GEVREY REGULARITY FOR A CLASS OF DISSIPATIVE EQUATIONS

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Abstract. In this paper, we establish Gevrey class regularity of solutions to a class of dissipative equations with a general quadratic nonlinearity for initial data in certain Besov type spaces. We then apply our result to the Navier-Stokes equations, the surface quasi-geostrophic equations, the Kuramoto-Sivashinsky equation and the barotropic quasi-geostrophic equation. In particular, we provide an alternate proof, as well as L^q extensions, of the results of Oliver and Titi ([38]) concerning temporal decay of solutions to the Navier-Stokes equations in higher Sobolev norms. We also obtain a new class of initial data where such decay holds for the 2D Navier-Stokes equations. Similar decay result is also proven for the 2D surface quasi-geostrophic equation.

1. Introduction

Regular solutions of many dissipative equations, such as the Navier-Stokes equations(NSE), the Kuramoto-Sivashinsky equation, the surface quasi-geostrophic equation and the Smoluchowski equation are in fact analytic, in both space and time variables ([37], [17], [4], [14], [46]). It is well-known that in case of the NSE, the space analyticity radius is an important physical object: at this length scale the viscous effects and the (nonlinear) inertial effects are roughly comparable. Below this length scale the Fourier spectrum decays exponentially ([16], [24], [25], [13]). Other applications occur in the study of long term dynamics of solutions ([38]), establishing geometric regularity criteria for the NSE and in measuring the spatial complexity of the flow (see [31], [34], [22]).

An effective approach for estimating the analyticity radius for the NSE via Gevrey norms was introduced by Foias and Temam ([18]). In this approach, one avoids cumbersome recursive estimation of higher order derivatives. Since its introduction, Gevrey class technique has become a standard tool for studying analytic properties of solutions for a wide class of dissipative equations (see [8], [15], [3], [32]). This was extended to establish analyticity of solutions to the NSE in L^p spaces ([32]), and subsequently to initial data in certain distributional spaces ([5]). In [38], it was shown how Gevrey norm estimates can be used to derive sharp upper and lower bounds for the (time) decay of higher order derivatives of solutions to the NSE.

We consider a nonlinear evolution equation of the form

$$\check{\mathbf{u}}_t + \check{\mathfrak{D}}\check{\mathbf{u}} = \check{B}(\check{\mathbf{u}}, \check{\mathbf{u}}) + \check{F}, \check{\mathbf{u}}_0 \in \mathcal{L},$$

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where $\check{\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^m$ is the solution we seek, $\check{F}$ is a given external "force" and $\mathcal{L}$ is an appropriate Banach space to which the initial data is assumed to belong. The operator $\check{\mathfrak{D}}$ is a densely defined "dissipative" operator while $\check{B}(\check{\mathbf{u}}, \check{\mathbf{v}})$ is a densely defined bilinear operator. Our assumption on the bilinear operator will include those having the form

$$\check{B}(\check{\mathbf{u}}, \check{\mathbf{v}}) = R(S\check{\mathbf{u}} \otimes T\check{\mathbf{v}}), \tag{1}$$

where R, S, T are operator matrices whose entries are Fourier multipliers with symbols $m_i(\xi)$, $i \in \{R, S, T\}$ satisfying $|m_i(\xi)| \leq C|\xi|^{\alpha_i}, \alpha_i \in \mathbb{R}, C > 0$. The type of the dissipative operators we consider will also be given by a Fourier multipliers of appropriate type. They will include the Laplacian, the fractional Laplacian and certain linear combination of them (for instance as in the Kuramoto-Sivashinsky equation).

It will be convenient for us to consider the mild formulation of the above mentioned dissipative equation, namely,

$$\check{\mathbf{u}} = e^{-t\check{\mathfrak{D}}}\check{\mathbf{u}}_0 + \int_0^t e^{-(t-s)\check{\mathfrak{D}}}\check{B}(\check{\mathbf{u}}, \check{\mathbf{u}})\, ds + \int_0^t e^{-(t-s)\check{\mathfrak{D}}}\check{F}(s)\, ds, \tag{2}$$

where $\{e^{-t\mathfrak{D}}\}_{t\geq 0}$ denotes the solution semi-group for the corresponding linear equation. The initial data will be assumed to belong to certain "homogeneous" Besov and potential spaces, which will include distributional spaces with negative regularity index. We will obtain solutions to (2) belonging to appropriate Gevrey classes. These solutions will be global (in time) for small initial data in the "critical space".

Our applications include the Navier-Stokes equations, the 2D surface quasi-geostrophic equations, the Kuramoto-Sivashinsky equation and the barotropic geostrophic equations. As a consequence of our result, we provide an alternate proof, as well as an L^p versions, of the results in [38], [40], [42] concerning large time decay of solutions to the 3D Navier-Stokes equations in higher (homogeneous) Sobolev norms. However, unlike [38], [40], [42], we do not assume any L^2 decay. In case of the 2D NSE, in addition to the class $L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, we obtain a new class (namely, $\dot{\mathbb{H}}^{-1}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$) where such a decay holds. See the Application section and the discussion there for details. We also obtain similar decay results for the subcritical 2D surface quasi-geostrophic equations. These applications hinge on the fact that we obtain Gevrey class solutions for initial data belonging to critical (homogeneous) Besov and potential spaces with negative regularity index.

2. Notation and Setting

Denote by $\mathbb{V}' = \{\mathbf{v} : \mathbb{R}^n \to \mathbb{C}^n\}$ the topological vector space of all functions endowed with the topology of point wise convergence and let $\mathbb{V}$ be a closed subspace of $\mathbb{V}'$. Let $\mathfrak{D} : \mathbb{V}_{\mathfrak{D}} \to \mathbb{V}$ and $B : \mathbb{V}_B \times \mathbb{V}_B \to \mathbb{V}$ respectively denote a linear and a bilinear operator defined on dense subspaces $\mathbb{V}_{\mathfrak{D}} \subset \mathbb{V}$ and $\mathbb{V}_B \subset \mathbb{V}$.

We will respectively denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and the inverse Fourier transform given by the formulas

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} f(x)e^{-\imath\xi\cdot x}\, dx \text{ and } (\mathcal{F}^{-1}f)(x) = \int_{\mathbb{R}^n} f(\xi)e^{\imath\xi\cdot x}\, d\xi, \quad (f \in L^1(\mathbb{R}^n)).$$

We briefly recall a few facts concerning the Fourier transform (see [27]). The Fourier inversion formula, namely $\mathcal{F}^{-1}\mathcal{F}(f) = \frac{1}{(2\pi)^n} f$, holds (directly) for all f such that both f and $\mathcal{F}f$ belong to $L^1(\mathbb{R}^n)$. The Fourier transform can be extended as a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ for $1 \le p \le 2$, where henceforth, for any $1 \le r \le \infty$, we will denote its Hölder conjugate r' by

$$r' := \frac{r}{r-1}, \quad 1 \le r \le \infty.$$

For $1 \le p \le 2$ and $f \in L^p(\mathbb{R}^n)$, the Hausdorff-Young inequality (see [27]) asserts

$$\|\mathcal{F}(f)\|_{L^{p'}} \le \|f\|_{L^p}.$$

For $f \in L^p(\mathbb{R}^n), 2 < p \le \infty$, one may define its distributional Fourier transform, which is in fact defined for any tempered distribution. With this extended definition, $\mathcal{F}L^p(\mathbb{R}^n)$ is a Banach space with the norm $\|\mathcal{F}f\|_{\mathcal{F}L^p} = \|f\|_{L^p}, 2 < p \le \infty$. The space $\mathcal{F}L^\infty$ is known as the space of *pseudomeasures* (see [27]).

Letting $\mathbf{u} = \mathcal{F}(\mathbf{u})$, we can reformulate (2) formally as

$$\mathbf{u}(t) = e^{-t\mathfrak{D}}\mathbf{u}_0 + \int_0^t e^{-(t-s)\mathfrak{D}} B[\mathbf{u},\mathbf{u}]\, ds + \int_0^t e^{-(t-s)\mathfrak{D}} F(s)\, ds \ (\mathbf{u}_0 \in \mathbb{V}), \tag{3}$$

where, in this case, $B[\mathbf{u},\mathbf{u}] = \mathcal{F}(\check{B}[\check{\mathbf{u}},\check{\mathbf{u}}])$ and $F = \mathcal{F}(\check{\mathbf{u}})$. The operator $\mathfrak{D}$ is assumed to be a densely defined "multiplication" operator on $\mathbb{V}$ of the form

$$(\mathfrak{D}_f\mathbf{v})(\xi) = (\mathfrak{D}\mathbf{v})(\xi) = f(\xi)\mathbf{v}(\xi) \tag{4}$$

where $f : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{C})$ is a given $n \times n$ matrix valued function. Although in our applications considered here, $f(\xi)$ is scalar valued (i.e., $f(\xi) = f(\xi)I_{n\times n}$) it may potentially be matrix-valued if for instance one considers the effect of rotation.

