\{ f(x) : X \to \mathbb{R} \mid \sum_{x\in X} \pi(x)\,|f(x)|^2 = \|f\|_2 < \infty \right\} \end{aligned} \tag{3.1}$$

and the dot product in $\ell^2(X,\pi)$ is given by the formula

$$(f,g)_\pi = \sum_{x\in X} f(x)\,\overline{g}(x)\,\pi(x).$$

Moreover, $\ell^2(X,\pi) \subset \ell^1(X,\pi)$ which means the subspace

$$\ell^2_{\mathbf{1}}(X,\pi) = \left\{ f(x) \in \ell^2(X,\pi) \mid (f,\mathbf{1})_\pi = \sum_{x\in X} \pi(x)\,f(x) = 0 \right\} \tag{3.2}$$

is $\mathbb{P}$-invariant and $\|\mathbb{P}\|_2 \leq 1$.

On a countable phase space, the ergodicity of the discrete-time Markov chain gives the Law of Large Numbers (LLN). First, we will give a lemma regarding the expectation and variance of the additive functional of the stationary ergodic chain.

Lemma 3.1 *Let $\overline{f}$ be the mean value of f, i.e., $\overline{f} = (f, \mathbf{1})_\pi = \sum_{x\in X} \pi(x) f(x)$, and let*

$$\widetilde{f}(x) = f(x) - (f, \mathbf{1})_\pi \tag{3.3}$$

be the centralization of the functional f. Then

$$\mathbb{E}_\pi [S(T)] = T\overline{f} + \underline{\underline{O}}(1)$$

and

$$Var_\pi [S(T)] = T\sigma^2(f) + \underline{\underline{O}}(1)$$

where

$$\sigma^2(f) = \pi\widetilde{f}^2 + 2\pi\left(\widetilde{f}, \Phi\widetilde{f}\right).$$

Here $\Phi = \sum_{t=1}^{\infty} \left(\mathbb{P}^t - \pi\right)$ is the fundamental matrix of the stationary ergodic chain.

Next theorem follows directly from Lemma 3.1.

Theorem 3.2 *If $f \in \ell^1(X, \pi)$, then*

$$\frac{S(T)}{T} \underset{T\to\infty}{\to} \overline{f} = (f, \mathbf{1})_\pi = \sum_{x\in X} \pi(x) f(x), \quad P-a.s.$$

i.e., for any $\varepsilon > 0$

$$\Pr\left\{\left|\frac{S(T)}{T} - \overline{f}\right| \geq \varepsilon\right\} \underset{T\to\infty}{\longrightarrow} 0.$$

Proof. *Using the Chebyshev's Inequality we have*

$$\Pr\left\{\left|\frac{S(T)}{T} - \overline{f}\right| \geq \varepsilon\right\} \leq \frac{1}{\varepsilon^2} Var_\pi \left[\frac{S(T)}{T}\right] = \frac{1}{\varepsilon^2 T^2} Var_\pi \left[S(T)\right] \underset{T\to\infty}{\longrightarrow} 0.$$

■

Note that, this result is similar to the LLN for the sequence of independent and identically distributed random variables $\mathcal{X}_i$, $i = 1, \ldots, n$, (see [3]), that is,

$$\frac{\mathcal{X}_1 + \cdots + \mathcal{X}_n}{n} \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\mathcal{X}_{\cdot}\right], \ \ P - a.s$$

The assumption of ergodicity only results in LLN for the normalized additive functionals of Markov chains. Because we want to prove the CLT in the case of Gaussian limiting law, we must study the asymptotic behavior of $S(T)$ after normalization. Therefore, for our purposes, we assume $f \in \ell^2(X, \pi)$. Next, consider the centralized additive functional

$$\widetilde{S}(T) = \sum_{t=0}^{T-1} \widetilde{f}\left(x\left(t\right)\right) = S(T) - T\left(f, \mathbf{1}\right)_\pi$$

for $\widetilde{f}\left(x\left(t\right)\right)$ given in (3.3). After normalization

$$\frac{\widetilde{S}(T)}{T} = \frac{S(T)}{T} - \left(f, \mathbf{1}\right)_\pi$$

one can expect that

$$\frac{\widetilde{S}(T)}{T} = \frac{S(T)}{T} - (f, \mathbf{1})_\pi \underset{T\to\infty}{\to} \mathcal{N}(0, \sigma^2), \tag{3.4}$$

which unfortunately is not always true. Here, calculation of the expectation of the normalized centralized sum (3.4) is straightforward by Theorem 3.2, i.e., for $f \in \ell^2(X, \pi)$

$$\mathbb{E}_\pi\left[\frac{\widetilde{S}(T)}{T}\right] = \mathbb{E}_\pi\left[\frac{S(T)}{T} - (f, \mathbf{1})_\pi\right] = \mathbb{E}_\pi\left[\frac{S(T)}{T}\right] - (f, \mathbf{1})_\pi \underset{T\to\infty}{\to} 0$$

since $\ell^2(X, \pi) \subset \ell^1(X, \pi)$. However, the situation with the variance is more complicated. Andrei Markov, who invented the Markov chain theory, proved the CLT for the *finite* Markov chains under the condition that

$$\begin{aligned}
\lim_{T\to\infty} TVar_\pi\left[\frac{\widetilde{S}(T)}{\sqrt{T}}\right] &= \lim_{T\to\infty} T\left\{\mathbb{E}_\pi\left[\left(\frac{\widetilde{S}(T)}{\sqrt{T}}\right)^2\right] - \left(\mathbb{E}_\pi\left[\frac{\widetilde{S}(T)}{\sqrt{T}}\right]\right)^2\right\} \\
&= \lim_{T\to\infty} T\mathbb{E}_\pi\left[\left(\frac{S(T)}{\sqrt{T}} - (f, \mathbf{1})_\pi\right)^2\right] \\
&= \lim_{T\to\infty} T\mathbb{E}_\pi\left[\left(\frac{S(T)}{\sqrt{T}} - \mathbb{E}_\pi\left[\frac{S(T)}{\sqrt{T}}\right]\right)^2\right] \\
&= \lim_{T\to\infty} TVar_\pi\left[\frac{S(T)}{\sqrt{T}}\right] \\
&= Var_\pi(f) + 2\sum_{k=1}^{\infty} Cov_\pi\left(f, p^{(k)} f\right) \\
&= 2\left(\widetilde{f}, \sum_{k=0}^{\infty} p^{(k)} \widetilde{f}\right)_\pi - \left(\widetilde{f}, \widetilde{f}\right)_\pi \\
&= \sigma^2(f) > 0.
\end{aligned}$$

The simple examples show that even in the case of the finite Markov chains, the limiting variance $\sigma^2(f)$ can vanish, i.e., $\sigma^2(f) = 0$. In the case of countable Markov chains, the situation is even more complicated. Therefore, to avoid this complication we apply two methods namely Döblin method and Martingale approach, to prove the CLT.

3.1 Döblin Method

Döblin method proposed by Chung [5,6], who established the method based on Döblin's idea in [9], in the following setting. Let $x_0 \in X$ be a fixed point and let $0 = \tau_0 \leq \tau_1 < \cdots < \tau_s < \cdots$ be the successive moments of returns of the chain from $x_0 \to x_0$, i.e.,

$$\begin{aligned} \tau_1 &= \min\{t > 0 \mid x(t) = x_0 \text{ given that } x(0) = x_0\}, \\ \tau_2 &= \min\{t > 0 \mid x(\tau_1 + t) = x_0 \text{ given that } x(\tau_1) = x_0\}, \ldots \end{aligned}$$

where τ_s is a random variable. Let $l(T)$ be a unique nonnegative integer satisfying $\tau_{l(T)} \leq t < \tau_{l(T)+1}$. The *dissection formula* for the additive functional is given by

$$S(T) = \sum_{t=0}^{T-1} f(x(t)) = \sum_{t=0}^{\tau_1 - 1} f(x(t)) + \sum_{t=\tau_1}^{\tau_2 - 1} f(x(t)) + \cdots + \sum_{t=\tau_{l(t)}}^{T-1} f(x(t))$$

$$= \sum_{t=0}^{\tau_1 - 1} f(x(t)) + \sum_{s=1}^{l(T)-1} \left\{ \sum_{t=\tau_s}^{\tau_{s+1}-1} f(x(t)) \right\} + \sum_{t=\tau_{l(T)}}^{T-1} f(x(t)).$$

Let

$$\mathcal{X}_s = \sum_{t=\tau_s}^{\tau_{s+1}-1} f(x(t)), \quad 1 \leq s \leq l(T). \tag{3.5}$$

Here $\mathcal{X}_s$'s are random variables which are independent and identically distributed with a common distribution (§I. *Thm*.14.3 in [6]). Then we can rewrite equation (3.5) as

$$S(T) = \sum_{t=0}^{\tau_1 - 1} f(x(t)) + \sum_{s=1}^{l(T)-1} \mathcal{X}_s + \sum_{t=\tau_{l(t)}}^{T-1} f(x(t)) \ . \tag{3.6}$$

Considering the first sum in equation (3.6), we have

$$\left| \sum_{t=0}^{\tau_1 - 1} f(x(t)) \right| \leq \sum_{t=0}^{\tau_1 - 1} |f(x(t))|$$

where the right-hand side is independent of T. Since τ_1 is finite with probability 1, then this sum is bounded by a fixed random variable with probability 1. Next consider the third sum in equation (3.6) . Chung proved that for a positively recurrent Markov chain, this sum is bounded in probability (§I. *Thm*.14.8). These two facts bring us to the conclusion that the asymptotic behavior of $S(T)$ depends on only the asymptotic behavior of random variables given in (3.5) . Before we continue, we must note that all the following statements are made for any real-valued functional f on the countable phase space X. However, they can be reduced to a particular, case of functionals, namely $g = \overline{f}$, for the positive recurrent class, (§I. *Thm*.15.1) by Chung [6], which is essential for our work.

Theorem 3.3 *Let $\Im(f) = \mathbb{E}(\mathcal{X})$. If $\Im(f)$ and $\Im(g)$ are both finite and not both zero, then we have*

$$\frac{\sum_{t=0}^{T-1} f(x(t))}{\sum_{t=0}^{T-1} g(x(t))} \underset{T\to\infty}{\longrightarrow} \frac{\Im(f)}{\Im(g)}$$

with probability 1.

In his work Chung proved that the first two moments of the random variable $\mathcal{X}_s$ exist and finite (see, §I. *Thm.* 14.5&14.7), and for a positively recurrent Markov chain they can be calculated. Therefore,

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{x_0}\left\{\sum_{s=1}^{l(t)-1} \mathcal{X}_s\right\} = \lim_{T\to\infty} \frac{\mathbb{E}\left[l(T)\right]}{T}\mathbb{E}_{x_0}\left(\mathcal{X}\right) = \pi\left(x_0\right)\mathbb{E}_{x_0}\left(\mathcal{X}\right).$$

Since the chain has a unique limiting distribution π, we have

$$\lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\sum_{t=1}^{T-1} f\left(x\left(t\right)\right) \mid x_0 \in X\right\} = \pi\left(x\right)\mathbb{E}_{x_0}\left(\mathcal{X}\right) = \sum_{x\in X} \pi\left(x\right) f\left(x\right),$$

hence,

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{S(T)}{\sqrt{T}}\right] = \sum_{x\in X} \pi\left(x\right) f\left(x\right).$$

Let $\zeta_s\left(x\right) = \tau_{s+1}\left(x\right) - \tau_s\left(x\right)$ be the time spent between leaving the state x and returning back there for the s^{th} time. Then define

$$\begin{aligned} M_s &= \frac{\mathbb{E}_\pi\left[\mathcal{X}_s\right]}{\mathbb{E}_\pi\left[\zeta_s\right]}; \\ \widehat{f} &= f - M_s; \\ Z_s(x) &= \sum_{[\tau_s,\tau_{s+1}]} \widehat{f}\left(x\left(T\right)\right); \end{aligned}$$

where

$$\frac{1}{T}\sum_{t=0}^{T-1} f\left(x\left(t\right)\right) \xrightarrow[T\to\infty]{P} M_s$$

for positive recurrent class. Moreover, in a positive recurrent class if M_s exists for any s, then it exists for every s. Therefore, it is independent of s and can be assumed constant, i.e., $M_s = M$. Now let

$$\sigma^2\left(\widehat{f}\right) = \mathbb{E}_\pi\left[(Z_s(x))^2\right],$$

and

$$B = \pi(x)\,\sigma^2\left(\widehat{f}\right) = \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\left(\sum_{s=1}^{l(t)-1} Z_s(x)\right)^2\right\}.$$

Then Chung states the *Döblin's Central Limit Theorem* as follows.

Theorem 3.4 *If* $\mathbb{E}_\pi\left[\zeta_s^2\right] < \infty$ *and* $0 < \sigma^2\left(\widehat{f}\right) < \infty$, *then* $S(T)$ *is asymptotically normally distributed with mean* $M{\cdot}T$ *and variance* $B{\cdot}T$, *that is, for every real* x

$$\lim_{T\to\infty}\Pr\left\{\frac{S(T) - M{\cdot}T}{\sqrt{B{\cdot}T}} \le x\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy.$$

Döblin proved in [9] that if $\sigma^2\left(\widehat{f}\right) = 0$ then $\dfrac{S(T)}{\sqrt{T}}$ is the sum of $M's$ for all $1 \le s \le l(T)$, and depends only on the values of $x(0)$ and $x(T)$.

Remark 3.5 *Note that* $\sigma^2\left(\widehat{f}\right)$ *is the variance of* Z_s, *not* $\mathcal{X}_s$. *However, under the assumption* $\mathbb{E}_\pi\left[\zeta_s^2\right] < \infty$, *condition* $\sigma^2\left(\widehat{f}\right) < \infty$ *yields to* $\mathbb{E}_\pi\left[\mathcal{X}_1^2\right] < \infty$.