We will make certain structural assumptions on both the linear operator $\mathfrak{D}$ and the bilinear operator B. Concerning the linear operator, we assume that there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties:

P1. There exist $\sigma, K > 0$ such that

$$\|e^{-tf(\xi)}x\| \le Ce^{-g(\sqrt{t}^{\sigma}\xi)}\|x\| \text{ for all } 0 \le t \le K, \xi \in \mathbb{R}^n, x \in \mathbb{C}^n.$$

P2. For all $m \ge 0$ we have $\sup_{\xi\in\mathbb{R}^n} |\xi|^m e^{-g(\xi)} < \infty$.

P3. There exists $0 < \gamma \le 1$ and $\lambda_0 > 0$ such that $\sup_{\xi\in\mathbb{R}^n} (\lambda_0|\xi|^\gamma - g(\xi)) < \infty$.

For example, in our applications to the Navier-Stokes and the quasi-geostrophic equations, $f(\xi) = g(\xi) = |\xi|^\kappa, \kappa \in [1,2]$, while for the Kuramoto-Sivashinsky equation, $f(\xi) = g(\xi) = |\xi|^4 - |\xi|^2$.

Remark 1. The properties P1 and P2 are necessary for existence theory while P3 is necessary for establishing Gevrey regularity. The property P1 is satisfied for instance if the matrix $f(\xi)$ is normal and its eigenvalues, denoted by $f_i(\xi), 1 \le i \le n$ satisfy $tf_i(\xi) \ge g(\sqrt{t}^{\sigma}\xi)$ for all $1 \le i \le n$.

For any $\beta \in \mathbb{R}$, define the linear operator

$$(\Lambda^\beta \mathbf{v})(\xi) = |\xi|^\beta \mathbf{v}(\xi), \;\; \mathbf{v} \in \mathbb{V}, \xi \in G.$$

Clearly, the operators Λ^β commute with $e^{-t\mathfrak{D}}$. This fact will be used throughout. We assume that for some $\alpha, \beta_1, \beta_2 \in \mathbb{R}$, the bilinear operator satisfies the estimate

$$|B[\mathbf{u}, \mathbf{v}](\xi)| \le C|\xi|^\alpha (|\Lambda^{\beta_1}\mathbf{u}| * |\Lambda^{\beta_2}\mathbf{v}|)(\xi) \text{ for all } \xi \in G. \tag{5}$$

Here, and henceforth, we will denote by C any generic constant which may depend only on the fixed parameters, like $1 \le p \le \infty$ or the ones occurring in P1–P3 or (5). Also, for $\mathbf{v} \in \mathbb{V}$, here (and henceforth) we denote by $|\mathbf{v}|$ the $\mathbb{R}_+$-valued function on $\mathbb{R}^n$ defined by $|\mathbf{v}|(\xi) = |\mathbf{v}(\xi)|$. Recall that $\mathcal{F}$ and $\mathcal{F}^{-1}$ will denote the Fourier transform and its inverse. If $\check{B}$ is as in (1), then (5) holds for $B(\mathbf{u}, \mathbf{v}) = \mathcal{F}\left(\check{B}(\check{\mathbf{u}}, \check{\mathbf{v}})\right)$, where $\check{\mathbf{u}} = \mathcal{F}^{-1}(\mathbf{u})$ and $\check{\mathbf{v}} = \mathcal{F}^{-1}(\mathbf{v})$.

Let $\theta \in \mathbb{R}$ and $1 \le p \le \infty$. We will denote

$$\mathbb{V}_{\theta,p} = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\|_{\theta,p} := \left\{\int_{\mathbb{R}^n} |\xi|^{\theta p} |\mathbf{v}(\xi)|^p \, d\xi\right\}^{1/p} < \infty\}.$$

When $\theta = 0$, we will simply write the corresponding space and norm as $\mathbb{V}_p$ and $\|\cdot\|_p$ respectively.

For $\sigma \in \mathbb{R}, 1 \le q \le \infty$, the homogeneous potential spaces $\dot{H}^\sigma_q$ are defined as

$$\dot{H}^\sigma_q = \{f : \|f\|_{\dot{H}^\sigma_q} := \|(-\Delta)^{\sigma/2} f\|_{L^q} < \infty\}.$$

For $q = 2$, we will write $\dot{H}^\sigma_q = \dot{\mathbb{H}}^\sigma$. Using the Fourier transform, it is easy to see that $\dot{\mathbb{H}}^\sigma = \mathbb{V}_{\sigma,2}$. Thus, for $1 \le q \le 2$, $\mathcal{F}(\dot{H}^\theta_q) \subset \mathbb{V}_{\theta,q'}$, while for $1 \le p \le 2$, we have $\mathcal{F}^{-1}(\mathbb{V}_{\theta,p}) \subset \dot{H}^\theta_{p'}$.

For $1 \le p \le \infty$, γ, σ as in P1–P3 and a fixed $\lambda \ge 0$, we define the Gevrey norm

$$\|\mathbf{v}\|_{G_\theta(\tau)} := \left\{\int_{\mathbb{R}^n} e^{\lambda p (\sqrt{\tau}^\sigma |\xi|)^\gamma} |\xi|^{\theta p} |\mathbf{v}(\xi)|^p \, d\xi\right\}^{1/p}, \;\; \tau \ge 0, \mathbf{v} \in \mathbb{V}. \tag{6}$$

Note that, in addition to θ and τ, this norm also depends on σ, γ, λ and p. However, in each of our applications, these values will be fixed and for notational simplicity, we omit them from the notation. In case $\theta = 0$ in a certain consideration, we will simply write $\|\mathbf{v}\|_{G(\tau)}$ instead of $\|\mathbf{v}\|_{G_\theta(\tau)}$. Since λ and p will be fixed subject to certain conditions, we will suppress the dependence of the Gevrey norm on these parameters.

Remark 2. Let $\gamma = 1$ in (6) and $\theta \in \mathbb{R}$ be such that $\theta p' < n$. If $\|\mathbf{v}\|_{G_\theta(\tau)} < \infty$, then for any λ_α with $0 < \lambda_\alpha < \lambda\sqrt{\tau}^\sigma$, there exists a corresponding positive constant C, independent of $\mathbf{v}$, such that

$$\int e^{\lambda_\alpha |\xi|} |\mathbf{v}(\xi)| \, d\xi \le C\|\mathbf{v}\|_{G_\theta(\tau)}.$$

This is due to Hölders inequality and the fact that $\int_{\mathbb{R}^n} |x|^{\theta p'} e^{-\eta|x|} dx < \infty$ for any $\eta > 0$ and $\theta p' < n$. The Paley-Wiener theorem (see e.g. [27]) implies that $\check{\mathbf{v}} = \mathcal{F}^{-1}(\mathbf{v})$ is the restriction to $\mathbb{R}^n$ of a holomorphic function on the domain $\{z = x + \imath y \in \mathbb{C}^n : |y| < \lambda_\alpha\}$.

In case $0 < \gamma < 1$, $\check{\mathbf{v}}$ belongs to the non-analytic Gevrey classes and is consequently smooth.

Definition 3. A mild solution of the dissipative equation we consider is a function $\mathbf{u} : [0,T] \to \mathbb{V}$ satisfying (3). The equation (3) is assumed to hold *a.e.* $0 \le t \le T$. We also require that $\mathbf{u}(\cdot)$ and F be such that the two integrals in (3) converge absolutely for *a.e.* $\xi \in \mathbb{R}^n$.

Remark 4. $\mathbf{u}(\cdot)$ is a solution of (3) on the interval $[0,T]$ such that

$$\int_0^t \|e^{-(t-s)\mathfrak{D}} B[\mathbf{u},\mathbf{u}]\|_{\mathbb{V}_1}\, ds < \infty \text{ and } \mathbf{u} \in \mathbb{V}_1 \, a.e.\, t \in (0,T),$$

then $\check{\mathbf{u}} = \mathcal{F}^{-1}\mathbf{u}$ exists *a.e.* t and satisfies (2). This is an easy consequence of Fubini's theorem on interchanging the order of integration.

Let $\mathcal{S}'$ denote the space of tempered distributions. We recall that homogeneous L^q-based Besov spaces with negative regularity index can be defined via the heat kernel (see [35]) as

$$\dot{B}_q^{-\delta,\infty} = \{f \in \mathcal{S}' : \sup_{t>0} \sqrt{t}^{\delta} \|e^{t\Delta} f\|_{L^q} < \infty\}.$$

Motivated by this, we will now define Besov type spaces with negative index in our setting.

Definition 5. For $1 \le p \le \infty, \delta \ge 0$, the (homogeneous) Besov type space with negative index is defined to be

$$\mathbb{B}_p^{-\delta,\infty} = \left\{ \mathbf{v} \in \mathbb{V} : \|\mathbf{v}\|_{\mathbb{B}_p^{-\delta,\infty}} := \sup_{0<t<T} \sqrt{t}^{\delta} \|e^{-t\mathfrak{D}}\mathbf{v}\|_{L^p} < \infty \right\}. \tag{7}$$

In case $\check{\mathfrak{D}} = -\Delta$, the Hausdorff-Young inequality shows that for $1 \le p \le 2$, we have $\mathcal{F}^{-1}(\mathbb{B}_p^{-\delta,\infty}) \subset \dot{B}_q^{-\delta,\infty}$ where $q = p'$ denotes the Hölder conjugate of p. On the other hand, $\mathcal{F}(\dot{B}_q^{-\delta,\infty}) \subset \mathbb{B}_p^{-\delta,\infty}$ for $1 \le q \le 2$ and $p = q'$ (in which case $2 \le p \le \infty$). The following proposition elucidates the relation between the spaces $\mathbb{V}_{\theta,r}$ and $\mathbb{B}_p^{-\delta,\infty}$.

Proposition 2.1. *Let $r > p$ and $\delta = n(\frac{1}{p} - \frac{1}{r}) - \theta > 0$, where n denotes the space dimension. Then we have*

$$\mathbb{V}_{\theta,r} \subset \mathbb{B}_p^{-\delta,\infty} \text{ and } \|\mathbf{u}\|_{\mathbb{B}_p^{-\delta,\infty}} \le C\|\mathbf{u}\|_{\mathbb{V}_{\theta,r}}.$$

Proof. Set $q = \frac{r}{p}, \theta_1 = \theta p$ and $q' = r/(r-p)$, the Hölder conjugate of q. By P1, Hölder's inequality and a change of variable, we have

$$\|e^{-t\mathfrak{D}}\mathbf{u}\|_p^p \le \int e^{-pg(\sqrt{t}^{\sigma}\xi)} |\mathbf{u}(\xi)|^p \, d\xi$$
$$\le \left(\int |\xi|^{-\theta_1 q'} e^{-q'pg(\sqrt{t}^{\sigma}\xi)}\, d\xi\right)^{1/q'} \left(\int |\xi|^{\theta_1 q} |\mathbf{u}(\xi)|^r \, d\xi\right)^{1/q} \le \frac{C}{\sqrt{t}^{\delta p}} \left(\int |\xi|^{\theta_1 q} |\mathbf{u}(\xi)|^r \, d\xi\right)^{1/q}.$$

The very last inequality is obtained by making a change of variable and noting that, due to P3, the integral

$$\int |\eta|^{-\theta_1 q'} e^{-q'pg(\eta)}\, d\eta = C < \infty \text{ provided } \theta_1 q' < n.$$

This condition on θ_1 translates to $\delta > 0$.

□

3. Main Results

Following the notation and setting of the previous section, we will state our main results.

Theorem 3.1. *Let* $\mathbf{u}_0 \in \mathbb{B}_p^{-\delta,\infty}$ *and* $F : [0, T'] \to \mathbb{V}$. *Let moreover*

$$\left.\begin{array}{rcl} M & := & \|\mathbf{u}_0\|_{\mathbb{B}_p^{-\delta,\infty}} + \sup_{0\le s\le T'} \|\sqrt{s}^{1-\delta} F(s)\|_{G(s)} < \infty \text{ and} \\ \mu & = & 2 - \delta - \sigma(\alpha + \beta_1 + \beta_2 + \frac{n}{p'}). \end{array}\right\} \quad (8)$$

Assume that the following conditions for the parameters hold:
i) $0 \le \delta < \min\{1, 1 - \frac{\sigma(\beta_1+\beta_2)}{2}, 1 - \frac{\sigma(\beta_1+\beta_2)}{2} - \frac{n\sigma}{2}(\frac{1}{p'} - \frac{1}{p})\}$, $\delta \le 2 - \sigma(\alpha + \beta_1 + \beta_2 + \frac{n}{p'})$
(ii) $\max\{\alpha + \beta_1, \alpha + \beta_2\} < \frac{2}{\sigma}$ *(iii)* $\min\{\beta_1, \beta_2\} > -\frac{n}{p'}$ *(iv)* $\frac{n}{\max\{p',2\}} - \frac{|\beta_1-\beta_2|}{2} > 0$
(v) $\alpha + \beta_1 + \beta_2 + \frac{n}{p'} \ge 0$ *(vi)* $\alpha + \frac{n}{p'} \ge 0$ *(vii)* $\alpha + \frac{n}{p} > 0$ *(viii)* $\alpha + \beta_1 + \beta_2 + \frac{n}{p'} < \frac{4}{\sigma}$.
With these assumptions, we have the following results.

(a) *If* $\mu > 0$, *there exists* $T > 0$ *and a solution* $\mathbf{u}$ *of (3) in* $C_c((0, T); \mathbb{V}_p)$ *which moreover satisfies*

$$\sup_{0<t<T} \sqrt{t}^{\delta} \|\mathbf{u}\|_{G(t)} \le 2M.$$

In fact, we may take $T < \min\{T', \left(\frac{1}{4CM}\right)^{2/\mu}\}$.

(b) *If* $\mu = 0$ *and* $T' = \infty$, *there exists a constant* $\epsilon > 0$, *independent of* $\mathbf{u}_0$ *and* F, *such that if* $M < \epsilon$ *then* $T = \infty$.

(c) *Let* $\check{\mathbf{u}} = \mathcal{F}^{-1}(\mathbf{u})$, *where* $\mathbf{u}$ *is as in parts (a) or (b). Then* $\check{\mathbf{u}}$ *satisfies (2).*

We will now state our result for initial data in $\mathbb{V}_{\theta_0,p}$ spaces, where $\theta_0 \in \mathbb{R}$. For simplicity, here we will take $F = 0$ in (3).

Theorem 3.2. *Let* $\mathbf{u}_0 \in \mathbb{V}_{\theta_0,p}$, $\theta_0 \in \mathbb{R}$. *Assume that the following conditions hold:*
(i) $\theta_0 > \max\{\frac{\beta_1+\beta_2}{2} - \frac{1}{\sigma}, \frac{\beta_1+\beta_2}{2} + \frac{n}{2}(\frac{1}{p'} - \frac{1}{p}) - \frac{1}{\sigma}\}$, $\theta_0 \ge \alpha + \beta_1 + \beta_2 + \frac{n}{p'} - \frac{2}{\sigma}$
(ii) $\theta_0 \le \alpha + \beta_1 + \beta_2 + \frac{n}{p'}$, $\theta_0 < \min\{\beta_1 + \frac{n}{p'}, \beta_2 + \frac{n}{p'}\}$
(iii) $\max\{\alpha + \beta_1, \alpha + \beta_2\} < \frac{2}{\sigma}$ *(iv)* $\frac{n}{\max\{p',2\}} - \frac{|\beta_1-\beta_2|}{2} > 0$
(v) $\alpha + \frac{n}{p'} + \vartheta_0 \ge 0$ *(vi)* $\alpha + \frac{n}{p} + \vartheta_0 > 0$ *(vii)* $\alpha + \beta_1 + \beta_2 + \frac{n}{p'} - \vartheta_0 < \frac{4}{\sigma}$.

$$M := \|\mathbf{u}_0\|_{\mathbb{V}_{\theta_0,p}} \text{ and } \mu := 2 - \sigma(\alpha + \beta_1 + \beta_2 + \frac{n}{p'} - \theta_0).$$

Then, there exists $T > 0$ *and a solution* $\mathbf{u}$ *of (3) in* $C([0, T]; \mathbb{V}_{\theta_0,p})$ *which also satisfies*

$$\sup_{0<t<T} \|\mathbf{u}\|_{G_{\theta_0}(t)} \le 2M.$$

In case $\mu > 0$, *we may take* $T < \left(\frac{1}{4CM}\right)^{2/\mu}$. *If* $\mu = 0$ *and* $M < \epsilon$, *where* $\epsilon > 0$ *is a suitable constant independent of* $\mathbf{u}_0$, *then we may take* $T = \infty$. *Moreover if* $\theta_0 p' < n$, *then* $\check{\mathbf{u}} = \mathcal{F}^{-1}(\mathbf{u})$ *satisfies (3).*

Remark 6. In Theorems 3.1 and 3.2, the spaces corresponding to $\mu = 0$ will be referred to as the critical spaces. Note that these are precisely the spaces where we obtain global existence for small initial data (provided $K = \infty$ in P1). In our applications to the Navier-Stokes and surface quasi-geostrophic equations, the critical spaces correspond to the "scale invariant spaces". See [35] for a discussion on scale invariant spaces pertaining to the Navier-Stokes equations.