In our setting, we assume that $(f,\mathbf{1})_\pi = 0$, i.e., $f \in \ell^2_{\mathbf{1}}(X,\pi)$, and

$$\mathbb{E}_\pi\left[\zeta_1^2\right] = \mathbb{E}_\pi\left[(\tau_2 - \tau_1)^2\right] < \infty.$$

Next we consider the random variable

$$\mathcal{X}_1 = \sum_{t=\tau_1}^{\tau_2 - 1} f\left(x\left(t\right)\right).$$

By definition of the inner product, we find the expectation of $\mathcal{X}_1$ as

$$\mathbb{E}_\pi\left[\mathcal{X}_1\right] = \mathbb{E}_\pi\left[\sum_{t=\tau_1}^{\tau_2-1} f\left(x\left(t\right)\right)\right] = \sum_{t=\tau_1}^{\tau_2-1} \pi\left(x\right) f\left(x\left(t\right)\right) = \left(f, \mathbf{1}\right)_\pi = 0.$$

Moreover, $\mathbb{E}_\pi\left[\mathcal{X}_1^2\right] < \infty$ by Remark 3.5. Then by Döblin's Central Limit Theorem we have

$$\frac{S\left(T\right)}{\sqrt{T}} \underset{T\to\infty}{\to} \mathcal{N}\left(0, D\right)$$

where $D = \dfrac{\sigma^2\left(f\right)}{\alpha}$ and $\alpha = \mathbb{E}_\pi\left[\zeta_1\right].$

Remark 3.6 *On the time interval* $\left[0, T\right]$ *, we have approximately* $\dfrac{T}{\alpha} = \dfrac{T}{\mathbb{E}_\pi\left[\zeta_1\right]} = \dfrac{T}{\mathbb{E}_\pi\left[\tau_{0\to 0}\right]}$ *loops due to Law of Large Numbers.*

The Döblin method can be extended in principle to the limit theorems on the convergence to the stable distribution. Assume that the random variable

$$\mathcal{X}_1 = \sum_{t=0}^{\tau_1 - 1} f\left(x\left(t\right)\right)$$

belongs to the domain of attraction of the stable distribution $St_{\alpha,\beta}$ with the parameters $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$. It means (due to classical theory) that

$$\begin{aligned} \mathbb{P}\left\{\mathcal{X}_1 > x\right\} &\sim c_1 \frac{L(x)}{x^{\alpha}}, \ x \to +\infty \\ \mathbb{P}\left\{\mathcal{X}_1 < -x\right\} &\sim c_2 \frac{L(|x|)}{|x|^{\alpha}}, x \to -\infty \end{aligned} \tag{3.7}$$

where $L(x)$ is slowly varying function, i.e., on $x \geq 0$ for all $a > 0$; $\lim\limits_{x\to\infty} \frac{L(ax)}{L(x)} = 1$. For instance, $L(x) = \log^{\beta} x$. Conditions given in (3.7) are called regularity conditions of the tails. Consider

$$S_n^* = \frac{1}{n^{\frac{1}{\alpha}} L_1(n)} \left(S_n - A_n\right),$$

where

$$A_n = \begin{cases} n\, \mathbb{E}[\mathcal{X}_\cdot] \, , \alpha > 1 \\ \ \ 0 \qquad \ , \alpha < 1 \end{cases}.$$

Then we have

$$S_n^* \ \underrightarrow{law} \ St_{\alpha,\beta}$$

with parameter $\beta = \dfrac{c_1 - c_2}{c_1 + c_2}$.

The main difficulty here is related to the justification of the regular tails condition given in (3.7). We can use, of course, the equations for the characteristic function of $\mathcal{X}_1$, i.e.,

$$\varphi(k) = \mathbb{E}\left[e^{ik\tau_1}\right]$$

but we cannot in general solve this equation. As a result, the theorems on the convergence of the distribution of the linear functional $S_{T,f}$ in law to the stable distribution can be proven only in some particular cases, including our main object "Loop Markov Chain."

3.2 Martingale Approach

This approach is an adaptation of Lindeberg method given in [4]. Later on P. Lévy [16] expanded the method to martingales. Since then this approach has been studied extensively (see [1, 12, 14, 13, 19]). The goal of this method is to ease the study of the asymptotic behavior of the additive functional by decomposing it into the sum of martingale differences with ergodic and stationary increments.

Now consider the arbitrary $f \in \ell^2_{\mathbf{1}}(X, \pi)$. Hence, we don't need the centralization. Next, consider the homological equation

$$g - \mathbb{P}g = f \tag{3.8}$$

for any given f, which will play the central role in this approach. The central moment here is to find the solution of the equation (3.8), namely g, in $\ell^2_{\mathbf{1}}(X, \pi)$. In general, without the Döblin condition (2.3), we can prove the CLT only in the case when equation (3.8) has solution in $\ell^2(X, \pi)$ which has the form

$$g(x) = (\mathbf{I} - \mathbb{P})^{-1} f(x) = \sum_{t=0}^{\infty} (\mathbb{P}^{(t)} f)(x) = f(x) + \sum_{t=1}^{\infty} \left(\sum_{y \in X} (p^{(t)}(x,y) - \pi(y)) f(y) \right)$$

Here it is clear to see that, under the Döblin condition (2.3) , solution g belongs to $\ell_{\mathbf{1}}^2(X, \pi)$. This result was based on the results of Ibragimov and Linnik [15] regarding the stationary processes. However, Döblin condition does not always exist. For example, in our case, we have a countable number of loops with varying lengths. If the lengths are not bounded, the Döblin condition does not hold.

Now assuming $g \in \ell_{\mathbf{1}}^2(X, \pi)$, we expand the additive sum as

$$\begin{aligned} S(T) &= f(x(0)) + f(x(1)) + \cdots + f(x(T)) \\ &= g(x(0)) \underbrace{+ g(x(1)) - (\mathbb{P}g)(x(0))}_{M_{\xi_1}} + \underbrace{g(x(2)) - (\mathbb{P}g)(x(1))}_{M_{\xi_2}} - \\ &\cdots + \underbrace{g(x(T)) - (\mathbb{P}g)(x(T-1))}_{M_{\xi_T}} - (\mathbb{P}g)(x(T)) \\ &= g(x(0)) - (\mathbb{P}g)(x(T)) + M_{\xi_1} + M_{\xi_2} + \cdots + M_{\xi_T} \end{aligned} \tag{3.9}$$

In this dissertation, we consider the Loop Markov chains where the chain returns to point 0 after transition on the loop is completed, i.e.,

$$(\mathbb{P}g)(x(s-1)) = g(x(s)), \quad 0 \leq s \leq T$$

which yields to

$$g(x(0)) - (\mathbb{P}g)(x(T)) = 0.$$

Consider the filtration $\mathcal{F}_s = \sigma\left(x\left(0\right), \ldots x\left(s\right)\right)$. Then we have

$$\begin{aligned}
\mathbb{E}\left[M_{\xi s} \mid \mathcal{F}_{s-1}\right] &= \mathbb{E}\left[g(x(s)) - (\mathbb{P}g)(x(s-1)) \mid \mathcal{F}_{s-1}\right] \\
&= \mathbb{E}\left[g(x(s)) \mid \mathcal{F}_{s-1}\right] - (\mathbb{P}g)(x(s-1)) \\
&= \mathbb{E}\left[(\mathbb{P}g)(x(s-1)) \mid \mathcal{F}_{s-1}\right] - (\mathbb{P}g)(x(s-1)) \\
&= (\mathbb{P}g)(x(s-1)) - (\mathbb{P}g)(x(s-1)) \\
&= g(x(s)) - (\mathbb{P}g)(x(s-1))
\end{aligned}$$

which means $M_{\xi_{(\cdot)}}$'s are stationary martingales adapted to the filtration $\mathcal{F}_s$ with expectation 0. Then the Central Limit Theorem in [18] is given as below.

Theorem 3.7 *Let $f \in \ell^2_{\mathbf{1}}(X, \pi)$. If the homological equation has the solution $g \in \ell^2_{\mathbf{1}}(X, \pi)$, then*

$$\frac{S(T)}{\sqrt{T}} = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} f\left(x\left(t\right)\right) \underset{T\to\infty}{\overset{law}{\to}} \mathcal{N}\left(0, \sigma^2\left(f\right)\right)$$

where

$$\sigma^2\left(f\right) = \lim_{T\to\infty} Var\left(S\left(T\right)\right) = \left(g, g\right)_\pi - \left(\mathbb{P}g, \mathbb{P}g\right)_\pi = \left(f, Bf\right)_\pi . \tag{3.10}$$

In this presentation, B is the covariance operator which is adjoint and non-negative. It is bounded only under the Döblin condition (2.3) , and given by

$$B = \mathbf{I} + (\mathbb{P}+\mathbb{P}^*) + \cdots + (\mathbb{P}^n+(\mathbb{P}^*)^n) + \cdots$$

where $\mathbb{P}^* = [p^*(x,y)]$ is the stochastic conjugate operator given by

$$p^*(x,y) = \frac{p(y,x)\,\pi(y)}{\pi(x)}.$$

We can extend this method to multidimensional case. Consider the vector function

$$\overrightarrow{f}(x) = \overrightarrow{f} = (f^{(1)}, \dots f^{(N)}) \tag{3.11}$$

where $\forall f_i(x) \in \ell^2_{\mathbf{1}}(X,\pi)$, under the assumption that homological equation

$$g_i - \mathbb{P}g_i = f_i \tag{3.12}$$

has solutions $g_i(x) \in \ell^2_{\mathbf{1}}(X,\pi)$, for all $i = \overline{1,N}$. The following CLT is a consequence of the Martingale approach.

Theorem 3.8 *Consider the normalized vector sum*

$$\frac{S(T)}{\sqrt{T}} = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} \overrightarrow{f}(x(t)).$$

where $\overrightarrow{f}(x)$ *is the vector function given in* (3.11) *. Assume that the homological equation* (3.12) *has the solution* $g_i(x) \in \ell^2_{\mathbf{1}}(X,\pi)$*. Then*

$$\frac{\overrightarrow{S}(T)}{\sqrt{T}} = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} \overrightarrow{f}(x(t)) \underset{T\to\infty}{\overset{law}{\longrightarrow}} \mathcal{N}(0,\mathcal{C}),$$

and the limiting covariance matrix is given by formula for $1 \leq m, l \leq N$

$$\begin{aligned} \mathcal{C} &= [c_{ij}] = (g_i, g_i)_\pi - (\mathbb{P}g_i, \mathbb{P}g_i)_\pi \\ &= \left[\sum_{i=1}^{\infty} p_i \overline{f}_i^{(m)} \overline{f_i}^{(l)} n_i^2 - \left(\sum_{i=1}^{\infty} p_i \overline{f_i}^{(m)} n_i \right) \left(\sum_{j=1}^{\infty} p_j \overline{f}_j^{(l)} n_j \right) \right]. \end{aligned}$$

Proof. Consider the sum

$$\begin{aligned} \overrightarrow{S_T} &= \overrightarrow{f}(x(0)) + \overrightarrow{f}(x(1)) + \cdots + \overrightarrow{f}(x(T)) \\ &= \overrightarrow{g}(x(0)) - \underbrace{(\mathbb{P}\overrightarrow{g})(x(0)) + \overrightarrow{g}(x(1))}_{\overrightarrow{M}_{\xi_1}} - \underbrace{(\mathbb{P}\overrightarrow{g})(x(1)) + \overrightarrow{g}(x(2))}_{\overrightarrow{M}_{\xi_2}} - \\ &\quad \cdots - \underbrace{(\mathbb{P}\overrightarrow{g})(x(T-1)) + \overrightarrow{g}(x(T))}_{\overrightarrow{M}_{\xi_T}} - (\mathbb{P}\overrightarrow{g})(x(T)) \\ &= \overrightarrow{g}(x(0)) - (\mathbb{P}\overrightarrow{g})(x(T)) + \overrightarrow{M}_{\xi_1} + \overrightarrow{M}_{\xi_2} + \cdots + \overrightarrow{M}_{\xi_T}. \end{aligned}$$

If the initial distribution of our chain is π, then the random vectors $\overrightarrow{M}_{\xi_1}, \overrightarrow{M}_{\xi_2}, \cdots, \overrightarrow{M}_{\xi_T}$ form an ergodic martingale-difference

$$\begin{aligned} \overrightarrow{M}_{\xi_i} &= \left((\mathbb{P}\overrightarrow{g}_i)(x(\xi_i)) - \overrightarrow{g}_i(x(\xi_{i+1})) \right) \\ s.t.\ \mathbb{E}[\overrightarrow{M}_{\xi_i}] &= 0 \text{ and } Cov\ (\overrightarrow{M}_{\xi_i}) = \left[(\mathbb{P}^*\mathbb{P} - \mathbf{I}) g_i, g_j \right]_\pi \end{aligned}$$

Therefore, $\overrightarrow{M}_{\xi_i}$ are uncorrelated, i.e., $\mathcal{C} = Cov\left(\overrightarrow{S}_T\right) = T\ Cov\ (\overrightarrow{M}_{\xi_i}),\ i = \overline{1, T}$. ∎

CHAPTER 4: LOOP MARKOV CHAINS

4.1 Discrete-time Loop Markov Chain on Countable Phase Space

Consider the connected, aperiodic and positively recurrent Markov chain $x(t)$, $t = 0, 1, 2, \ldots$ The phase space of our Loop Markov chain is given with $X = \{0\} \cup \{(i,j), i = 1, 2, \ldots; j = 1, 2, \ldots, n_i\}$, i.e., there is a countable collection of loops that are pinned at a central position 0, and each loop i has a varying finite length n_i. Here aperiodicity implies $GCD(n_i,\ i \geq 1) = 1$. The transition of the chain occurs as follows. From the central position 0, the chain $x(t)$ moves to the first position $(i,1)$ on the i^{th} loop with probability p_i. Since the selection of the loop which the chain $x(t)$ will move is random, we have $\sum_i p_i = 1$. From there the process moves to the second position $(i,2)$ on the i^{th} loop with probability 1 and so forth until it reaches position (i, n_i), that is, it is deterministic. Finally, the movement of the chain $x(t)$ on the i^{th} loop comes to an end when it reaches the central position 0 with probability 1. Afterward, the chain $x(t)$ jumps to another loop, and a similar scheme occurs with the same probabilities given above. Hence, the transition probabilities for discrete-time Loop Markov chain are

$$\begin{aligned} p(0,(i,1)) &= p_i \text{ and } p(0,x) = 0 \text{ if } x \neq (\cdot,1) \\ p((i,j),(i,j+1)) &= 1 \text{ and } p((i,j),x) = 0 \text{ if } x \neq (i,j+1),\ j < n_i \\ p((i,n_i),0) &= 1 \text{ and } p((i,n_i),x) = 0 \text{ if } x \neq 0 \end{aligned}$$

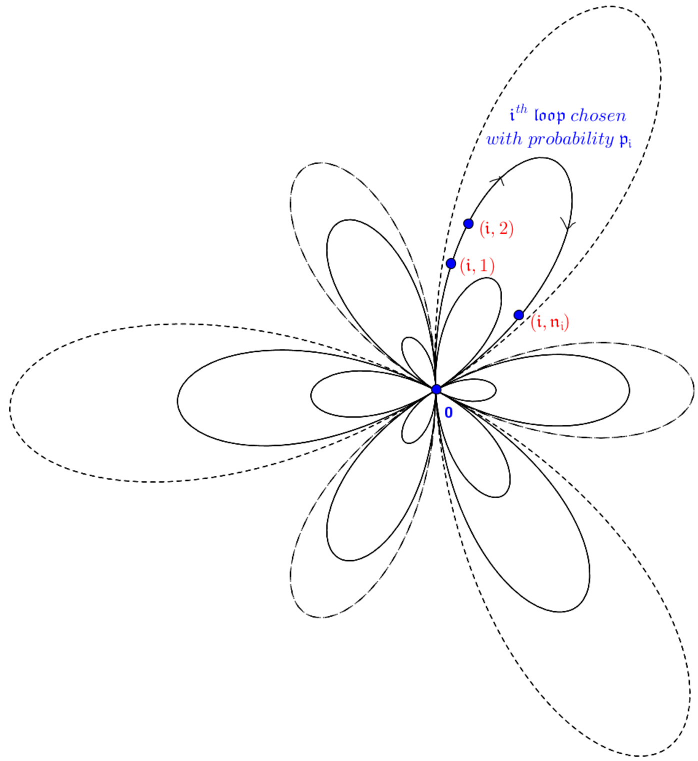

Figure 4.1: Discrete-time Loop Markov chain with countable phase space

The discrete-time Loop Markov chain is ergodic so there exists a unique stationary distribution $\pi(\cdot)$ satisfying the system (2.5) which is given by the following lemma.