We will now consider the borderline case $p = 1$. Here we will only consider two particular cases: the 3D Navier-Stokes equations and the critical quasi-geostrophic equation.

Theorem 3.3. *Let* $\mathbf{u}_0 \in \mathbb{V}_{-1,1}$. *Then there exists solution* $\mathbf{u}$ *to (3) for the 3D Navier-Stokes equations for adequate* $T > 0$ *such that the solution* $\mathbf{u}(\cdot)$ *satisfies* $\sup_{t\in[0,T]} \|\mathbf{u}(t)\|_{G_{-1}(t)} < \infty$. *If* $\|\mathbf{u}_0\|_{-1,1}$ *is suitably small, then the existence time* T *can be taken to be infinity.*

We have the following result for the critical 2D surface quasi-geostrophic equation.

Theorem 3.4. *Let* $\mathbf{u}_0 \in \mathbb{V}_1$. *Then there exists solution* $\mathbf{u}$ *to (3) for the critical quasi-geostrophic equations for adequate* $T > 0$ *such that the solution* $\mathbf{u}(\cdot)$ *satisfies* $\|\mathbf{u}\|_\Sigma < \infty$. *If* $\|\mathbf{u}_0\|_1$ *is suitably small, then the existence time* T *can be taken to be infinity.*

Remark 7. The same results hold for sequences $\mathbf{f} = (f_k)_{k\in\mathbb{Z}^n}$ where the norm $\|\mathbf{f}\|_{\theta,p} = \left\{\sum_k (1+|k|)^{\theta p}|f_k|^p\right\}^{1/p}$.

4. Applications

In this section, we give applications of our results to various dissipative equations. In all these cases, for simplicity, we will take the force F to be zero.

4.1. Navier-Stokes equations.

The incompressible Navier-Stokes equations (henceforth NSE) in fluid dynamics are given by

$$\check{\mathbf{u}}_t - \Delta\check{\mathbf{u}} + \nabla p + \nabla\cdot(\check{\mathbf{u}}\otimes\check{\mathbf{u}}) = 0,\ \nabla\cdot\check{\mathbf{u}} = 0, \check{\mathbf{u}}(0) = \check{\mathbf{u}}_0,$$

where $\check{\mathbf{u}} : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ is the velocity vector field, p is the pressure and $\check{\mathbf{u}}_0$ is the initial velocity. The pressure can be regarded as a Lagrangian multiplier which imposes the divergence free condition. Due to the presence of pressure, these equations are nonlocal. It is customary to apply the Leray projection operator on the Navier-Stokes equations to eliminate pressure. If one does that, then the mild formulation can be rewritten as

$$\check{\mathbf{u}} = e^{t\Delta}\check{\mathbf{u}}_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\nabla\cdot(\check{\mathbf{u}}\otimes\check{\mathbf{u}})\,ds, \tag{9}$$

where $\mathbb{P}$ is the Leray projection operator on divergence free vector fields. Here we have used the fact that in the absence of boundary, the Leray projection and the Laplacian commute. It is in fact a Fourier multiplier with the symbol given by

$$\mathbb{P}(\xi) = I - \frac{\xi\otimes\xi}{|\xi|^2}.$$

There is a vast body of literature on local and global existence of weak, mild and strong solutions to the Navier-Stokes equations; we refer the reader to [44], [11], [35] and the references there in. The largest space in which the 3D NSE is thus far known to be locally well-posed is the space BMO^{-1} ([30]). Roughly speaking, the space BMO^{-1} comprises of all functions that can be written as sums of BMO functions and their derivatives.

Applying the Fourier transform and letting $\mathbf{u} = \mathcal{F}(\check{\mathbf{u}})$, (9) can be (formally) reformulated in the form (3), where

$$(\mathfrak{D}\mathbf{u})(\xi) = |\xi|^2\mathbf{u}(\xi) \text{ and } B(\mathbf{u},\mathbf{v})(\xi) = -\imath\mathbb{P}(\xi)\int \xi\cdot(\mathbf{u}(\xi-\eta)\otimes\mathbf{v}(\eta))\,d\eta.$$

A mild solution of the NSE in our setting will be a solution to (3) with B and $\mathfrak{D}$ as defined above.

Theorem 4.1. *Consider the Navier-Stokes equations on $\mathbb{R}^n, n = 2, 3$. In this case, assumptions P1–P3 are satisfied with $\sigma = 1, \gamma = 1, K = \infty$ and any $\lambda_0 > 0$ and (5) is satisfied with $\alpha = 1$ and $\beta_1 = \beta_2 = 0$. With these values of the parameters, we have the following results.*

(i) *Let $1 < p \le n'$ and $0 \le \delta \le 1 - \frac{n}{p'}$ and with $\mathbf{u}_0 \in \mathbb{B}_p^{-\delta,\infty}$. Let $\mu = (1 - \frac{n}{p'}) - \delta$. In case $\delta < 1 - \frac{n}{p'}$ (i.e., $\mu > 0$), then there exists a unique mild solution $\mathbf{u}$ of the NSE belonging to $C((0,T);\mathbb{V}_p)$ where $T = \frac{C}{\|\mathbf{u}_0\|^{2/\mu}_{\mathbb{B}_p^{-\delta,\infty}}}$. Moreover, we also have $\sup_{0<t<T}\sqrt{t}^{\delta}\|\mathbf{u}(t)\|_{Gv(t)} < \infty$ and $\check{\mathbf{u}} = \mathcal{F}^{-1}(\mathbf{u})$ satisfies (9). In case $\delta = 1 - \frac{n}{p'}$ and $\|\mathbf{u}_0\|_{\mathbb{B}_p^{-\delta,\infty}} < \epsilon$ for an adequate constant $\epsilon > 0$, the same result holds with $T = \infty$.*

(ii) *Let $1 < p \le \infty$, $\theta_0 \ge \frac{n}{p'} - 1$ and $\mu = \theta_0 - (\frac{n}{p'} - 1)$. If $\mathbf{u}_0 \in \mathbb{V}_{\theta_0,p}$, then there exists a $T > 0$ and an unique mild solution $\mathbf{u}$ of the NSE with $\mathbf{u} \in C([0,T];\mathbb{V}_{\theta_0,p})$ satisfying $\sup_{0<t<T}\|\mathbf{u}(t)\|_{Gv_{\theta_0}(t)} < \infty$. If $\theta_0 > \frac{n}{p'} - 1$, then T can be taken to be $T = \frac{C}{\|\mathbf{u}_0\|^{2/\mu}_{\theta_0,p}}$. On the other hand, in case $\theta_0 = \frac{n}{p'} - 1$, there exists an $\epsilon > 0$ such that if $\|\mathbf{u}_0\|_{\theta_0,p} < \epsilon$, then $T = \infty$. In all these cases, $\check{\mathbf{u}} = \mathcal{F}^{-1}(\mathbf{u})$ solves (9).*

(iii) *Let $n = 3$, $\check{\mathbf{u}}_0 \in L^2(\mathbb{R}^3)$ and $\check{\mathbf{u}}$ be a Leray-Hopf weak solution of the NSE satisfying the energy inequality*

$$\|\check{\mathbf{u}}(t)\|^2_{L^2} + \int_0^t \|(\Delta)^{1/2}\check{\mathbf{u}}(s)\|^2_{L^2}\,ds \le \|\check{\mathbf{u}}_0\|^2_{L^2}. \tag{10}$$

Let $\epsilon > 0$ is as in part (ii). There exists $t_0 > 0$ such that $\check{\mathbf{u}}$ is a classical solution of the NSE for all $t \ge t_0$ which, for all $\zeta > \frac{1}{2}$, satisfies the estimate

$$\|(-\Delta)^{\zeta/2}\check{\mathbf{u}}(t)\|^2_{L^2} \le \frac{(2\zeta-1)^{2\zeta-1}}{(2e)^{2\zeta-1}}\;\frac{\epsilon}{(t-t_0)^{\zeta-\frac{1}{2}}}. \tag{11}$$

Moreover, for $3 < q < \infty$ and $\zeta > 0$ we also have the estimate

$$\|(-\Delta)^{\zeta/2}\check{\mathbf{u}}(t)\|_{L^q} \le \frac{\epsilon}{\sqrt{(t-t_0)}^{\zeta+\delta}}\frac{\zeta^\zeta}{e^\zeta} \text{ where } \delta = 1 - \frac{3}{q}. \tag{12}$$

(iv) *For $n = 2$, let $\check{\mathbf{u}}_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ or $\check{\mathbf{u}}_0 \in \dot{\mathbb{H}}^{-\frac{1}{2}}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $\epsilon > 0$ be as in part (ii). Let $\check{\mathbf{u}}$ be the global classical solution of the NSE with initial data $\check{\mathbf{u}}_0$. There exists $t_0 > 0$ such that for all $t > t_0$ and $\zeta > 0$, satisfies the estimate*

$$\|(-\Delta)^{\zeta/2}\check{\mathbf{u}}(t)\|^2_{L^2} \le \frac{(2\zeta-1)^{2\zeta-1}}{(2e)^{2\zeta-1}}\;\frac{\epsilon}{(t-t_0)^{\zeta}}. \tag{13}$$

Moreover, for $2 < q < \infty$ and $\zeta > 0$, we also have the estimate

$$\|(-\Delta)^{\zeta/2}\check{\mathbf{u}}(t)\|_{L^q} \le \frac{\epsilon}{\sqrt{(t-t_0)}^{\zeta+\delta}}\frac{\zeta^\zeta}{e^\zeta} \text{ where } \delta = 1 - \frac{2}{q}. \tag{14}$$

Existence of solutions to the NSE in Gevrey classes was first proven for the periodic boundary condition by Foias and Temam ([18]) (for initial data in $\mathbb{H}^1$) and subsequently by Oliver and Titi on the whole space, with initial data in $\mathbb{H}^s, s > n/2, n = 2, 3$ (see also [35] for initial data in $\dot{\mathbb{H}}^{1/2}$ for 3D NSE). By following a slightly different approach, Grujic and Kukavica proved analyticity of solutions to the 3D NSE for initial data in $L^q, q > 3$. On the other hand, existence (local in time for arbitrary data and global for small data in suitable critical spaces) of mild (and in fact, classical) solution to the NSE for initial data in L^p and Morrey spaces goes back to the work in [19], [26], [47], [20], [21] and more recently, on homogeneous Besov spaces $\dot{B}_q^{-\delta,\infty}, \delta = \frac{3}{q} - 1, 3 < q < \infty$ ([7]) for the 3D NSE. Part (i) of the above theorem establishes Gevrey regularity for solutions with initial data in the related spaces $\mathbb{B}_p^{-\delta,\infty}, p = q'$ while part (ii) was obtained previously in [5]. Concerning Theorem 3.3, the spaces $\mathbb{V}_{-1,1}$ are (by taking inverse Fourier transform) contained in the homogeneous potential spaces $\dot{H}_\infty^{-1}$ and consequently, in BMO^{-1}. Analyticity of the Koch-Tataru solutions constructed for small initial data in BMO^{-1} has been proven in [23] by first establishing analyticity for L^∞ initial data and then invoking an uniqueness result ([33]). Thus for small data, Theorem 3.3 is implied by [23]. However, our approach is more direct and applies more generally, although we chose to present it ony for the NSE.

The decay in L^2-based (homogeneous) Sobolev norms $\|\mathbf{u}\|_{\dot{\mathbb{H}}^\zeta}$ for the NSE as in (11) and (13) were, to the best of our knowledge, first given in [40] and [42]. However, the constants C_ζ were not explicitly identified. The sharp (and optimal, in the sense of providing lower bounds as well) decay results were provided by Oliver and Titi ([38]) following the Gevrey class approach. The constants C_ζ identified there is of the same order as provided here. In the above mentioned results however, there is an assumption of the decay of the L^2 norm of the solution. This is circumvented for the 3D NSE here due to our working in the "critical" space $\dot{\mathbb{H}}^{1/2}$. In the 2D setting, we provide a new space of initial data (namely, $\dot{\mathbb{H}}^{-1/2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$) where such decay result holds. In fact, as a corollary to (13) and (47), by interpolation, it follows that for initial data in this class, the L^2 norm decays like $t^{-\frac{1}{4}}$. It can be easily seen (via Fourier transform) that the class $\dot{\mathbb{H}}^{-1/2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ is different from the class $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ where such a decay result was previously known ([41]). The L^q based decay results presented here ((12) and (14)) are new to the best of our knowledge and is obtained from our result in part (i).

4.2. **Surface quasi-geostrophic equation.** In $\mathbb{R}^2$, we consider the sub-critical surface quasi-geostrophic equation given by

$$\eta_t + \nabla\cdot((\mathcal{R}\eta)\eta) + (-\Delta)^{\kappa/2}\eta = 0,\ 1 < \kappa \le 2, \tag{15}$$

where $\mathcal{R} = (-\mathcal{R}_2, \mathcal{R}_1)$ denote the Riesz transform. The critical and super-critical case correspond to $\kappa = 1$ and $0 < \kappa < 1$ respectively. This equation is an important model in geophysical fluid dynamics and has received considerable attention recently; see for instance [12] or [14] and the references there in. The critical quasi-geostrophic equation, corresponding to $\alpha = \frac{1}{2}$ is the correct dimensional analogue of the 3D Navier-Stokes equations. The global well-posedness of this equation has been proven only recently ([6], [29]).

Long time behavior of sub-critical quasi-geostrophic equations was studied in [12], [10]. Analyticity, as well as time decay rate of $\|(-\Delta)^{\zeta/2}\eta\|_{L^q}, q > \frac{2}{\kappa-1}$ of solutions, was obtained in [14]. The initial data in [14] was assumed to be in $L^{\frac{2}{\kappa-1}}$. The decay result provided here on L^2-based homogeneous Sobolev spaces do not follow from the result there and appears to be new. The analyticity result provided here in Besov (type) spaces with negative regularity index, also appears to be new. For local (and global) existence results in corresponding Besov spaces, see [2].

Global well-posedness for the critical quasi-geostrophic equation (i.e., $\kappa = 1$) in the borderline homogeneous Besov space $\dot{B}^{0,1}_{\infty}$ was proven in [1]. However, higher regularity of solution is not established there. Note that this space is contained in $\dot{B}^{0,\infty}_{\infty}$. The well-posedness of the critical quasi-geostrophic equation in this class is open. Our class in Theorem 3.4, namely $\mathbb{V}_1$, where we established global well-posedness (for small data) in the Gevrey class (and thus in fact the solutions are analytic) is contained in $\dot{B}^{0,\infty}_{\infty}$, but is distinct from the class $\dot{B}^{0,1}_{\infty}$ considered in [1].

Theorem 4.2. *Consider the sub-critical 2D quasi-geostrophic equation (i.e., $1 < \kappa \leq 2$). In this case, assumptions P1–P3 and condition (5) are satisfied with $\sigma = \frac{2}{\kappa}, \gamma = 1, K = \infty, \alpha = 1, \beta_1 = \beta_2 = 0$ and any $\lambda_0 > 0$. With these values of the parameters, we have the following results.*

(i) *Let $1 < p < \infty$ and $\delta \leq 2 - \frac{2}{\kappa}(1 + \frac{2}{p'})$ and denote $\mu = 2 - \frac{2}{\kappa}(1 + \frac{2}{p'}) - \delta$. In case $\delta < 2 - \frac{2}{\kappa}(1 + \frac{2}{p'})$, setting $T = \frac{C}{\|\eta_0\|^{2/\mu}_{\mathbb{B}_p^{-\delta,\infty}}}$, there exists a mild solution η to (15) which belongs to $C((0,T);\mathbb{V}_p)$ which moreover satisfies $\sup_{0<t<T} \sqrt{t}^{\delta}\|\eta(t)\|_{G(t)} < \infty$. Moreover, if $\delta = 2 - \frac{2}{\kappa}(1 + \frac{2}{p'})$, then $T = \infty$ provided $\|\eta_0\|_{\mathbb{B}_p^{-\delta,\infty}} < \epsilon$ for adequate $\epsilon > 0$.*

(ii) *Let $1 < p < \infty$, $\theta_0 \geq 1 + \frac{2}{p'} - \kappa$ and denote $\mu = 2(1 - \frac{1}{\kappa}(1 + \frac{2}{p'} - \theta_0))$. If $\eta_0 \in \mathbb{V}_{\theta_0,p}$, then there exists a $T > 0$ and an unique mild solution η of (15) with $\eta \in C([0,T];\mathbb{V}_{\theta_0,p})$ which moreover satisfies $\sup_{0<t<T} \|\mathbf{u}(t)\|_{Gv_{\theta_0}(t)} < \infty$. If $\theta_0 > 1 + \frac{2}{p'} - \kappa$, then T can be taken to be $T = \frac{C}{\|\eta_0\|^{2/\mu}_{\theta_0,p}}$. On the other hand, in case $\theta_0 = 1 + \frac{2}{p'} - \kappa$, there exists an $\epsilon > 0$ such that if $\|\eta_0\|_{\theta_0,p} < \epsilon$, then $T = \infty$.*

(iii) *Let $\frac{4}{3} \leq \kappa \leq 2$. If $\check{\eta}_0 \in \dot{\mathbb{H}}^{2-\kappa}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, then there exists $t_0 > 0$ such that for $\zeta > 2 - \kappa$, the following decay holds:*

$$\|(-\Delta)^{\zeta/2}\eta(t)\|^2_{L^2} \leq \left(\frac{\zeta + \kappa - 2}{\lambda e}\right)^{2(\zeta+\kappa-2)} \frac{\epsilon}{(t - t_0)^{\frac{2(\zeta+\kappa-2)}{\kappa}}}.$$

In [12], it was shown that for initial data in the class $\mathbb{H}^s (s > 2 - \kappa)$, $\|\eta\|_{\mathbb{H}^s}$ is bounded on any interval $[0,T]$ (although the spatial domain there was the torus, a similar argument can be carried out in the whole space). Part (ii) of our result then immediately establishes space analyticity of solutions on $[0,T]$. However, since a uniform global bound on $[0,\infty)$ is not in general available, the lower bound on the analyticity radius from our method will shrink.