Lemma 4.1 *The invariant distribution for the discrete-time Loop Markov chain with countable number of loops is given by*

$$\pi(0) = \frac{1}{1+\sum_{i=1}^{\infty} p_i n_i}$$

$$\pi(i,j) = p_i \, \pi(0) \ , \ j = 1, 2, ..., n_i.$$

Proof. Assume that $\pi : X \to [0,1]$ is stationary distribution satisfying the system (2.5) . Therefore, for all $y \in X$, we have

$$\pi(y) = \sum_{x \in X} \pi(x)\, p(x,y).$$

This means for any position $(i,j) \in X$, $i = 1,2,...$ and $j = 1,2,...,n_i$, we have

$$\pi(i,j+1) = \pi(i,j)\, p((i,j),(i,j+1)).$$

By using the transition probabilities for any loop i, we obtain

$$\begin{cases} \pi(i,1) = \pi(0)\, p(0,(i,1)) = p_i\, \pi(0) \\ \pi(i,2) = \pi(i,1)\, p((i,1),(i,2)) = \pi(i,1) = p_i\, \pi(0) \\ \vdots \\ \pi(i,n_i-1) = \pi(i,n_i-2)\, p((i,n_i-2),(i,n_i-3)) = \pi(i,n_i-2) = p_i\, \pi(0) \\ \pi(i,n_i) = \pi(i,n_i-1)\, p((i,n_i-1),(i,n_i)) = \pi(i,n_i-1) = p_i\, \pi(0). \end{cases}$$

Hence we have

$$\pi(i,j) = p_i\, \pi(0), \; j = 1,2,...,n_i. \tag{4.1}$$

Furthermore, π is a probability distribution which means

$$\sum_{\{0\} \cup (i,j) \in X} \pi(i,j) = 1.$$

This fact along with equation (4.1) implies that

$$\pi(0) + \sum_{i=1}^{\infty} n_i p_i \pi(0) = 1$$

Therefore, the invariant distribution at the control position is

$$\pi(0) = \frac{1}{1 + \sum\limits_{i=1}^{\infty} n_i p_i}. \tag{4.2}$$

■

Remark 4.2 *Discrete time countable Loop Markov chain is ergodic iff* $\sum\limits_{i=1}^{\infty} p_i n_i < \infty$.

Remark 4.3 *The Loop Markov chain does not satisfy the Döblin condition if the sequence* $\{n_i\}$ *is unbounded.*

Next, formulate the CLT for the discrete-time Loop Markov chain. For this particular chain, the functional spaces $\ell^1(X,\pi)$, $\ell^2(X,\pi)$ and the subspace $\ell^2_{\mathbf{1}}(X,\pi)$ have the following forms:

$$\ell^1(X,\pi) = \left\{ f(x) : X \to \mathbb{R} \mid |f(0)| + \sum_{i=1}^{\infty} p_i \sum_{j=1}^{n_i} |f(i,j)| < \infty \right\}$$

$$\ell^2(X,\pi) = \left\{ f(x) : X \to \mathbb{R} \mid f^2(0) + \sum_{i=1}^{\infty} p_i \sum_{j=1}^{n_i} |f(i,j)|^2 < \infty \right\}$$

$$\ell^2_{\mathbf{1}}(X,\pi) = \left\{ f \in \ell^2(X,\pi) \mid f(0) + \sum_{i=1}^{\infty} p_i \sum_{j=1}^{n_i} |f(i,j)| = 0 \right\}$$

For each function $f \in \ell^2_{\mathbf{1}}(X,\pi)$, we will introduce the new one which is constant on each loop i, $i = 1, 2, \dots$:

$$\overline{f}(i,j) = \overline{f}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} f(i,j)$$

i.e., averaging over the loops. So we will assume $f(i,j) = \overline{f}_i$ on each loop $i = 1, 2, \ldots$ First, we will explore the Martingale approach.

Theorem 4.4 *Let $f = \overline{f}_i \in \ell^2_{\mathbf{1}}(X, \pi)$. The homological equation has solution $g \in \ell^2_{\mathbf{1}}(X, \pi)$, and*

$$\frac{S(T)}{\sqrt{T}} = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} f(x(t)) \underset{T\to\infty}{\overset{law}{\to}} \mathcal{N}\left(0, \sigma^2(f)\right),$$

if $\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^3 < \infty$, then

$$\sigma^2(f) = (g,g)_\pi - (\mathbb{P}g, \mathbb{P}g)_\pi = \frac{\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^2 - \left(\sum_{i=1}^{\infty} p_i \overline{f}_i n_i\right)^2}{1 + \sum_{i=1}^{\infty} p_i n_i}.$$

Proof. Assume that $f(i,j) = \overline{f}_i$ constant on each loop. We already proved that martingale differences have zero means. Next, we consider the homological equation

$$g - \mathbb{P}g = f$$

to attain the variance of the martingale differences. Since $(\mathbb{P}g)(i,j) = g(i, j+1)$, for any $i = 1, 2, \ldots$ we have

$$\begin{aligned} g(i,1) - g(i,2) &= f(i,1) \\ g(i,2) - g(i,3) &= f(i,2) \\ &\vdots \\ g(i,n_i) - g(0) &= f(i,n_i) \end{aligned}$$

which yields to

$$g\left(i,1\right)=g\left(0\right)+\sum_{j=1}^{n_i}f\left(i,j\right)=g\left(0\right)+\overline{f}_i n_i \tag{4.3}$$

Since $g\in\ell^2_{\mathbf{1}}\left(X,\pi\right)\subset\ell^2\left(X,\pi\right)$, it is clear to see that

$$\sum_{i=1}^{\infty}p_i\sum_{j=1}^{n_i}\left|\overline{f}_i n_i\right|^2=\sum_{i=1}^{\infty}p_i\overline{f}_i^2 n_i^3<\infty. \tag{4.4}$$

Next we will calculate $g\left(0\right)$. We can generalize equation $\left(4.3\right)$ for any $j=1,2,...,n_i$ as

$$g\left(i,j\right)=g\left(0\right)+\sum_{s=j}^{n_i}f\left(i,s\right) \tag{4.5}$$

Solution $\left(4.5\right)$ belongs to $\ell^2_{\mathbf{1}}\left(X,\pi\right)$; therefore, we have

$$g\left(0\right)+\sum_{i=1}^{\infty}p_i\sum_{s=1}^{n_i}\left(g\left(0\right)+s\overline{f}_i\right)=0$$

$$g\left(0\right)+\sum_{i=1}^{\infty}p_i n_i g\left(0\right)+\sum_{i=1}^{\infty}p_i\overline{f}_i\frac{n_i\left(n_i+1\right)}{2}=0$$

$$g\left(0\right)\left[1+\sum_{i=1}^{\infty}p_i n_i\right]+\sum_{i=1}^{\infty}p_i\overline{f}_i\frac{n_i\left(n_i+1\right)}{2}=0$$

$$g\left(0\right)=-\frac{1}{2}\frac{\sum_{i=1}^{\infty}p_i\overline{f}_i n_i\left(n_i+1\right)}{\left[1+\sum_{i=1}^{\infty}p_i n_i\right]} \tag{4.6}$$

Here note that assumption $f=\overline{f}_i\in\ell^2_{\mathbf{1}}\left(X,\pi\right)$ is equivalent to

$$f\left(0\right)+\sum_{i=1}^{\infty}p_i\overline{f}_i n_i=0$$

which gives

$$f(0) = -\sum_{i=1}^{\infty} p_i \overline{f}_i n_i \tag{4.7}$$

Then by substituting this in (4.6) , we have

$$g(0) = \frac{1}{2}\,\frac{f(0) - \sum_{i=1}^{\infty} p_i \overline{f}_i n_i^2}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]}.$$

Next, we calculate the variance

$$\sigma^2(f) = (g,g)_\pi - (\mathbb{P}g, \mathbb{P}g)_\pi$$

explicitly. Again we consider the homological equation

$$g - \mathbb{P}g = f \quad \Longleftrightarrow \quad \mathbb{P}g = g - f.$$

Since $(\mathbb{P}g)(i,j) = g(i,j+1)$, for $i = 1,2,\ldots$ we have

$$\begin{aligned}
\sigma^2(f) &= (g,g)_\pi - (\mathbb{P}g, \mathbb{P}g)_\pi \\
&= \frac{g^2(0) - [g(0) - f(0)]^2}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]} \\
&\qquad + \frac{1}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]} \sum_{i=1}^{\infty} p_i \sum_{s=1}^{n_i} \left[\left(g(0) + s\overline{f}_i\right)^2 - \left(g(0) + (s-1)\,\overline{f}_i\right)^2\right]
\end{aligned}$$

$$= \frac{-f^2(0) + 2g(0) f(0)}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]} + \frac{1}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]} \sum_{i=1}^{\infty} p_i \sum_{s=1}^{n_i} \left[\overline{f}_i^2 (2s-1) + 2g(0)\overline{f}_i\right]$$

$$= \frac{-f^2(0) + 2g(0) f(0)}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]} + \frac{1}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]} \sum_{i=1}^{\infty} p_i \left[\overline{f}_i^2 \left(2\frac{n_i(n_i+1)}{2} - n_i\right) + 2g(0)\overline{f}_i n_i\right]$$

$$= \frac{-f^2(0) + 2g(0) f(0) + \sum_{i=1}^{\infty} p_i \left[\overline{f}_i^2 n_i^2 + 2g(0)\overline{f}_i n_i\right]}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]}$$

$$= \frac{-f^2(0) + 2g(0) f(0) + \sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^2 + 2g(0) \sum_{i=1}^{\infty} p_i \overline{f}_i n_i}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]}$$

Here substituting $f(0)$ given by (4.7), we get

$$= \frac{-f^2(0) + 2g(0) f(0) + \sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^2 - 2g(0) f(0)}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]}$$

$$= \frac{\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^2 - \left(\sum_{i=1}^{\infty} p_i \overline{f}_i n_i\right)^2}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]}.$$

■

Remark 4.5 *The proof of Theorem 4.4 is based on the direct calculations of the solution $g(\cdot)$ of the homological equation. Here we observe that the limiting variance contains $\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^2$ but not $\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^3$, which indicates that probably the condition $\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^3 < \infty$ is too strong, that is, assumption $g \in \ell_{\mathbf{1}}^2 (X, \pi)$ is not necessary for CLT by this approach.*

Next, we will show that, by using Döblin method one can prove the following extension of Theorem 4.4.

Theorem 4.6 *If $f = \overline{f}_i \in \ell_{\mathbf{1}}^2 (X, \pi)$ and $\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^2 < \infty$, then*

$$\frac{S(T)}{\sqrt{T}} = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} f(x(t)) \underset{T\to\infty}{\overset{law}{\to}} \mathcal{N}\left(0, \sigma^2(f)\right)$$

where $\sigma^2(f)$ is the same variance given in Theorem 4.4.

Proof. Consider the random variable

$$\mathcal{X}_1 = \sum_{t=\tau_1}^{\tau_2 - 1} f\left(x\left(t\right)\right)$$

In section 3.1, we stated that asymptotic behavior of the additive functional is related to asymptotic behavior of the random variable $\mathcal{X}_1$. Assume that transition of the chain starts at point 0. Then it jumps to the i^{th} loop with probability p_i where it spends

$$\tau_{0\to 0} = \zeta_1\left(x\right) = \tau_2\left(x\right) - \tau_1\left(x\right) = 1_{point(0)} + n_i \tag{4.8}$$

time on the loop before it reaches central position 0 again. For the Loop Markov chain, where we assume $f = \overline{f}_i \in \ell_{\mathbf{1}}^2 (X, \pi)$ is constant on the i^{th} loop, $\mathcal{X}_1$ has the form

$$\mathcal{X}_1 = \sum_{t=\tau_1}^{\tau_2 - 1} f\left(x\left(t\right)\right) = f\left(0\right) + \overline{f}_i n_i$$

since $\sum\limits_{j=1}^{n_i} f(i,j) = \overline{f}_i n_i$. By assumption $\overline{f}_i \in \ell^2_{\mathbf{1}}\left(X, \pi\right)$, we have

$$f\left(0\right) + \sum_{i=1}^{\infty} p_i \overline{f}_i n_i = 0.$$

Then expectation of $\mathcal{X}_1$ is

$$\begin{aligned}
\mathbb{E}_\pi\left[\mathcal{X}_1\right] &= \sum_{x \in X} \pi\left(i,j\right) \mathcal{X}_1 = \pi\left(0\right) \sum_{i=0}^{\infty} p_i \left[f\left(0\right) + n_i \overline{f}_i\right] \\
&= \frac{1}{\left[1 + \sum\limits_{i=1}^{\infty} p_i n_i\right]} \left\{f\left(0\right) \sum_{i=0}^{\infty} p_i + \sum_{i=0}^{\infty} p_i \overline{f}_i n_i\right\} \\
&= \frac{1}{\left[1 + \sum\limits_{i=1}^{\infty} p_i n_i\right]} \left\{f\left(0\right) - f\left(0\right)\right\} \\
&= 0
\end{aligned}$$

for $\sum\limits_{i=0}^{\infty} p_i = 1$. This implies that variance is equivalent to the second moment, so we have

$$Var_\pi\left[\mathcal{X}_1\right] = \mathbb{E}_\pi\left[\mathcal{X}_1^2\right] = \sum_{x \in X} \pi\left(i,j\right) \mathcal{X}_1^2$$

$$= \pi(0)\sum_{i=0}^{\infty} p_i \left[f(0) + n_i \overline{f}_i\right]^2$$

$$= \pi(0)\sum_{i=0}^{\infty} p_i \left[f^2(0) + 2f(0)\, n_i \overline{f}_i + n_i^2 \overline{f}_i^2\right]$$

$$= \frac{1}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]} \left\{f^2(0)\sum_{i=0}^{\infty} p_i + 2f(0)\sum_{i=0}^{\infty} p_i n_i \overline{f}_i + \sum_{i=0}^{\infty} p_i n_i^2 \overline{f}_i^2\right\}$$

$$= \frac{1}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]} \left\{f^2(0) + 2f(0)(-f(0)) + \sum_{i=0}^{\infty} p_i n_i^2 \overline{f}_i^2\right\}$$

$$= \frac{1}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]} \left\{\sum_{i=0}^{\infty} p_i n_i^2 \overline{f}_i^2 - f^2(0)\right\}$$

$$= \frac{\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^2 - \left(\sum_{i=1}^{\infty} p_i \overline{f}_i n_i\right)^2}{\left[1 + \sum_{i=1}^{\infty} p_i n_i\right]}$$

i.e., the same variance $\sigma^2(f)$ from Theorem 4.4. ∎

Conclusion 4.7 *In Theorem 4.4 (Martingale approach) we assumed* $\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^3 < \infty$*, and in Theorem 4.6 (Döblin method) we assumed* $\sum_{i=1}^{\infty} p_i \overline{f}_i^2 n_i^2 < \infty$*. Therefore, we can conclude that the Döblin method is stronger than Martingale approach in the case of Loop Markov chain.*