However, appealing to our result for the critical case $\dot{\mathbb{H}}^{2-\kappa}$, we can prove that the analyticity radius in fact increases (at the rate $t^{1/\kappa}$) for large times provided $\kappa \geq 4/3$.

4.3. **Kuramoto-Sivashinsky equation.** The Kuramoto-Sivashinsky equation (KSE) is

$$\check{\mathbf{u}}_t + \Delta^2 \check{\mathbf{u}} + \Delta \check{\mathbf{u}} + \frac{1}{2}|\nabla \check{\mathbf{u}}|^2 = 0, \ \check{\mathbf{u}}(x,0) = \check{\mathbf{u}}_0(x) \tag{16}$$

where $\check{\mathbf{u}}(x,t) : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, and the initial data is $\check{\mathbf{u}}_0$. The KSE models pattern formations on unstable flame fronts and thin hydrodynamic films. There is a large literature surrounding the one-dimensional KSE subject to the space periodic boundary condition; see [45] and the references therein. For more recent results, see [9], [43], [39] and the references therein. As before, by taking Fourier transform, the mild formulation can be written in the form (3) where

$$(\mathfrak{D}\mathbf{v})(\xi) = |\xi|^4 - |\xi|^2 \text{ and } B(\mathbf{u},\mathbf{v})(\xi) = \frac{1}{2}\int \eta \cdot (\xi - \eta)\mathbf{u}(\xi-\eta)\mathbf{v}(\eta)\, d\eta.$$

Theorem 4.3. *Consider the (16) in dimension $n \geq 2$. The assumptions P1 and P2 are satisfied with $f(\xi) = g(\xi) = |\xi|^4 - |\xi|^2, \sigma = \frac{1}{2}$ and $K = 1$ while P3 is satisfied with $\gamma = 1$ and any $\lambda_0 > 0$. The condition in (5) is satisfied with $\beta_1 = \beta_2 = 1$ and $\alpha = 0$. Consequently, we have the following results.*

(i) *Let $\mathbf{u}_0 \in \mathbb{V}_{\theta_0,p}$ with $\max\{-1, \frac{n}{p'} - 2\} < \theta_0 < \frac{n}{p'} + 1$ and $1 < p < \infty$. There exists $T > 0$ and a solution $\mathbf{u} \in C([0,T]; \mathbb{V}_{\theta_0,p})$ of the (16) such that $\sup_{t\in(0,T]} \|\mathbf{u}\|_{G_{\theta_0}(t)} < \infty$.*

(ii) *Let $0 \leq \delta < \min\{\frac{1}{2}, 1 - \frac{n}{2p'}\}, 1 < p < \infty$ and $\mathbf{u}_0 \in \mathbb{B}_p^{-\delta,\infty}$. There exists $T > 0$ and a solution $\mathbf{u} \in C((0,T); \mathbb{V}_p)$ of the (16) such that $\sup_{t\in(0,T]} \sqrt{t}^{\delta} \|\mathbf{u}\|_{G(t)} < \infty$.*

Part (i) of the result was previously obtained in [3].

4.4. **Barotropic quasi-geostrophic equation.** We will now provide an application to the barotropic quasi-geostrophic equation with Newtonian (eddy) viscosity (see [36]). For simplicity, we will assume both the beta plane effect and the bottom topography to be zero. It is possible to include nonzero values of these in our approach at the expense of complicating the statement of the following result. This equation is given by

$$\check{\mathbf{u}}_t + \nabla \cdot \big((\nabla^\perp \Delta^{-1} \check{\mathbf{u}})\check{\mathbf{u}}\big) = \Delta \check{\mathbf{u}}, \tag{17}$$

where $\check{\mathbf{u}} : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ denotes potential vorticity. As before, by taking Fourier transform, we can reformulate the mild version as (3) where

$$(\mathfrak{D}\mathbf{u})(\xi) = |\xi|^2 \mathbf{u}(\xi) \text{ and } B(\mathbf{u},\mathbf{v})(\xi) = \int_{\mathbb{R}^2} \left(\frac{\xi \cdot \eta^\perp}{|\eta|^2} \mathbf{u}(\eta)\mathbf{v}(\xi - \eta)\right) d\eta,$$

where for $\eta = (\eta_1, \eta_2)$ we denote $\eta^\perp = (-\eta_2, \eta_1)$.

Theorem 4.4. *Consider the barotropic quasi-geostrophic equation (17) with Newtonian eddy viscosity on $\mathbb{R}^2$. In this case, assumptions P1-P3 are satisfied with $\sigma = 1, \gamma = 1, K = \infty$ and any $\lambda_0 > 0$. Furthermore, (5) is satisfied with $\alpha = 1, \beta_1 = -1$ and $\beta_2 = 0$. We have the following results.*

(i) *Let* $0 \le \delta < \frac{2}{p}, p > 2$ *and* $\mathbf{u}_0 \in \mathbb{B}_p^{-\delta,\infty}$. *Then there exists a* $T > 0$ *and a solution* $\mathbf{u}$ *of (17) on* $C((0,T);\mathbb{V}_p)$ *such that* $\sup_{t\in(0,T)} \|\mathbf{u}\|_{G_{\theta_0}(t)} < \infty$. *Moreover, if* $\delta = \frac{2}{p}$ *and* $\|\mathbf{u}_0\|_{\mathbb{B}_p^{-\delta,\infty}}$ *is sufficiently small, we can take* $T = \infty$.

(ii) *Let* $\mathbf{u}_0 \in \mathbb{V}_{\theta_0,p}, \theta_0 \ge -\frac{2}{p},\ 4/3 < p < \infty$. *Then there exists a solution* $\mathbf{u}$ *of (17) on* $[0,T]$ *satisfying* $\sup_{t\in(0,T)} \|\mathbf{u}\|_{G_{\theta_0}(t)} < \infty$. *In case* $\|\mathbf{u}_0\|_{-\frac{2}{p},p}$ *is sufficiently small, then* $T = \infty$.

The author is not aware of a similar analyticity result for this equation on the whole space, particularly for initial data in such a low regularity space.

5. Proof of Main Results and Applications

5.1. **Proof of Main Results.** We will need the following convolution inequality due to Kerman (Theorem 3.1, [28]).

Let $1 < p < \infty$ *and recall that* $p' = p/(p-1)$ *denotes the Hölder conjugate of* p. *Assume*

$$\max(\theta_1,\theta_2) < n/p', \theta_1+\theta_2 \ge 0, \theta_1+\theta_2 > n(\frac{1}{p'}-\frac{1}{p}).$$

Then, we have

$$\|f * g\|_{\theta_1+\theta_2-\frac{n}{p'}} \le C\|f\|_{\theta_1,p}\|g\|_{\theta_2,p}. \tag{18}$$

In case $p = \infty$, (18) holds provided $\frac{n}{2} < \theta_1,\ \theta_2 < n$ ([5]). We will also need the following crucial lemma throughout.

Lemma 5.1. *Let* γ, λ_0 *and* σ *be as in assumptions P1–P3. For any*

$$c > 0,\ 0 \le \lambda \le \frac{\lambda_0}{\sqrt{c}^{\sigma\gamma}},\ 0 \le s \le t,\ 0 \le \gamma \le 1 \text{ and } \sigma\gamma \le 2,$$

and $\xi, \eta \in \mathbb{C}^d$, *we have the following estimates*

$$|\xi|^\gamma \le |\xi-\eta|^\gamma + |\eta|^\gamma \text{ and } \quad e^{\lambda(\sqrt{t}^\sigma|\xi|)^\gamma} e^{-g(\sqrt{(t-s)/c}^\sigma\xi)} \le Ce^{\lambda(\sqrt{s}^\sigma|\xi|)^\gamma}. \tag{19}$$

Proof. For $x, y \ge 0$ and $\gamma \in [0,1]$, recall first the elementary inequality $(x+y)^\gamma \le x^\gamma + y^\gamma$. This follows easily from the fact that the function $f(\zeta) = 1 + \zeta^\gamma - (1+\zeta)^\gamma,\ \zeta > 0$ is non-negative. Thus, by triangle inequality,

$$|\xi|^\gamma \le (|\xi-\eta| + |\eta|)^\gamma \le |\xi-\eta|^\gamma + |\eta|^\gamma,$$

and the first inequality in (19) follows.

For the second, note that we have

$$e^{\lambda(\sqrt{t}^\sigma|\xi|)^\gamma} e^{-g(\sqrt{(t-s)/c}^\sigma\xi)} = e^{\lambda((\sqrt{t}^\sigma|\xi|)^\gamma - (\sqrt{s}^\sigma|\xi|)^\gamma)} e^{-g(\sqrt{(t-s)/c}^\sigma\xi)} e^{\lambda(\sqrt{s}^\sigma|\xi|)^\gamma}$$
$$\le e^{\lambda(\sqrt{t-s}^\sigma|\xi|)^\gamma - g(\sqrt{(t-s)/c}^\sigma\xi)} e^{\lambda(\sqrt{s}^\sigma|\xi|)^\gamma} = e^{\lambda\sqrt{c}^{\sigma\gamma}(\sqrt{(t-s)/c}^\sigma|\xi|)^\gamma - g(\sqrt{(t-s)/c}^\sigma\xi)} e^{\lambda(\sqrt{s}^\sigma|\xi|)^\gamma}$$
$$\le e^{\lambda_0(\sqrt{(t-s)/c}^\sigma|\xi|)^\gamma - g(\sqrt{(t-s)/c}^\sigma\xi)} e^{\lambda(\sqrt{s}^\sigma|\xi|)^\gamma} \le Ce^{\lambda(\sqrt{s}^\sigma|\xi|)^\gamma},$$

where, to obtain the first inequality in the second line above chain of inequalities, we used $\sqrt{t}^{\sigma\gamma} - \sqrt{s}^{\sigma\gamma} \le \sqrt{t-s}^{\sigma\gamma}$ (which, in turn, follows from the first inequality in (19)), while the inequalities in the last line follow from the assumption on the range of λ as well as P3.

□

The following proposition is useful for establishing Gevrey regularity.

Proposition 5.2. *Let $0 \le \lambda \le \frac{\lambda_0}{\sqrt{2}^{\sigma\gamma}}$ and $\sigma\gamma \le 2$, where $\lambda_0, \sigma, \gamma$ are as in assumptions P1–P3. For $0 \le s \le t$, we have*

$$\|e^{-(t-s)\mathfrak{D}}\mathbf{u}\|_{G(t)} \le C\|e^{-\frac{t-s}{2}\mathfrak{D}}\mathbf{u}\|_{G(s)}. \tag{20}$$

Proof. We have the following estimates

$$\begin{aligned}
\|e^{-(t-s)\mathfrak{D}}\mathbf{u}\|^p_{G(t)} &= \int e^{\lambda p(\sqrt{t}^\sigma|\xi|)^\gamma}|e^{-\frac{t-s}{2}f(\xi)}e^{-\frac{t-s}{2}f(\xi)}\mathbf{u}(\xi)|^p\,d\xi \\
&\le C\int e^{\lambda p(\sqrt{t}^\sigma|\xi|)^\gamma}e^{-pg(\sqrt{(t-s)/2}^\sigma\xi)}|e^{-\frac{t-s}{2}f(\xi)}\mathbf{u}(\xi)|^p\,d\xi \\
&\le C\int e^{\lambda p(\sqrt{s}^\sigma|\xi|)^\gamma}|e^{-\frac{t-s}{2}f(\xi)}\mathbf{u}(\xi)|^p\,d\xi = C\|e^{-\frac{t-s}{2}\mathfrak{D}}\mathbf{u}\|^p_{G(s)},
\end{aligned}$$

where the first inequality follows from P1 and P2 and the second from the two inequalities in (19). □

Proposition 5.3. *Let $\vartheta \ge 0, \delta \ge 0$ and $0 \le \lambda \le 2^{-\sigma\gamma}\lambda_0$ be fixed. If $\mathbf{u}_0 \in \mathbb{B}_p^{-\delta,\infty}$, then*

$$\sup_{0<t<T}\sqrt{t}^{\delta+\sigma\vartheta}\|\Lambda^\vartheta e^{-t\mathfrak{D}}\mathbf{u}_0\|_{G(t)} \le C\|\mathbf{u}_0\|_{\mathbb{B}_p^{-\delta,\infty}}.$$

Moreover, if $\delta < 1$ and $F : [0,T] \to \mathbb{V}$ be such that

$$\sup_{0\le s\le T}\|\sqrt{s}^{1-\delta}F(s)\|_{G(s)} < \infty,$$

then we have

$$\sup_{0\le t\le T}\sqrt{t}^{\delta+\sigma\vartheta}\|\Lambda^\vartheta\int_0^t e^{-(t-s)\mathfrak{D}}F(s)\,ds\|_{G(t)} \le C\sup_{0\le s\le T}\|\sqrt{s}^{1-\delta}F(s)\|_{G(s)}.$$

Proof. The proof essentially follows from the definition of the Besov norm and using properties P1-P3.

$$\begin{aligned}
&\|\Lambda^\vartheta e^{-t\mathfrak{D}}\mathbf{u}_0\|^p_{G(t)} = \int e^{\lambda p(\sqrt{t}^\sigma|\xi|)^\gamma}|e^{-t\mathfrak{D}}\Lambda^\vartheta\mathbf{u}_0|^p\,d\xi = \int e^{\lambda p(\sqrt{t}^\sigma|\xi|)^\gamma}|e^{-\frac{t}{4}\mathfrak{D}}e^{-\frac{t}{4}\mathfrak{D}}\Lambda^\vartheta e^{-\frac{t}{2}\mathfrak{D}}\mathbf{u}_0|^p\,d\xi \\
&\le C\int e^{\lambda p(\sqrt{t}^\sigma|\xi|)^\gamma}e^{-pg(\sqrt{t/4}^\sigma\xi)}e^{-pg(\sqrt{t/4}^\sigma\xi)}|\Lambda^\vartheta e^{-\frac{t}{2}\mathfrak{D}}\mathbf{u}_0|^p\,d\xi \\
&\le C\int e^{\lambda p(\sqrt{t}^\sigma|\xi|)^\gamma}e^{-pg(\sqrt{t/4}^\sigma\xi)}e^{-pg(\sqrt{t/4}^\sigma\xi)}|\xi|^{p\vartheta}|e^{-\frac{t}{2}\mathfrak{D}}\mathbf{u}_0|^p\,d\xi \\
&\le C\sqrt{t}^{-p\sigma\vartheta}\int|e^{-\frac{t}{2}\mathfrak{D}}\mathbf{u}_0|^p\,d\xi \le C\sqrt{t}^{-p(\sigma\vartheta+\delta)}\|\mathbf{u}_0\|^p_{\mathbb{B}_p^{-\delta,\infty}},
\end{aligned}$$

provided $0 \leq \lambda \leq 2^{-\sigma\gamma}\lambda_0$. The first inequality in the last line above follows using property P2 and subsequently proceeding as in Proposition 5.2.

The second part of the proposition concerning F can be proven in a similar manner.

□

Proof of Theorem 3.1

The strategy of the proof is similar to [47]. Let $T \leq K$ be fixed for now, where K is as in P1. Let $\mathbb{V}_p = L^p \cap \mathbb{V}$ and $C_c((0,T);\mathbb{V}_p)$ denote the space of all functions from $(0,T)$ to $\mathbb{V}_p$ that are continous (in L^p norm) when restricted to any compact subspace of $(0,T)$. Consider the path space endowed with the path space norm

$$\Sigma := \{\mathbf{u} \in C_c((0,T);\mathbb{V}_p) : \\ \|\mathbf{u}\|_\Sigma := \sup_{0<t<T} \max\{\sqrt{t}^{\delta}\|\mathbf{u}\|_{G(t)},\ \sqrt{t}^{\delta+\sigma\vartheta}\|\Lambda^{\vartheta}\mathbf{u}\|_{G(t)}\} < \infty\Big\}, \tag{21}$$

for adequate $\vartheta \geq 0$. Here $G(\cdot)$ denotes the Gevrey norm as defined in (6) and the number $\vartheta \geq 0$ will be specified later. Clearly, Σ is a Banach space.