There are results on the convergence in law to the Gaussian distribution without the assumption that $f \in \ell_{\mathbf{1}}^2(X, \pi)$. The typical result (similar to the corresponding theorem i.i.d.r.v.) is given below:

Theorem 4.8 *Assume that* $f = \overline{f}_i$ *and* $\sum_{i=1}^{N} p_i \overline{f}_i^2 n_i^2 = \infty$ *but* $\sum_{i=1}^{N} p_i \overline{f}_i^2 n_i^2 = L(N) \uparrow \infty$, *as* $N \to \infty$, *where* $L(N)$ *is slowly varying function. Then for appropriately chosen slowly varying function* $L_1(N)$, *we have*

$$\frac{S(T)}{\sqrt{T}L_1(N)} = \frac{1}{\sqrt{T}L_1(N)} \sum_{t=0}^{T} f x(t)) \underset{T\to\infty}{\overset{law}{\longrightarrow}} \mathcal{N}(0,1).$$

Remark 4.9 *Theorem 4.8 cannot be proven by Martingale approach, and if the series* $\sum_{i=1}^{N} p_i \overline{f}_i^2 n_i^2$ *diverges faster, one can expect (under some regularity conditions) convergence to the stable distribution. We will discuss this topic in Chapter 5.*

4.2 Continuous-time Loop Markov Chain on Countable Phase Space

In the previous section, we discussed the Loop Markov chain with discrete time $t = 0, 1, \ldots$ Discrete-time model required a crucial technical condition like $GCD(n_i, i \geq 1) = 1$, and the local limit theorems were not simple. Now we will introduce the continuous-time Loop Markov chain. Assume that on the phase space $X = \{0\} \cup \{(i,j), i = 1, 2, \ldots; j = 1, 2, \ldots, n_i\}$ the continuous-time chain $x(t)$, $t \geq 0$, moves deterministically on each loop; however, it spends exponentially distributed time with parameter $\lambda > 0$ at each position on every loop. Therefore, each transition occurs with density $\pi(t) = \lambda e^{-\lambda t} \mathcal{I}_{t \geq 0}$. At the moment of the jump, chain $x(t)$ moves from central position 0 to the first position $(i, 1)$ on the i^{th} loop with probability p_i, i.e., $\sum_i p_i = 1$. Then it moves to the next position on the i^{th} loop, that is, moves from (i,j) to $(i, j+1)$, $j = 1, 2, \ldots, n_i - 1$, with probability 1. Lastly, it moves from position (i, n_i) to 0 with probability 1.

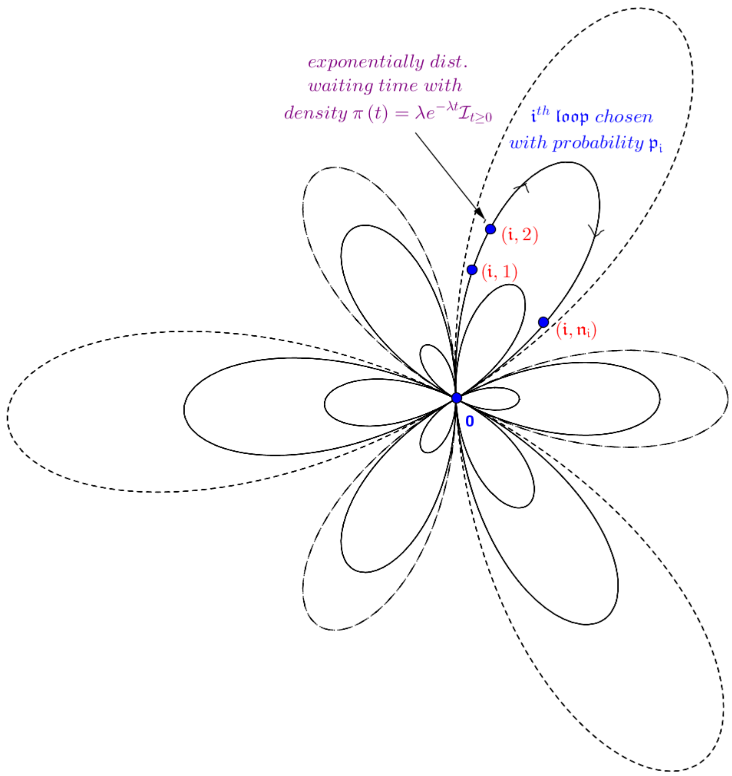

Figure 4.2: Continuous-time Loop Markov chain with countable phase space

Therefore, the transition probabilities at the moments of jumps are the same as the discrete-time Loop Markov chain, i.e.,

$$\begin{aligned} p\left(0,(i,1)\right) &= p_i \text{ and } p\left(0,x\right)=0 \text{ if } x\neq(\cdot,1) \\ p\left((i,j),(i,j+1)\right) &= 1 \text{ and } p\left((i,j),x\right)=0 \text{ if } x\neq(i,j+1),\ j<n_i \\ p\left((i,n_i),0\right) &= 1 \text{ and } p\left((i,n_i),x\right)=0 \text{ if } x\neq 0. \end{aligned}$$

Now recall from chapter 2 that $\mathbb{A} = \left[\lambda_{x\to y},\ x,y\in X\right]$ is the generator matrix for the chain. Then, we define the infinitesimal generator $\mathcal{L}$ of the continuous-time

Loop Markov chain as

$$(\mathcal{L}f)(x) = \lim_{\Delta t \to 0^+} \frac{\mathbb{P}_{\Delta t} f(x) - f(x)}{\Delta t} = (\mathbb{P}f)'(0) = \mathbb{A} \tag{4.9}$$

which is given by the formula

$$\begin{aligned} &(i)\ \ (\mathcal{L}f)(0) = \lambda \sum_{i=1}^{\infty} p_i \left[f(i,1) - f(0) \right] \\ &(ii)\ \ (\mathcal{L}f)(i,j) = \lambda \left[f(i,j+1) - f(i,j) \right], \quad j = 1, 2, \ldots, n_i - 1 \\ &(iii)\ \ (\mathcal{L}f)(i,n_i) = \lambda \left[f(0) - f(i,n_i) \right]. \end{aligned} \tag{4.10}$$

Remark 4.10 *Transition of the chain is similar to the discrete case. The only difference is that each jump requires the exponentially distributed waiting time.*

The invariant distribution $\pi(\cdot)$ now exists without arithmetical conditions on $\{n_i,\ i \geq 1\}$, and has the same form as in the discrete case:

$$\pi(0) = \frac{1}{1 + \sum_{i=1}^{\infty} p_i n_i}$$

$$\pi((i,j)) = p_i\ \pi(0),\ j = 1, 2, ..., n_i.$$

In the discrete case, the random variable $\tau_{0 \to 0}$, moment of first return from position 0 to 0, had values $(1 + n_i)$, $i \geq 1$, $j = \overline{1, n_i}$ with probabilities p_i. Differently from the first model, it has the density

$$p_{\tau_{0 \to 0}}(t) = \sum_{i=1}^{\infty} p_i \left(\pi_\lambda(t) \right)^{(1+n_i)}$$

where $\pi_\lambda(t) = \lambda e^{-\lambda t}\mathcal{I}_{t\geq 0}$, i.e.,

$$p_{\tau_{0\to 0}}(t) = \mathcal{I}_{t\geq 0}\left\{\sum_{i=1}^{\infty} p_i \frac{\lambda^{(1+n_i)}\ t^{n_i}}{n_i!} e^{-\lambda t}\right\}.$$

Then the corresponding Laplace transform

$$\widehat{p}_{\tau_{0\to 0}}(k) = \mathbb{E}\left[e^{-k\tau_{0\to 0}}\right] = \sum_{i=1}^{\infty} p_i \left(\frac{\lambda}{\lambda+k}\right)^{n_i}$$

Note that

$$\mathbb{E}_\pi\left[\tau_{0\to 0}\right] = \mathbb{E}_\pi\left[\zeta_1\right] = \sum_{i=1}^{\infty} p_i \frac{(n_i+1)}{\lambda} = \frac{1}{\lambda}\left(1+\sum_{i=1}^{\infty} p_i n_i\right) = \frac{1}{\lambda\pi(0)}. \tag{4.11}$$

Both Döblin method and Martingale approach apply to this model. Since the phase space X is still countable, we have the functional spaces $\ell^1(X,\pi)$, $\ell^2(X,\pi)$, and $\ell^2_{\mathbf{1}}(X,\pi)$:

$$\ell^1(X,\pi) = \left\{f(x) : X \to \mathbb{R} \mid |f(0)| + \sum_{i=1}^{\infty} p_i \sum_{j=1}^{n_i} |f(i,j)| < \infty\right\}$$

$$\ell^2(X,\pi) = \left\{f(x) : X \to \mathbb{R} \mid f^2(0) + \sum_{i=1}^{\infty} p_i \sum_{j=1}^{n_i} |f(i,j)|^2 < \infty\right\}$$

$$\ell^2_{\mathbf{1}}(X,\pi) = \left\{f \in \ell^2(X,\pi) \mid f(0) + \sum_{i=1}^{\infty} p_i \sum_{j=1}^{n_i} |f(i,j)| = 0\right\}$$

Consider the additive functional of continuous-time Loop Markov chain on the time interval $[0,T]$ given by

$$S(T) = \int_0^T f\left(x\left(t\right)\right)\,dt.$$

For each function $f \in \ell^2_{\mathbf{1}}\left(X, \pi\right)$, if we again assume

$$f\left(i, j\right) = \overline{f}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} f(i, j)$$

on each loop $i = 1, 2, \ldots$, then the asymptotic behavior of $S(T)$ is similar to the discrete case since we have the same countable phase space X. Therefore, the presentation of CLT looks virtually almost identical to the CLT in the case of the discrete-time Loop Markov chain. However, for the general functions, the situation is different.

First, we will explore the Döblin method. Assume that $f(x) \in \ell^2_{\mathbf{1}}\left(X, \pi\right)$ is a general function. The *dissection formula* has the form

$$S\left(T\right) = \int_0^T f\left(x\left(t\right)\right)\,dt = \int_0^{\tau_1} f\left(x\left(t\right)\right)\,dt + \int_{\tau_1}^{\tau_2} f\left(x\left(t\right)\right)\,dt + \cdots + \int_{\tau_{l(t)}}^{T} f\left(x\left(t\right)\right)\,dt$$

$$= \int_0^{\tau_1} f\left(x\left(t\right)\right)\,dt + \sum_{s=1}^{l(T)-1} \left\{ \int_{\tau_s}^{\tau_{s+1}} f\left(x\left(t\right)\right)\,dt \right\} + \int_{\tau_{l(t)}}^{T} f\left(x\left(t\right)\right)\,dt$$

Let

$$\mathcal{X}_s = \int_{\tau_s}^{\tau_{s+1}} f\left(x\left(t\right)\right)\,dt, \quad 1 \le s \le l(T).$$

We already know that $\mathcal{X}_s$'s are random variables which are independent and identically distributed with a common distribution. Therefore, we may represent $S(T)$

as

$$S\left(T\right)=\int\limits_{0}^{\tau_1} f\left(x\left(t\right)\right)\,dt+\sum_{s=1}^{l(T)-1}\mathcal{X}_s+\int\limits_{\tau_{l(t)}}^{T} f\left(x\left(t\right)\right)\,dt$$

where $l(T)$ be a unique nonnegative integer satisfying $\tau_{l(T)}\le t<\tau_{l(T)+1}$. Since asymptotic behavior of $S(T)$ is related to asymptotic behavior of $\mathcal{X}_s$, we will examine this variable to construct the CLT.

Theorem 4.11 *If $f(x)\in\ell_{\mathbf{1}}^2\left(X,\pi\right)$, then*

$$\frac{S(T)}{\sqrt{T}}=\frac{1}{\sqrt{T}}\int\limits_{0}^{T} f(x(t)) \underset{T\to\infty}{\overset{law}{\longrightarrow}} \mathcal{N}\left(0,\sigma^2(f)\right)$$

where

$$\sigma^2(f)=\frac{f^2\left(0\right)+\sum\limits_{i=0}^{\infty}p_i\sum\limits_{j=1}^{n_i}f^2\left(i,j\right)}{\lambda^2\left[1+\sum\limits_{i=1}^{\infty}p_i n_i\right]}.$$

Proof. Assume that transition of the chain starts at point 0. Then it jumps to the i^{th} loop with probability p_i where it spends exponentially distributed waiting time $\tau_{0\to 0}$ on the loop before it reaches central position 0 again. Then for any $f(x)\in\ell_{\mathbf{1}}^2\left(X,\pi\right)$ under condition $x\left(0\right)=0$, random variable

$$\mathcal{X}_1=\int\limits_{\tau_1}^{\tau_2} f\left(x\left(t\right)\right)\,dt=f(0)+\sum_{j=1}^{n_i} f\left(i,j\right)$$

can be expressed in terms of independent exponential random variables with parameter λ, i.e., $Exp\left(\lambda\right)$. Here assumption $f(x)\in\ell_{\mathbf{1}}^2\left(X,\pi\right)$ is equivalent to

$$f(0)+\sum_{i=1}^{\infty}p_i\sum_{j=1}^{n_i}f(i,j)=0 \iff \sum_{i=1}^{\infty}p_i\sum_{j=1}^{n_i}f(i,j)=-f(0) \tag{4.12}$$

Then, it is clear to see that

$$\begin{aligned}
\mathbb{E}_\pi[\mathcal{X}_1] &= \sum_{x\in X}\pi(i,j)\,\mathcal{X}_1=\frac{\pi(0)}{\lambda}\sum_{i=1}^{\infty}p_i\left[f(0)+\sum_{j=1}^{n_i}f(i,j)\right]\\
&= \frac{1}{\lambda\left[1+\sum\limits_{i=1}^{\infty}p_in_i\right]}\left\{f(0)\sum_{i=1}^{\infty}p_i+\sum_{i=1}^{\infty}p_i\sum_{j=1}^{n_i}f(i,j)\right\}\\
&= \frac{1}{\lambda\left[1+\sum\limits_{i=1}^{\infty}p_in_i\right]}\{f(0)-f(0)\}\\
&= 0
\end{aligned}$$

for $\sum\limits_{i=0}^{\infty}p_i=1$. This implies that variance is equivalent to the second moment and, we have

$$\begin{aligned}
Var_\pi[\mathcal{X}_1] &= \mathbb{E}_\pi\left[\mathcal{X}_1^2\right]=\sum_{x\in X}\pi(i,j)\,\mathcal{X}_1^2\\
&= \frac{\pi(0)}{\lambda^2}\sum_{i=0}^{\infty}p_i\left[f(0)+\sum_{j=1}^{n_i}f(i,j)\right]^2\\
&= \frac{1}{\lambda^2\left[1+\sum\limits_{i=1}^{\infty}p_in_i\right]}\sum_{i=0}^{\infty}p_i\left[f^2(0)+2f(0)\sum_{j=1}^{n_i}f(i,j)+\left(\sum_{j=1}^{n_i}f(i,j)\right)^2\right].
\end{aligned}$$

By substituting $f(0)$ found in (4.12) in the last equation, we get

$$= \frac{1}{\lambda^2\left[1+\sum\limits_{i=1}^{\infty} p_i n_i\right]} \sum_{i=0}^{\infty} p_i \left[f^2(0) - 2\sum_{i=1}^{\infty} p_i \sum_{j,k=1}^{n_i} f(i,j) f(i,k) \right.$$
$$\left. + \sum_{j=1}^{n_i} f^2(i,j) + 2 \sum_{j,k=1;k\neq j}^{n_i} f(i,j) f(i,k) \right]$$

$$= \frac{1}{\lambda^2\left[1+\sum\limits_{i=1}^{\infty} p_i n_i\right]} \left[f^2(0) \sum_{i=0}^{\infty} p_i - 2\left(\sum_{i=1}^{\infty} p_i\right)^2 \sum_{j,k=1}^{n_i} f(i,j) f(i,k) \right. \tag{4.13}$$
$$\left. + \sum_{i=0}^{\infty} p_i \sum_{j=1}^{n_i} f^2(i,j) + 2\sum_{i=0}^{\infty} p_i \sum_{j,k=1;k<j}^{n_i} f(i,j) f(i,k) \right].$$

Since $\sum\limits_{i=0}^{\infty} p_i = 1$, by substituting (4.11), equation (4.13) yields to

$$Var_{\pi}[\mathcal{X}_1] = \frac{1}{\lambda^2\left[1+\sum\limits_{i=1}^{\infty} p_i n_i\right]} \left[f^2(0) + \sum_{i=0}^{\infty} p_i \sum_{j=1}^{n_i} f^2(i,j) \right]$$
$$= \frac{f^2(0) + \sum\limits_{i=0}^{\infty} p_i \sum\limits_{j=1}^{n_i} f^2(i,j)}{\lambda^3 \mathbb{E}_{\pi}[\tau_{0\to 0}]}.$$

Here we have $\sigma^2(f) > 0$ because even tough transitions are deterministic, the time spent in each position is random. Therefore, the contribution of each loop into variance is strictly positive. Also, $\sigma^2(f) < \infty$ for any $f \in \ell^2_{\mathbf{1}}(X,\pi)$. ■

Secondly, we will examine the Martingale approach to prove CLT. The existence of the martingale differences and the variance $\sigma^2(f)$ for the continuous case are studied extensively by Bhattacharya [1] and Holzmann [13]. Consider the homological equation for the continuous-time model which has the form

$$f(x) = -(\mathcal{L}g)(x) \tag{4.14}$$

where $f(x) \in \ell^2_{\mathbf{1}}(X, \pi)$. Equation (4.14) has a unique solution but this solution is not necessarily from $\ell^2_{\mathbf{1}}(X, \pi)$. In the case $g(x) \in \ell^2_{\mathbf{1}}(X, \pi)$, we have the CLT in the following form.

Theorem 4.12 *If $f(x) \in \ell^2_{\mathbf{1}}(X, \pi)$ and the solution $g(x)$ of (4.14) is in $\ell^2_{\mathbf{1}}(X, \pi)$, then*

$$\frac{S(T)}{\sqrt{T}} = \frac{1}{\sqrt{T}} \left(\int_0^T f(x(t))\, dt \right) \underset{T \to \infty}{\overset{law}{\to}} \mathcal{N}(0, \sigma^2(f))$$

and

$$\sigma^2(f) = -(\mathcal{L}g(x), g(x)).$$

Proof. Consider the additive functional

$$S(T) = \int_0^T f(x(t))\, dt.$$

Since $f(x) \in \ell^2_{\mathbf{1}}(X, \pi)$, we have $(f, \mathbf{1})_\pi = 0$. Then due to ergodicity of the chain $x(t)$ with continuous time $0 \le t \le T$, we get

$$\frac{S(T)}{T} \underset{T \to \infty}{\to} (f, \mathbf{1})_\pi = 0 \qquad P - a.s.$$

The additive functional $S\left(T\right)$ can be approximated by the integral sum

$$S_{\triangle}\left(T\right)=\sum_{k=0}^{\left[\frac{T}{\triangle}\right]}\triangle f\left(x_{k\triangle}\right)$$

The sequence $x_{k\triangle}$, $k=0,1,\ldots,$ for fixed $\triangle$ is a Markov chain with discrete time, therefore we can apply the standard Martingale approach introduced in section 4.1.

Firstly, we will solve the homological equation for fixed $\triangle$:

$$\frac{g\left(x\right)-\left(\mathbb{P}_{\triangle}g\right)\left(x\right)}{\triangle}=-\left(\mathcal{L}_{\triangle}g\right)\left(x\right)=f(x)$$

for any $x\in X$. Then using this presentation of the homological equation, we expand the additive sum as

$$\begin{aligned}
S_{\triangle}\left(T\right) &= f(x\left(0\right))+f(x_{\triangle})+\cdots+f(x_{(N-1)\triangle}) \\
&= \frac{1}{\triangle}\left\{g(x\left(0\right))-(\mathbb{P}_{\triangle}g)(x\left(0\right))+g(x_{\triangle})-(\mathbb{P}_{\triangle}g)(x_{\triangle})\right. \\
&\qquad \left.+\cdots+g(x_{(N-1)\triangle})-(\mathbb{P}_{\triangle}g)(x_{(N-1)\triangle})\right\} \\
&= \frac{g(x\left(0\right))}{\triangle}+\underbrace{\frac{\left[g(x_{\triangle})-(\mathbb{P}_{\triangle}g)(x_0)\right]}{\triangle}}_{M_{\xi_{\triangle}}}+\underbrace{\frac{\left[g(x_{2\triangle})-(\mathbb{P}_{\triangle}g)(x_{\triangle})\right]}{\triangle}}_{M_{\xi_{2\triangle}}} \\
&\quad \cdots+\underbrace{\frac{\left[g(x_{(N-1)\triangle})-(\mathbb{P}_{\triangle}g)(x_{(N-2)\triangle})\right]}{\triangle}}_{M_{\xi_{(N-1)\triangle}}}-\frac{(\mathbb{P}_{\triangle}g)(x_{(N-1)\triangle})}{\triangle}
\end{aligned}$$

$$= \left[\frac{g(x(0)) - (\mathbb{P}_{\triangle} g)(x_{(N-1)\triangle})}{\triangle} \right] + M_{\xi_{\triangle}} + M_{\xi_{2\triangle}} + \cdots + M_{\xi_{(N-1)\triangle}}$$

for $N = \left[\frac{T}{\triangle} \right]$. The first term in (**??**) tends to $g(x(0)) - g(x(t))$ as $\triangle \to 0$, and the remaining terms yield to

$$-(\mathcal{L}_{\triangle} g)(x) = \frac{g(x) - (\mathbb{P}_{\triangle} g)(x)}{\triangle} \underset{\triangle \to 0}{\longrightarrow} -(\mathcal{L} g)(x)$$

for any $x \in X$, where $\mathcal{L}$ is the generator given by (4.9). Here we observe that, the sequence $M_{\xi_{k\triangle}}$, $k = 1, 2, \ldots,$ forms a martingale difference which yields to

$$-\sum_{k=1}^{N} \triangle (\mathcal{L}_{\triangle} g)(x_{k\triangle}) = -\int_{0}^{T} (\mathcal{L} g)(x(s))\, ds + \overline{\overline{O}}(1) \qquad P - a.s.$$

Finally, we can rewrite the additive sum as

$$\begin{aligned} S(t) &= \int_{0}^{t} f(x(s))\, ds = g(x(0)) - g(x(t)) - \int_{0}^{t} (\mathcal{L} g)(x(s))\, ds \\ &= g(x(0)) - g(x(t)) - \widetilde{S}(t) \end{aligned}$$

Here the additive functional

$$\widetilde{S}(t) = \int_{0}^{t} (\mathcal{L} g)(x(s))\, ds$$

is a martingale with continuous time $0 \leq t \leq T$, and $g(x)$ is the solution of the homological equation given by (4.14), that is,

$$-\left(\mathcal{L}g\right)(x) = -\lim_{\triangle\to 0}(\mathcal{L}_{\triangle}g)(x) = f(x)$$

This is true due to the fact that, the conjugated equation $\mathcal{L}^*\pi = 0$, equivalently $\pi\mathcal{L} = 0$, has a unique solution up to the constant factor since for invariant distribution $\pi(x)$, the assumption $f \in \ell^2_{\mathbf{1}}(X, \pi)$ implies

$$(f, \mathbf{1})_\pi = -\left(\mathcal{L}g, \mathbf{1}\right)_\pi = -\left(g, \mathcal{L}^*\mathbf{1}\right)_\pi = 0$$

Note that, we use the Fredholm alternative here.

Now consider the discrete-time homological equation given by

$$g - \mathbb{P}_{\triangle}g = f$$

which has a solution in $\ell^2_{\mathbf{1}}(X, \pi)$ given as

$$g = \left(\mathbf{I} - \mathbb{P}_{\triangle}\right)^{-1} f$$

Now adapting this form for the current case, we have

$$\left(\frac{\mathbf{I} - \mathbb{P}_{\triangle}}{\triangle}\right) g_{\triangle} = f \tag{4.15}$$

where

$$g_{\triangle} = \triangle f + \triangle\mathbb{P}_{\triangle} f + \triangle\mathbb{P}_{2\triangle} f + \cdots \tag{4.16}$$

for $\mathbb{P}^k_\triangle = \mathbb{P}_{k\triangle}$, $k = 1, 2, \ldots$ Then as $\triangle \to 0$, equation (4.15) yields to

$$-\left(\mathcal{L}g\right)(x) = f(x)$$

by the definition of the infinitesimal generator, and the solution in (4.16) has the form

$$g = \int_0^\infty \mathbb{P}_s f \; ds.$$

■

4.3 Continuous-time Loop Markov Chain on Continuous Phase Space

Construction of this model is the same as the first two models. In this model, the phase space X is a subset of $\mathbb{R}^2$ and time is continuous with deterministic motion along each loop. Consider the family of semicircles of length $\ell_a = \frac{\pi a}{2}$, $a \geq 0$. The transition of the Markov chain occurs as follows. It starts from the central position 0 and jumps along the $a - axis$ with the distribution density function $p(a)$. Here we will assume that the function $p(a)$ is continuous and fast decreasing function for $a \downarrow 0$ to avoid the possibility of having a large number of short jumps. Then from point a, it moves along the circle S_a with constant speed $v = \frac{2}{\pi}$, and then returns to central position 0 after time $\tau_{0\to 0} = \tau_1 = \frac{\ell_a}{v} = \frac{a\pi^2}{4}$. Therefore, $p(a)$ is the distribution of the moment of first return τ_1 , too. Once the chain $x(t)$ returns to central position 0, it jumps to another semicircle and a similar scheme occurs. We call this process the continuous-time Markov chain on continuous phase space.

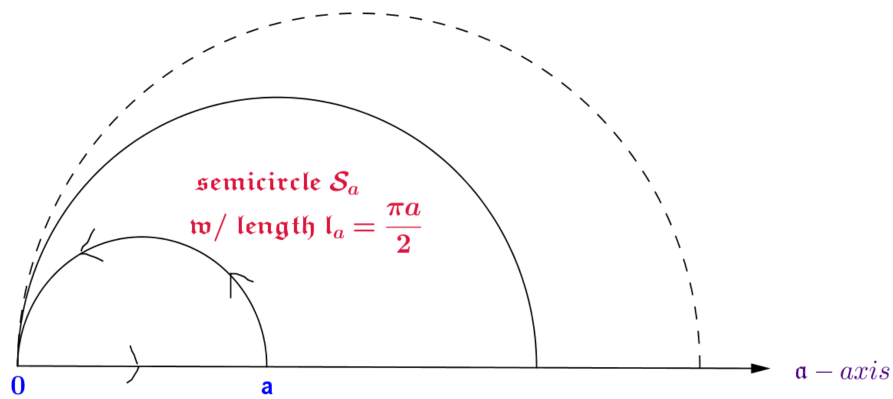

Figure 4.3: Continuous-time Loop Markov chain with continuous phase space

Now consider the function $f(\cdot)$ defined on the set of semicircles $\{S_a,\ a > 0\}$. Without loss of generality, one can assume that this function is constant on each S_a. Therefore, we assume it equals to the mean value of $f(\cdot)$ on S_a, i.e.,

$$f = \overline{f}(a) = \frac{2}{\pi a} \int_0^{\frac{\pi a}{2}} f(x(t))dt$$

Consider the integral on time interval $[0, T]$

$$\int_0^T f(x(t))dt = \int_0^T \overline{f}(x(t))dt - \int_0^T \left[\overline{f}(x(t)) - f(x(t)) \right] dt \tag{4.17}$$

The last integral in equation (4.17) is the only part that has a non-zero contribution to the process since it is the last and the only incomplete loop. Thus, it has no significant contribution to the limiting distribution. Now consider the *dissection formula* has the form

$$S(T) = \int_0^T f(x(t))\,dt = \int_0^{\tau_1} f(x(t))\,dt + \int_{\tau_1}^{\tau_2} f(x(t))\,dt + \cdots + \int_{\tau_{l(t)}}^{T} f(x(t))\,dt$$

where $0 < \tau_i < T$, $i = 1, \ldots, n$, are consecutive return times to the central position 0, namely motion times on semicircles. Then, for finite number of functions $f_1(a), \ldots, f_n(a)$ and n loops with motion times $\tau_1, \ldots, \tau_n$, we have the additive functionals

$$S_i(n) = \int_0^{\tau_1 + \ldots + \tau_n} f_i(x(t))dt = \overline{f}_i(\tau_1) + \ldots + \overline{f}_i(\tau_n) \quad \text{for} \quad i = \overline{1, n}$$

which are the sums of i.i.d. random vectors. These i.i.d. random vectors form the following system

$$\overrightarrow{X_i} = \left(\overline{f}_1(\tau_i), \ldots, \overline{f}_n(\tau_i)\right) \quad \text{for} \quad i = 1, \ldots, n$$

which depends only on the single random variable τ_i. This random variable τ_i has a very degenerated distribution in $\mathbb{R}^n$, i.e., it only takes a single value in $\mathbb{R}^n$. However, under some non-degeneracy conditions of the curvature of the curve

$$\overrightarrow{\gamma}(t) = (\overline{f}_1(t), i = 1, 2, \ldots, n) \in \mathbb{R}^n$$

the sum $\overrightarrow{S}_n = \sum_{i=1}^{n} \overrightarrow{X_i}$ will satisfy the local CLT for the densities which opens the possibility to prove the limit theorem with stable limiting distribution for the vast class of the additive functionals on the Loop Markov chain. The central technical tool here is the stationary phase method.

4.4 Random Number Generators

A Random Number Generator (RNG) is a computational or physical device that outputs sequences of numbers that can be predicted only by a random chance. It has applications in gambling, cryptography (security), statistical sampling, computer simulation, etc. The most efficient LCG's have a modulo equal to a power of 2, most often 2^{32} or 2^{64}, because this allows the modulus operation to be computed by merely truncating all but the right-most 32 or 64 bits. Microsoft Visual C++ uses modulo 2^{32}, and MMIX (by Donald Knuth) uses modulo 2^{64} while Java uses 2^{48}.

In the case of Loop Markov chain with countable phase space, we will have the following situation. Consider the phase space $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{32}\}$ such that

$$X = \varepsilon_1.2^{31} + \varepsilon_2.2^{30} + \ldots + \varepsilon_{31}.2^1 + \varepsilon_{32}.$$

Note that, here $Card\{X\} = 2^{32}$. Next, consider the random number generator given by the recursive formula

$$x_{n+1} = (ax_n + b) \mod 2^{32} \tag{4.18}$$

under the condition $x_0 = 0$ This RNG provided by equation (4.18) is called Linear Congruential Generator (LCG). In this generator, if $b = 0$ then the generator is often called a Multiplicative Congruential Generator (MCG) or Lehmer RNG. If $b \neq 0$ then it is referred to as a Mixed Congruential Generator (MCG).

However, the problem even with the most efficient generators is that after long runs of randomness, the process repeats itself or the memory usage grows without bound. In the case of discrete-time Loop Markov chain with countable phase space, we can consider the Compound RNG's. Compound RNG's compounds or intermixes different sequences of random numbers to create a new sequence. Therefore, this type of generators has very efficient randomness properties. Moreover, they have longer periods which are not periodic. Hence, they are the best fit for our model.

Now assume that we have a family of the generators $\mathcal{G}^{(i)}$ that is defined by the recurrence relation

$$x_{n+1}^{(i)} = \left(a_i x_n^{(i)} + b_i\right) \mod 2^N, \ \ N = 32, 36, 48$$

for countable loops $i = 1, 2, \ldots$ The sequence of the real random numbers generated by this particular model is uniformly distributed on $(0, 1)$. Then by choosing an appropriate transformation, we can map these real values to the positions on the i^{th} loop.

CHAPTER 5: STABLE LIMITING LAWS FOR THE LOOP MARKOV CHAINS

5.1 Stable Limiting Laws on Countable Phase Space

The major property of Loop Markov chain is the fact that all additive functionals are the functions of the moment of first return τ. We will illustrate this idea on the following example. Assume that $n_k = k$, $k \geq 1$, i.e.; the successive loops have linearly increasing length. Now let p_k, $k \geq 1$, be the transition probabilities from 0 to $(k, 1)$. Here for our purposes, we will consider the Pareto type model which is given in the form

$$p_k = \frac{c(\alpha)}{k^{\alpha+1}} \text{ for } \alpha > 0$$

where

$$c(\alpha) = \frac{1}{\zeta(\alpha+1)}$$

and

$$\zeta(\alpha+1) = \sum_{k=1}^{\infty} \frac{1}{k^{\alpha+1}} \tag{5.1}$$

is the Riemann $\zeta-$function. Then random variable τ will have the distribution

$$p_k = \mathbb{P}\{\tau = k\} = \frac{1}{\zeta(\alpha+1)\, k^{\alpha+1}}, \quad k \geq 1 \tag{5.2}$$

Next consider the function $f(k,j)$ on the states of the Loop Markov chain. It can be presented, as we did in Chapter 4, in the form

$$f = \overline{f} + \widetilde{f} \quad \Longleftrightarrow \quad \widetilde{f} = f - \overline{f}$$

which yields to

$$\overline{f}(k,j) = \overline{f}(k) = \frac{1}{k}\sum_{i=1}^{k} f(k,i)$$

i.e., the mean value of $f(\cdot)$ on the k^{th} loop. This implies that

$$\sum_{j=1}^{k} \widetilde{f}(k,j) = 0$$

and for the additive functional

$$\widetilde{S}(T) = \sum_{t=0}^{T-1} \widetilde{f}(x(t)),$$

the only part that has non-zero contribution is the last incomplete loop. However, under the minimal condition of the boundedness of f, this contribution will be neglected. Therefore, we will assume $f(k,j) = f(k)$ from now on.

Next, consider the moment of returns $\tau_{(\cdot)}$ from 0 to 0. If $\tau_1, \ldots, \tau_m$ are successive returns, then

$$\sum_{t=1}^{\tau_1+\cdots+\tau_m} f(x_t) = \tau_1 f(\tau_1) + \cdots + \tau_m f(\tau_m).$$

For example, if $f(k) = 1$ then

$$\sum_{t=1}^{\tau_1+\cdots+\tau_m} f(x_t) = \tau_1 + \cdots + \tau_m$$

and for $f(k) = k^{\beta}$, we have

$$\sum_{t=1}^{\tau_1+\cdots+\tau_m} f(x_t) = k^{1+\beta}, \text{ etc.}$$

Now we will assume that the first return time $\tau_1 = \tau$ belongs to the domain of attraction of the Gaussian law $\mathcal{N}(0,1)$, i.e., it has the finite second moment. Let

$$\mu = \mathbb{E}[\tau] = \sum_{k=1}^{\infty} k\, p_k \tag{5.3}$$

and

$$\sigma^2 = \sum_{k=1}^{\infty} (k-\mu)^2\, p_k = \sum_{k=1}^{\infty} k^2\, p_k - \mu^2. \tag{5.4}$$

Especially, in discrete Pareto law under the condition $\alpha > 2$, we have the following results. By plugging (5.1) and (5.2) in equation (5.3), we get the first and the second moments as

$$\mu = \mathbb{E}[\tau] = \sum_{k=1}^{\infty} k \frac{1}{\zeta(\alpha+1)\, k^{\alpha+1}} = \sum_{k=1}^{\infty} \frac{1}{\zeta(\alpha+1)\, k^{\alpha}} = \frac{\zeta(\alpha)}{\zeta(\alpha+1)} \tag{5.5}$$

and

$$\mathbb{E}\left[\tau^2\right] = \sum_{k=1}^{\infty} k^2 \frac{1}{\zeta(\alpha+1)\, k^{\alpha+1}} = \sum_{k=1}^{\infty} \frac{1}{\zeta(\alpha+1)\, k^{\alpha-1}} = \frac{\zeta(\alpha-1)}{\zeta(\alpha+1)}$$

Thus, the variance in (5.4) becomes

$$
\begin{aligned}
\sigma^2 &= \mathbb{E}\left[\tau^2\right] - \left(\mathbb{E}\left[\tau\right]\right)^2 \\
&= \frac{\zeta\left(\alpha-1\right)}{\zeta\left(\alpha+1\right)} - \left(\frac{\zeta\left(\alpha\right)}{\zeta\left(\alpha+1\right)}\right)^2 \qquad (5.6) \\
&= \frac{\zeta\left(\alpha-1\right)\zeta\left(\alpha+1\right) - \zeta^2\left(\alpha\right)}{\zeta^2\left(\alpha+1\right)}.
\end{aligned}
$$

Next, we will introduce some notations and basic facts about the Renewal theory by Cox [7] . Consider the sum of moment of returns

$$S_n = \tau_1 + \cdots + \tau_n.$$

Let

$$N\left(T\right) = \min\left\{n \mid S_n < T \leq S_{n+1}\right\},$$

that is, $N\left(T\right)$ is the number of returns from central position 0 to 0 at the time interval $[0, T]$ under the condition that the chain started transition at point 0. For the random variable $N\left(T\right)$, one can apply the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT), i.e.,

$$\frac{S_n}{n} \underset{n\to\infty}{\overset{P-a.s.}{\longrightarrow}} \mu \qquad (5.7)$$

$$\frac{N(T)}{T} \underset{T\to\infty}{\overset{P-a.s.}{\longrightarrow}} \frac{1}{\mu} \tag{5.8}$$

and

$$\lim_{T\to\infty} \Pr\left\{\frac{S_n - \mu\cdot n}{\sqrt{\sigma^2 n}} \le x\right\} \longrightarrow \varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

These results are well known. However, the CLT for $N(T)$ is less trivial. In this case, we have the following result. First of all, note that from (5.7) and (5.8), $N(T) = \frac{nT}{S_n}$ we have

$$N(T) < r_T \iff \frac{nT}{S_n} < r_T \iff \frac{S_n}{n} r_T > T \iff S_{r_T} > T$$

This leads to

$$\begin{aligned} \Pr\{N(T) < r_T\} &= \Pr\{S_{r_T} > T\} \\ &= \Pr\left\{\frac{S_{r_T} - \mu\cdot r_T}{\sigma\sqrt{r_T}} > \frac{T - \mu\cdot r_T}{\sigma\sqrt{r_T}}\right\} \\ &= \Pr\left\{\frac{S_{r_T} - \mu\cdot r_T}{\sigma\sqrt{r_T}} > \frac{\frac{T}{\mu} - r_T}{\frac{\sigma}{\mu}\sqrt{r_T}}\right\} \end{aligned} \tag{5.9}$$

By letting $r_T = \frac{T}{\mu} + y\,\sigma\sqrt{\frac{T}{\mu^3}}$ in (5.9), we have

$$\begin{aligned} \Pr\{N(T) < r_T\} &= \Pr\left\{N(T) - \frac{T}{\mu} < y\,\sigma\sqrt{\frac{T}{\mu^3}}\right\} \\ &= \Pr\left\{\frac{S_{r_T} - \mu\cdot r_T}{\sigma\sqrt{r_T}} > -y\right\} \end{aligned}$$

It means that

$$\frac{N(T) - \frac{T}{\mu}}{\sigma\sqrt{\frac{T}{\mu^3}}} \underset{T\to\infty}{\overset{law}{\to}} \mathcal{N}(0,1)$$

i.e., the random variable $N(T)$ for large T is asymptotically normal with mean value $\frac{T}{\mu}$ and the variance $\frac{\sigma^2}{\mu^3}T$.

Now consider the random variable τ along with the functions of τ which are given as

$$\begin{aligned} \varphi_1(T) &= \sum_{k=1}^{N(T)} \tau_k f_1(\tau_k) \\ \varphi_2(T) &= \sum_{k=1}^{N(T)} \tau_k f_2(\tau_k) \\ &\vdots \\ \varphi_m(T) &= \sum_{k=1}^{N(T)} \tau_k f_m(\tau_k) \end{aligned}$$

We want to find their joint distribution for $T \to \infty$. In chapter 4, we have already discussed the cases of the attraction to the Gaussian law. Now we will study the case of stable limiting laws. First of all, we will study the behavior of the polynomial vector (τ, τ^2, τ^3) in detail. To do this, we let

$$\begin{aligned} \varphi_1(T) &= \sum_{k=1}^{N(T)} \tau_k \\ \varphi_2(T) &= \sum_{k=1}^{N(T)} \tau_k^2 \\ \varphi_3(T) &= \sum_{k=1}^{N(T)} \tau_k^3 \end{aligned}$$

and assume that $\tau.$ has the discrete Pareto law with parameter α, that is,

$$\Pr\{\tau. = k\} = \frac{1}{\zeta(\alpha+1)\,k^{\alpha+1}}, \;\; k \geq 1$$

Note that for $a \to \infty$, we have

$$\Pr\{\tau > a\} \sim \frac{1}{\zeta(\alpha+1)} \int_a^\infty \frac{dx}{x^{\alpha+1}} = \frac{1}{\zeta(\alpha+1)} \left(\frac{1}{\alpha a^\alpha}\right) = \frac{c_0}{a^\alpha}$$

$$\Pr\left\{\tau^2 > a\right\} = \Pr\left\{\tau > a^{\frac{1}{2}}\right\} \sim \frac{1}{\zeta(\alpha+1)} \int_{a^{\frac{1}{2}}}^\infty \frac{dx}{x^{\alpha+1}} = \frac{1}{\zeta(\alpha+1)} \left(\frac{1}{\alpha a^{\frac{\alpha}{2}}}\right) = \frac{c_0}{a^{\frac{\alpha}{2}}}$$

and

$$\Pr\left\{\tau^3 > a\right\} = \Pr\left\{\tau > a^{\frac{1}{3}}\right\} \sim \frac{1}{\zeta(\alpha+1)} \int_{a^{\frac{1}{3}}}^\infty \frac{dx}{x^{\alpha+1}} = \frac{1}{\zeta(\alpha+1)} \left(\frac{1}{\alpha a^{\frac{\alpha}{3}}}\right) = \frac{c_0}{a^{\frac{\alpha}{3}}}$$

Here note that, for negative a, that is $a = -b$ for $b > 0$,

$$\Pr\left\{\tau^2 < -b\right\} = \Pr\left\{\tau^3 < -b\right\} = 0$$

Due to the general theory of stable limiting law (see [11]), the random variable τ^2 belongs to the domain of attraction $St\left(\frac{\alpha}{2}, 1\right)$ with parameters $\frac{\alpha}{2}$ and $\beta = 1$ only if $\frac{\alpha}{2} < 2 \iff \alpha < 4$. Similarly, τ^3 belongs to the domain of attraction $St\left(\frac{\alpha}{3}, 1\right)$ with the parameter $\frac{\alpha}{3} < 2 \iff \alpha < 6$.

Next assume that $\alpha \in (2, 3)$. Then for any real number x, we have

$$\Pr\left\{\frac{\tau_1 + \cdots + \tau_n - n\,\mu_1}{\sqrt{n\sigma_1^2}} < x\right\} \underset{n\to\infty}{\to} \varphi(x)$$

where

$$\mu_1 = \mathbb{E}\left[\tau\right] = \frac{\zeta\left(\alpha\right)}{\zeta\left(\alpha+1\right)} \quad \text{and} \quad \sigma_1^2 = \frac{\zeta\left(\alpha-1\right)\zeta\left(\alpha+1\right)-\zeta^2\left(\alpha\right)}{\zeta^2\left(\alpha+1\right)}$$

by equations (5.5) and (5.6) . Similarly,

$$\Pr\left\{\frac{\tau_1^2+\cdots+\tau_n^2-n\,\mu_2}{n^{\frac{2}{\alpha}}}<x\right\}\underset{n\to\infty}{\longrightarrow} St_{\frac{\alpha}{2},1}\left(x\right)$$

and

$$\Pr\left\{\frac{\tau_1^3+\cdots+\tau_n^3-n\,\mu_3}{n^{\frac{3}{\alpha}}}<x\right\}\underset{n\to\infty}{\longrightarrow} St_{\frac{\alpha}{3},1}\left(x\right)$$

Next, we have to find the joint distribution of

$$S_{1,n}=\sum_{k=1}^{n}\tau_k\ ,\ S_{2,n}=\sum_{k=1}^{n}\tau_k^2\ \text{ and }\ S_{3,n}=\sum_{k=1}^{n}\tau_k^3$$

First of all, we have to find corresponding Lévy measure £. For τ^2 and τ^3, we consider

$$n\lim_{n\to\infty}\Pr\left\{\frac{\tau^2}{n^{\frac{2}{\alpha}}}\in\left[a_2,b_2\right],\ \frac{\tau^3}{n^{\frac{3}{\alpha}}}\in\left[a_3,b_3\right]\right\}$$

$$=\ \lim_{n\to\infty}\Pr\left\{\tau\in\left[a_2n^{\frac{1}{\alpha}},b_2n^{\frac{1}{\alpha}}\right],\ \tau\in\left[a_3n^{\frac{1}{\alpha}},b_3n^{\frac{1}{\alpha}}\right]\right\}$$

$$=\ c\left(\alpha\right)\left(a_2\wedge a_3\right)\left(b_2\wedge b_3\right)$$

Then the limiting Lévy measure £ contains atom $\sigma_1^2\delta_0(\lambda_1)$ at the central position 0 which corresponds to the Gaussian component ζ generated by τ, and singular continuous part, i.e.,

$$£([a_2, b_2] \times [a_3, b_3]) = (b_2 \wedge b_3) - (a_2 \wedge a_3) \tag{5.10}$$

The limiting distribution is not a standard stable law since we used different normalization for the various components; instead, it is called operator's stable distribution. It has the Gaussian component ζ with variance σ_1^2, which is independent on the vector (ζ_2, ζ_3). Note that vector (ζ_2, ζ_3) has the infinite-divisible distribution occupied with the Lévy measure £ given by (5.10). This measure is singular, and there is the question of the joint distribution density for (ζ_2, ζ_3).

The following lemma is based on general results by Yurinskii [20].

Lemma 5.1 *The operator's stable distribution occupied with the Lévy measure £ in* (5.10) *has bounded, continuous, and even of the class* $\mathbf{C}^\infty$ *distribution density.*

Proof. Consider the sequence of the i.i.d. random variables $Y_1, \ldots Y_n, \ldots$ which have the distribution $St\left(\frac{\alpha}{2}, 1\right)$. It is well known that the density $p_{\frac{\alpha}{2},1}(x)$ of this law satisfies all three conditions from the fundamental Lemmas 5 and 6 by Yurinskii [20] :

$$\left|p_{\frac{\alpha}{2},1}(x)\right| \leq c_1$$

$$\int_{\mathbb{R}} \left|p'_{\frac{\alpha}{2},1}(x)\right| dx < \infty$$

and

$$\int_{\mathbb{R}} |x|^{\beta} \left(p_{\frac{\alpha}{2},1}(x)\right) dx < \infty \quad \text{for } \ 0 < \beta < \frac{\alpha}{2}$$

Now apply these results to the sequence of the random vectors

$$\overrightarrow{Y}_i = \left(Y_i, Y_i^{\frac{3}{2}}\right) \quad \text{for} \quad i = 1, 2, \ldots$$

Note that the vector

$$\left(\frac{Y_1 + \cdots + Y_n - n\,\mu}{n^{\frac{2}{\alpha}}}, \frac{Y_1^{\frac{3}{2}} + \cdots + Y_n^{\frac{3}{2}}}{n^{\frac{3}{\alpha}}}\right)$$

has the limiting law with the Lévy measure £. Moreover, it has C^{∞} limit distribution due to Yurinskii [20]. ■

Here by combining the result from Lemma 5.1 and the fact that random walk on $\mathbb{Z}^3$ given by $\sum_{k=0}^{n-1} \left(\tau_k, \tau_k^2, \tau_k^3\right)$, we will obtain the local limit theorem.

Theorem 5.2 *For any integer k_1, k_2 and k_3 such that*

$$\left|\frac{k_1 - n\,\mu_1}{\sigma_1\sqrt{n}}\right| \le A_1, \qquad \left|\frac{k_2 - n\,\mu_2}{n^{\frac{2}{\alpha}}}\right| \le A_2, \qquad \left|\frac{k_3}{n^{\frac{3}{\alpha}}}\right| \le A_3$$

for fixed constants A_1, A_2, A_3. Then as $n \to \infty$

$$\Pr\left\{\sum_{i=1}^{n} \tau_i = k_1, \sum_{i=1}^{n} \tau_i^2 = k_2, \sum_{i=1}^{n} \tau_i^3 = k_3\right\} = \frac{1}{n^{\frac{1}{2}+\frac{2}{\alpha}+\frac{3}{\alpha}}}\,\varphi\left(\frac{k_1 - n\,\mu_1}{\sigma_1\sqrt{n}}\right) St_*\left(\frac{k_2 - n\,\mu_2}{n^{\frac{2}{\alpha}}}, \frac{k_3}{n^{\frac{3}{\alpha}}}\right)$$

where $St_*\left(\frac{k_2 - n\,\mu_2}{n^{\frac{2}{\alpha}}}, \frac{k_3}{n^{\frac{3}{\alpha}}}\right)$ is two dimensional operator's stable density with the Lévy mea-

sure £.

All previous results are related to the summation of i.i.d. random vectors with the degenerated distribution. However, our primary goal here is to construct the limit theorem for the functionals $\varphi_1(T)$, $\varphi_2(T)$, $\varphi_3(T)$ where the summation of the random vectors is performed up to the random moment $N(T)$ which has its Gaussian limit distribution. Fortunately, the distribution is relatively simple due to asymptotic independence of $S_{1,n}$ from the pair $(S_{2,n}, S_{3,n})$, that is, the factorization of the limit distribution on Gaussian component for $S_{1,n}$ and operator's stable law for $(S_{2,n}, S_{3,n})$.

Theorem 5.3 *Consider the additive functionals*

$$\varphi_2(T) = \sum_{k=1}^{N(T)} \tau_k^2 \quad \textit{and} \quad \varphi_3(T) = \sum_{k=1}^{N(T)} \tau_k^3$$

Then for $T \to \infty$ the random vector

$$\left(\frac{\varphi_2(T) - \mu_2 \dfrac{T}{\mu_1}}{\left(\dfrac{T}{\mu_1}\right)^{\frac{2}{\alpha}}}, \frac{\varphi_3(T)}{\left(\dfrac{T}{\mu_1}\right)^{\frac{3}{\alpha}}} \right)$$

converges in law to the operator's stable distribution with the Lévy measure £.

Proof. Consider the probability

$$\Pr\left\{\varphi_2\left(T\right) \le \mu_2 \frac{T}{\mu_1} + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}},\ \varphi_3\left(T\right) \le a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\right\}$$

$$= \sum_{n\ge 1} \Pr\left\{N\left(T\right)=n, \sum_{k=1}^{N(T)} \tau_k^2 \le \mu_2 \frac{T}{\mu_1} + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}}, \sum_{k=1}^{N(T)} \tau_k^3 < a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\right\}$$

$$= \sum_{n\ge 1} \Pr\left\{\sum_{k=1}^{n} \tau_k \le T, \sum_{k=1}^{n+1} \tau_k > T\,, \sum_{k=1}^{n} \tau_k^2 \le \mu_2 \frac{T}{\mu_1} + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}}, \right.$$
$$\left. \sum_{k=1}^{n} \tau_k^3 < a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\right\}$$

$$= \sum_{n\ge 1}\sum_{x\ge n}^{T} \Pr\left\{\tau_1+\cdots+\tau_n = x,\ \ \tau_{n+1} > T-x\,,\right. \tag{5.11}$$
$$\left. \sum_{k=1}^{n} \tau_k^2 - n\,\mu_2 \le \mu_2 \frac{T}{\mu_1} - n\,\mu_2 + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}},\ \ \sum_{k=1}^{n} \tau_k^3 < a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\right\}$$

Due to asymptotic independence of $S_{1,n}$ and $\left(S_{2,n}, S_{3,n}\right)$, the probability in (5.11) is equivalent to

$$= \sum_{n\ge 1}\sum_{x\ge n}^{T} \left\{\Pr\left\{\tau_1+\cdots+\tau_n = x\right\} \Pr\left\{\tau_{n+1} = T-x\right\}\right.$$
$$\left. \Pr\left\{\sum_{k=1}^{n} \tau_k^2 - n\,\mu_2 \le \mu_2 \frac{T}{\mu_1} - n\,\mu_2 + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}},\ \ \sum_{k=1}^{n} \tau_k^3 < a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\right\}\right\}$$

But we know that with arbitrary probability close to 1, we have

$$N\left(T\right) = \frac{T}{\mu_1} + \underline{\underline{O}}\left(\sqrt{T}\right)$$

i.e.,

$$\mu_2\left(\frac{T}{\mu_1}-n\right)=\underline{\underline{O}}\left(\sqrt{T}\right).$$

Since $T^{\frac{2}{\alpha}} \gg \sqrt{T}$, we can neglect the term $\mu_2\dfrac{T}{\mu_1} - n\,\mu_2$ in the second factor which leads finally to the desired form of the theorem. ■

Remark 5.4 *These results can be extended for the general polynomial vectors* $\left(\tau,\tau^2,\tau^3,\tau^4\right)$ *;* $\left(\tau,\tau^2,\tau^3,\tau^4,\tau^5\right)$ *, etc.*

5.2 Stable Limiting Laws on Continuous Phase Space

In this case, we assume that semicircles s have linearly increasing lengths s_ℓ, $\ell \geq 1$. Then the chain starts from the central position 0, jumps along the $a - axis$, see Figure 4.3, with the distribution density function $p(\ell)$ such that

$$p(\ell)\,d\ell = \Pr\left\{\tau_{0\to 0}\in(\ell,\ell+d\ell)\right\}$$

and moves along the semicircle with constant speed $v=\frac{2}{\pi}$, i.e., after some time it returns to the point 0, etc. Here for our purposes, we will consider the continuous symmetric Pareto type model which is given in the form

$$p(\ell)=\frac{c(\alpha)}{\ell^{\alpha+1}}\ \text{ for }\ \alpha>0$$

where

$$c(\alpha)=\frac{1}{\zeta(\alpha+1)}$$

and

$$\zeta(\alpha+1)=\int_{1}^{\infty}\frac{1}{\ell^{\alpha+1}}d\ell \tag{5.12}$$

is the Riemann $\zeta-$function. Then random variable τ will have the distribution

$$p(\ell)=p(-\ell)=\mathbb{P}\{\tau=\ell\}=\frac{c(\alpha)}{\ell^{\alpha+1}}\mathcal{I}_{|\ell|\geq 1}\ \text{ for }\ \alpha>0 \text{ and } \ell\geq 1 \tag{5.13}$$

Next consider the function $f(\ell,s)$ on the semicircles of the continuous-time Loop Markov chain on the continuous phase space. Similar to the situation in Section 5.1, it can be written in the form

$$f=\overline{f}+\widetilde{f} \iff \widetilde{f}=f-\overline{f}$$

which yields to

$$\overline{f}(\ell,s)=\overline{f}(\ell)=\frac{1}{s_\ell}\int_{0}^{s_\ell}f(\ell,t)\,dt$$

i.e., mean value of $f(\cdot)$ on the s^{th} loop. This implies that

$$\int_{0}^{s_\ell}\widetilde{f}(\ell,t)\,dt=0$$

and for $\widetilde{S}(T)$ given as

$$\widetilde{S}(T)=\int_{0}^{T}\widetilde{f}(\ell,t)\,dt+\mathcal{R}$$

where $\mathcal{R}$ is the only part that has non-zero contribution since it is the last incomplete semicircle. However, under minimal condition of the boundedness of f this contribution will be neglected. Therefore, we will assume $f(\ell, s) = f(\ell)$ from now on.

Next, consider the moment of returns $\tau_{(\cdot)}$ from 0 to 0. If $\tau_1, \ldots, \tau_m$ are successive returns, then for $f(\ell) = \pm \ell^{\beta}$, we have

$$\int_{1}^{\tau_1 + \cdots + \tau_m} f(\ell)\, d\ell = \sum_{i=1}^{s_\ell} \varepsilon_i \tau_i^{\beta}$$

where ε_i are symmetric Bernoulli random variables. By assuming that the first return time $\tau_1 = \tau$ belongs to the domain of attraction of the Gaussian law $\mathcal{N}(0,1)$, we can calculate the mean and the variance since the second moment exists. Let

$$\mu = \mathbb{E}[\tau] = \int_{\ell=1}^{\infty} \ell\, p(\ell)\, d\ell \tag{5.14}$$

and

$$\sigma^2 = \int_{\ell=1}^{\infty} (\ell - \mu)^2\, p(\ell)\, d\ell = \int_{\ell=1}^{\infty} \ell^2 p(\ell)\, d\ell - \mu^2. \tag{5.15}$$

Especially, in continuous Pareto law under the condition $\alpha > 2$, we have the following results. By plugging (5.12) and (5.13) in equation (5.14), we get the first and the second moments as

$$\mu = \mathbb{E}[\tau] = \int_{\ell=1}^{\infty} \ell\, \frac{1}{\zeta(\alpha+1)\, \ell^{\alpha+1}} d\ell = \int_{\ell=1}^{\infty} \frac{1}{\zeta(\alpha+1)\, \ell^{\alpha}} d\ell = \frac{\zeta(\alpha)}{\zeta(\alpha+1)} \tag{5.16}$$

$$\mathbb{E}\left[\tau^2\right] = \int_{\ell=1}^{\infty} \ell^2\, \frac{1}{\zeta(\alpha+1)\, \ell^{\alpha+1}} d\ell = \int_{\ell=1}^{\infty} \frac{1}{\zeta(\alpha+1)\, \ell^{\alpha-1}} d\ell = \frac{\zeta(\alpha-1)}{\zeta(\alpha+1)}$$

Thus, the variance in (5.15) becomes

$$\begin{aligned}\sigma^2 &= \mathbb{E}\left[\tau^2\right] - \left(\mathbb{E}\left[\tau\right]\right)^2 \\ &= \frac{\zeta\left(\alpha-1\right)}{\zeta\left(\alpha+1\right)} - \left(\frac{\zeta\left(\alpha\right)}{\zeta\left(\alpha+1\right)}\right)^2 \\ &= \frac{\zeta\left(\alpha-1\right)\zeta\left(\alpha+1\right) - \zeta^2\left(\alpha\right)}{\zeta^2\left(\alpha+1\right)}.\end{aligned} \tag{5.17}$$

Next, we will introduce some notations and basic facts about the Renewal theory by Cox [7] . Consider the sum of moment of returns

$$S_{s_\ell} = \tau_1 + \cdots + \tau_{s_\ell}.$$

Let

$$N\left(T\right) = \min\left\{s_\ell \mid S_{s_\ell} < T \leq S_{s_\ell+1}\right\}$$

that is, $N\left(T\right)$ is the number of returns from central position 0 to 0 at the time interval $[0, T]$ under the condition that the chain started transition at point 0. Recall from Section 5.1; we already know that

$$\frac{N\left(T\right) - \frac{T}{\mu}}{\sigma\sqrt{\frac{T}{\mu^3}}} \underset{T\to\infty}{\overset{law}{\longrightarrow}} \mathcal{N}(0,1)$$

i.e., the random variable $N\left(T\right)$ for large T is asymptotically normal with mean value $\frac{T}{\mu}$ and the variance $\frac{\sigma^2}{\mu^3}T$.

Next, consider the random variable τ along with the functions of τ which are given as

$$\varphi_1(T) = \int_{\ell=1}^{N(T)} \tau_\ell\, f_1(\tau_\ell)\ d\ell$$
$$\varphi_2(T) = \int_{\ell=1}^{N(T)} \tau_\ell\, f_2(\tau_\ell)\ d\ell$$
$$\vdots$$
$$\varphi_m(T) = \int_{\ell=1}^{N(T)} \tau_\ell\, f_m(\tau_\ell)\ d\ell$$

Similar to the previous case, we want to find their joint distribution for $T \to \infty$, and then we want to study the case of stable limiting laws. We will start with the analysis of the behavior of the polynomial vector (τ, τ^2, τ^3) in detail. We let

$$\varphi_1(T) = \int_{\ell=1}^{N(T)} \tau_\ell\, d\ell$$
$$\varphi_2(T) = \int_{\ell=1}^{N(T)} \tau_\ell^2\, d\ell$$
$$\varphi_3(T) = \int_{\ell=1}^{N(T)} \tau_\ell^3\, d\ell$$

and assume that $\tau.$ has the symmetric Pareto law with parameter $\alpha > 0$, that is,

$$p(\ell) = p(-\ell) = \mathbb{P}\{\tau. = \ell\} = \frac{c(\alpha)}{\ell^{\alpha+1}}\mathcal{I}_{|\ell|\geq 1} \ \text{ for } \ell \geq 1$$

Note that for $a \to \infty$, we have

$$\Pr\{\tau > a\} \sim \frac{1}{\zeta(\alpha+1)} \int_a^{\infty} \frac{dx}{x^{\alpha+1}} = \frac{1}{\zeta(\alpha+1)} \left(\frac{1}{\alpha a^{\alpha}}\right) = \frac{c_0}{a^{\alpha}}$$

$$\Pr\left\{\tau^2 > a\right\} = \Pr\left\{\tau > a^{\frac{1}{2}}\right\} \sim \frac{1}{\zeta(\alpha+1)} \int_{a^{\frac{1}{2}}}^{\infty} \frac{dx}{x^{\alpha+1}} = \frac{1}{\zeta(\alpha+1)} \left(\frac{1}{\alpha a^{\frac{\alpha}{2}}}\right) = \frac{c_0}{a^{\frac{\alpha}{2}}}$$

and

$$\Pr\left\{\tau^3 > a\right\} = \Pr\left\{\tau > a^{\frac{1}{3}}\right\} \sim \frac{1}{\zeta(\alpha+1)} \int_{a^{\frac{1}{3}}}^{\infty} \frac{dx}{x^{\alpha+1}} = \frac{1}{\zeta(\alpha+1)} \left(\frac{1}{\alpha a^{\frac{\alpha}{3}}}\right) = \frac{c_0}{a^{\frac{\alpha}{3}}}$$

Here note that, for negative a, that is $a = -b$ for $b > 0$,

$$\Pr\left\{\tau^2 < -b\right\} = \Pr\left\{\tau^3 < -b\right\} = 0$$

Due to the general theory of $\alpha-$stable limiting law, the random variable τ^2 belong to the domain of attraction $St\left(\frac{\alpha}{2}, 0\right)$ with parameters $\frac{\alpha}{2}$ and $\beta = 1$ only if $\frac{\alpha}{2} < 2 \iff \alpha < 4$. Similarly, τ^3 belongs to the domain of attraction $St\left(\frac{\alpha}{3}, 0\right)$ with the parameter $\frac{\alpha}{3} < 2 \iff \alpha < 6$.

Remark 5.5 *Since $\tau.$ has the symmetric Pareto law we have $\beta = 0$. This means τ^2 and τ^3 has $\alpha-$stable limiting law.*

Next assume that $\alpha \in (2, 3)$. Then for any real number x, we have

$$\Pr\left\{\frac{\tau_1 + \cdots + \tau_{s_\ell} - s_\ell\, \mu_1}{\sqrt{s_\ell \sigma_1^2}} < x\right\} \underset{s_\ell \to \infty}{\to} \varphi(x)$$

where

$$\mu_1 = \mathbb{E}\left[\tau\right] = \frac{\zeta\left(\alpha\right)}{\zeta\left(\alpha+1\right)} \quad \text{and} \quad \sigma_1^2 = \frac{\zeta\left(\alpha-1\right)\zeta\left(\alpha+1\right)-\zeta^2\left(\alpha\right)}{\zeta^2\left(\alpha+1\right)}$$

by equations (5.5) and (5.6) . Similarly,

$$\Pr\left\{\frac{\tau_1^2+\cdots+\tau_{s_\ell}^2-s_\ell\,\mu_2}{s_\ell^{\frac{2}{\alpha}}}<x\right\}\underset{s_\ell\to\infty}{\to} St_{\frac{\alpha}{2},0}\left(x\right)$$

and

$$\Pr\left\{\frac{\tau_1^3+\cdots+\tau_{s_\ell}^3-s_\ell\,\mu_3}{s_\ell^{\frac{3}{\alpha}}}<x\right\}\underset{s_\ell\to\infty}{\to} St_{\frac{\alpha}{3},0}\left(x\right)$$

Next, we have to find the joint distribution of

$$S_{1,s_\ell}=\int_0^{s_\ell}\tau_\ell\,d\ell\ ,\ S_{2,s_\ell}=\int_0^{s_\ell}\tau_\ell^2\,d\ell\ \text{ and }\ S_{3,s_\ell}=\int_0^{s_\ell}\tau_\ell^3\,d\ell$$

Note that, the Lévy measure £ has the form

$$£\left(\left[a_2,b_2\right]\times\left[a_3,b_3\right]\right)=\left(b_2\wedge b_3\right)-\left(a_2\wedge a_3\right) \tag{5.18}$$

similar to the first case. Then Lemma 5.1 can be restated as follows.

Lemma 5.6 *The operator's stable distribution occupied with the Lévy measure £ in* (5.18) *has bounded, continuous, and even of the class* $\boldsymbol{C}^\infty$ *distribution density.*

Proof. Consider the sequence of the i.i.d. random variables $Y_1,\ldots Y_{s_\ell},\ldots$ which have the distribution $St\left(\frac{\alpha}{2}, 0\right)$. Then by Yurinskii [20] :

$$\left|p_{\frac{\alpha}{2},1}(x)\right| \leq c_1$$

and

$$\int_{\mathbb{R}} \left|p'_{\frac{\alpha}{2},1}(x)\right| dx < \infty$$

Now apply these results to the sequence of the random vectors

$$\overrightarrow{Y}_i = \left(Y_i, Y_i^{\frac{3}{2}}\right) \quad \text{for} \quad i = 1, 2, \ldots$$

Note that the vector

$$\left(\frac{Y_1 + \cdots + Y_{s_\ell} - s_\ell\,\mu}{s_\ell^{\frac{2}{\alpha}}}, \frac{Y_1^{\frac{3}{2}} + \cdots + Y_{s_\ell}^{\frac{3}{2}}}{s_\ell^{\frac{3}{\alpha}}}\right)$$

has the limiting law with the Lévy measure £. Moreover, it has C^∞ limit distribution. ■

Here by combining the result from Lemma 5.6 and the fact that random walk on $\mathbb{Z}^3$ given by $\int_0^{s_\ell} \left(\tau_k, \tau_k^2, \tau_k^3\right) d\ell$, we will obtain the local limit theorem.

Theorem 5.7 *For any integer k_1, k_2 and k_3 such that*

$$\left|\frac{k_1 - s_\ell\,\mu_1}{\sigma_1\sqrt{s_\ell}}\right| \leq A_1, \qquad \left|\frac{k_2 - s_\ell\,\mu_2}{s_\ell^{\frac{2}{\alpha}}}\right| \leq A_2, \qquad \left|\frac{k_3}{s_\ell^{\frac{3}{\alpha}}}\right| \leq A_3$$

for fixed constants A_1, A_2, A_3. *Then as* $s_\ell \to \infty$

$$\Pr\left\{\int_0^{s_\ell}\tau_i\,di = k_1, \int_0^{s_\ell}\tau_i^2\,di = k_2, \int_0^{s_\ell}\tau_i^3\,di = k_3\right\} = \frac{1}{s_\ell^{\frac{1}{2}+\frac{2}{\alpha}+\frac{3}{\alpha}}}\varphi\left(\frac{k_1 - s_\ell\,\mu_1}{\sigma_1\sqrt{s_\ell}}\right) St_*\left(\frac{k_2 - s_\ell\,\mu_2}{s_\ell^{\frac{2}{\alpha}}}, \frac{k_3}{s_\ell^{\frac{3}{\alpha}}}\right)$$

where $St_*\left(\frac{k_2 - s_\ell\,\mu_2}{s_\ell^{\frac{2}{\alpha}}}, \frac{k_3}{s_\ell^{\frac{3}{\alpha}}}\right)$ *is two dimensional operator's stable density with the Lévy measure £.*

Finally, we want to construct the limit theorem for the functionals $\varphi_1(T)$, $\varphi_2(T)$, $\varphi_3(T)$ where the summation of the random vectors is performed up to the random moment $N(T)$ which has its Gaussian limit distribution. Recall that, the distribution is relatively simple due to asymptotic independence of S_{1,s_ℓ} from the pair $(S_{2,s_\ell}, S_{3,s_\ell})$, that is, the factorization of the limit distribution on Gaussian component for S_{1,s_ℓ} and operator's stable law for $(S_{2,s_\ell}, S_{3,s_\ell})$.

Theorem 5.8 *Consider the functionals*

$$\varphi_2(T) = \int_{\ell=1}^{N(T)} \tau_\ell^2\,d\ell \quad \text{and} \quad \varphi_3(T) = \int_{\ell=1}^{N(T)} \tau_\ell^3\,d\ell$$

Then for $T \to \infty$ *the random vector*

$$\left(\frac{\varphi_2(T) - \mu_2\dfrac{T}{\mu_1}}{\left(\dfrac{T}{\mu_1}\right)^{\frac{2}{\alpha}}}, \frac{\varphi_3(T)}{\left(\dfrac{T}{\mu_1}\right)^{\frac{3}{\alpha}}}\right)$$

converges in law to the operator's stable distribution with the Lévy measure £.

Proof. Consider the probability

$$\Pr\left\{\varphi_2(T) \le \mu_2 \frac{T}{\mu_1} + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}},\ \ \varphi_3(T) \le a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\right\}$$

$$= \sum_{s_\ell \ge 1} \Pr\left\{N(T) = s_\ell, \int_{\ell=1}^{N(T)} \tau_\ell^2 \, d\ell \ \le \mu_2 \frac{T}{\mu_1} + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}}, \int_{\ell=1}^{N(T)} \tau_\ell^3 \, d\ell < a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\right\}$$

$$= \sum_{s_\ell \ge 1} \Pr\left\{\int_0^{s_\ell} \tau_\ell \, d\ell \le T, \int_0^{s_\ell+1} \tau_\ell \, d\ell > T\ , \int_0^{s_\ell} \tau_\ell^2 \, d\ell \le \mu_2 \frac{T}{\mu_1} + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}}\ , \int_0^{s_\ell} \tau_\ell^3 \, d\ell < a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\right\}$$

$$= \sum_{s_\ell \ge 1} \int_{x \ge s_\ell}^{T} \Pr\Bigg\{\tau_1 + \cdots + \tau_{s_\ell} = x,\ \ \tau_{s_\ell+1} > T - x\ , \quad (5.19)$$

$$\int_0^{s_\ell} \tau_\ell^2 \, d\ell - s_\ell\, \mu_2 \le \mu_2 \frac{T}{\mu_1} - s_\ell\, \mu_2 + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}},\ \ \int_0^{s_\ell} \tau_\ell^3 \, d\ell < a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\Bigg\}$$

Due to asymptotic independence of S_{1,s_ℓ} and $\left(S_{2,s_\ell}, S_{3,s_\ell}\right)$, the probability in (5.19) is equivalent to

$$= \sum_{s_\ell \ge 1} \int_{x \ge s_\ell}^{T} \Big\{\Pr\left\{\tau_1 + \cdots + \tau_{s_\ell} = x\right\} \Pr\left\{\tau_{s_\ell+1} = T - x\right\}$$

$$\Pr\left\{\int_0^{s_\ell} \tau_\ell^2 \, d\ell - s_\ell\, \mu_2 \le \mu_2 \frac{T}{\mu_1} - s_\ell\, \mu_2 + a_1\left(\frac{T}{\mu_1}\right)^{\frac{2}{\alpha}},\ \ \int_0^{s_\ell} \tau_\ell^3 \, d\ell < a_2\left(\frac{T}{\mu_1}\right)^{\frac{3}{\alpha}}\right\}\Big\}$$

But we know that with arbitrary probability close to 1, we have

$$N(T) = \frac{T}{\mu_1} + \underline{\underline{O}}\left(\sqrt{T}\right)$$

i.e.,

$$\mu_2 \left(\frac{T}{\mu_1} - s_\ell \right) = \underline{\underline{O}} \left(\sqrt{T} \right).$$

Since $T^{\frac{2}{\alpha}} \gg \sqrt{T}$, we can neglect the term $\mu_2 \dfrac{T}{\mu_1} - s_\ell\, \mu_2$ in the second factor which leads finally to the desired form of the theorem. ■

Remark 5.9 *These results can be extended for the general polynomial vectors* $\left(\tau, \tau^2, \tau^3, \tau^4\right)$; $\left(\tau, \tau^2, \tau^3, \tau^4, \tau^5\right)$ *, etc.*

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