Let $[T_1, T_2] \subset (0,T)$. For any $t \in [T_1, T_2]$, we may write $e^{-tf(\xi)}\mathbf{u}_0 = e^{-sf(\xi)}e^{-T_1 f(\xi)}\mathbf{u}_0$, where $s \in (0, T_2 - T_1]$. By the definition of the space $B_p^{-\delta}$, we have $e^{-T_1 f(\xi)}\mathbf{u}_0 \in \mathbb{V}_p$. Since the map $t \to e^{-tf(\xi)}$ is continuous for each fixed ξ, by the Dominated Convergence Theorem, $e^{-t\mathfrak{D}}\mathbf{u}_0$ belongs to $C_c((0,T);\mathbb{V}_p)$. By a similar argument, $\int_0^t e^{-(t-s)\mathfrak{D}}F(s)\,ds$ belongs to $C_c((0,T);\mathbb{V}_p)$ as well. Now applying Proposition 5.3, it follows that both $e^{-t\mathfrak{D}}\mathbf{u}_0$ and $\int_0^t e^{-(t-s)\mathfrak{D}}F(s)\,ds$ belong to Σ (which, in particular, shows that $\Sigma \neq \{0\}$) and moreover, with M as defined in Theorem 3.1,

$$\|e^{-t\mathfrak{D}}\mathbf{u}_0\|_\Sigma + \left\|\int_0^t e^{-(t-s)\mathfrak{D}}F(s)\,ds\right\|_\Sigma \leq M < \infty. \tag{22}$$

For $\mathbf{u}, \mathbf{v} \in \Sigma$, define

$$b(\mathbf{u},\mathbf{v}) = \int_0^t e^{-(t-s)\mathfrak{D}}B[\mathbf{u}(s),\mathbf{v}(s)]\,ds. \tag{23}$$

Let

$$\mathcal{E} = \{\mathbf{u} \in \Sigma : \|\mathbf{u} - e^{-t\mathfrak{D}}\mathbf{u}_0 - \int_0^t e^{-(t-s)\mathfrak{D}}F(s)\,ds\|_\Sigma \leq M\}.$$

We define the map

$$S\mathbf{u} = e^{-t\mathfrak{D}}\mathbf{u}_0 + b(\mathbf{u},\mathbf{u}) + \int_0^t e^{-(t-s)\mathfrak{D}}F(s)\,ds, \quad \mathbf{u} \in \mathcal{E}, \tag{24}$$

and show that it is a contractive self map of $\mathcal{E}$. This then implies by contraction mapping principle that there is a unique fixed point which solves (3). In order to do this, it will be enough to obtain an estimate of the form

$$\|b(\mathbf{u},\mathbf{v})\|_\Sigma \leq C\sqrt{T}^{\mu}\|\mathbf{u}\|_\Sigma\|\mathbf{v}\|_\Sigma \text{for all } \mathbf{u},\mathbf{v} \in \mathcal{E}, \tag{25}$$

where μ is as in (8). If $\mu > 0$, then we may choose $T < \left(\frac{1}{4CM}\right)^{2/\mu}$, in which case S will turn out to be a contractive self map of $\mathcal{E}$. On the other hand, if $\mu = 0$, then we are in the critical space where we can obtain a solution with $T = \infty$ if M is sufficiently small.

We now proceed to obtain (25). We have the estimate

$$\begin{aligned}
&\sqrt{t}^{\delta+\sigma\vartheta}\|\Lambda^{\vartheta} b(\mathbf{u},\mathbf{v})\|_{G(t)} \\
&= \sqrt{t}^{\delta+\sigma\vartheta}\left\{\int e^{\lambda p(\sqrt{t}^{\sigma}|\xi|)^{\gamma}}|\xi|^{p\vartheta}\left|\int_0^t e^{-(t-s)\mathfrak{D}}B[\mathbf{u}(s),\mathbf{v}(s)]\,ds\right|^p d\xi\right\}^{1/p} \\
&\le \sqrt{t}^{\delta+\sigma\vartheta}\int_0^t\left\{\int e^{\lambda p(\sqrt{t}^{\sigma}|\xi|)^{\gamma}}|\xi|^{p\vartheta}|(e^{-(t-s)\mathfrak{D}}B[\mathbf{u}(s),\mathbf{v}(s)])(\xi)|^p\,d\xi\right\}^{1/p} ds \\
&= \sqrt{t}^{\delta+\sigma\vartheta}\int_0^t \|e^{-(t-s)\mathfrak{D}}\Lambda^{\vartheta}B[\mathbf{u}(s),\mathbf{v}(s)]\|_{G(t)}\,ds \\
&\le C\sqrt{t}^{\delta+\sigma\vartheta}\int_0^t \|e^{-\frac{t-s}{2}\mathfrak{D}}\Lambda^{\vartheta}B[\mathbf{u}(s),\mathbf{v}(s)]\|_{G(s)}\,ds,
\end{aligned} \tag{26}$$

where the inequality in the third line above is obtained using Minkowski's inequality while for the last inequality above, we used (20). We will now estimate the term

$$\|e^{-\frac{t-s}{2}\mathfrak{D}}\Lambda^{\vartheta}B[\mathbf{u}(s),\mathbf{v}(s)]\|_{G(s)}$$

in (26). To that end,

$$\begin{aligned}
&\|e^{-\frac{t-s}{2}\mathfrak{D}}\Lambda^{\vartheta}B[\mathbf{u}(s),\mathbf{v}(s)]\|_{G(s)}^p = \int e^{\lambda p(\sqrt{s}^{\sigma}|\xi|)^{\gamma}}|\xi|^{p\vartheta}|e^{-\frac{t-s}{2}\mathfrak{D}}B[\mathbf{u}(s),\mathbf{v}(s)]|^p\,d\xi \\
&\le C\int e^{\lambda p(\sqrt{s}^{\sigma}|\xi|)^{\gamma}}e^{-pg(\sqrt{\frac{t-s}{2}}^{\sigma}\xi)}|\xi|^{p\vartheta}|B[\mathbf{u}(s),\mathbf{v}(s)]|^p \\
&\le C\int e^{\lambda p(\sqrt{s}^{\sigma}|\xi|)^{\gamma}}e^{-pg(\sqrt{\frac{t-s}{2}}^{\sigma}\xi)}|\xi|^{p(\vartheta+\alpha)}\left(|\Lambda^{\beta_1}\mathbf{u}|*|\Lambda^{\beta_2}\mathbf{v}|\right)^p\,d\xi \\
&= C\int e^{\lambda p(\sqrt{s}^{\sigma}|\xi|)^{\gamma}}e^{-pg(\sqrt{\frac{t-s}{2}}^{\sigma}\xi)}|\xi|^{p(\vartheta+\alpha)}\left|\int|\xi-\eta|^{\beta_1}|\mathbf{u}(\xi-\eta)||\eta|^{\beta_2}|\mathbf{v}(\eta)|\,d\eta\right|^p d\xi \\
&\le C\int e^{-pg(\sqrt{\frac{t-s}{2}}^{\sigma}\xi)}|\xi|^{p(\vartheta+\alpha)} \\
&\qquad\qquad\left|\int e^{\lambda(\sqrt{s}^{\sigma}|\xi-\eta|)^{\gamma}}|\xi-\eta|^{\beta_1}|\mathbf{u}(\xi-\eta)|e^{\lambda(\sqrt{s}^{\sigma}|\eta|)^{\gamma}}|\eta|^{\beta_2}|\mathbf{v}(\eta)|\,d\eta\right|^p d\xi \\
&= C\int e^{-pg(\sqrt{\frac{t-s}{2}}^{\sigma}\xi)}|\xi|^{p(\vartheta+\alpha)}(f_{\mathbf{u}}*f_{\mathbf{v}})^p(\xi)\,d\xi,
\end{aligned} \tag{27}$$

where $f_{\mathbf{u}}(\xi) = e^{\lambda(\sqrt{s}^{\sigma}|\xi|)^{\gamma}}|\xi|^{\beta_1}|\mathbf{u}(\xi,s)|$ and $f_{\mathbf{v}}(\xi) = e^{\lambda(\sqrt{s}^{\sigma}|\xi|)^{\gamma}}|\xi|^{\beta_2}|\mathbf{v}(\xi,s)|$. In the above chain of inequalities, in order to obtain the first inequality we used property P1, to obtain the second we used (5) and finally, to get the third, we used the fact that $|\xi|^{\gamma} \le |\xi-\eta|^{\gamma} + |\eta|^{\gamma}$ for all $\xi,\eta \in \mathbb{R}^n$ and $\gamma \in [0,1]$. Note now that from the definition of the path space norm

$\|\cdot\|_\Sigma$, we have

$$\|f_{\mathbf{u}}(\cdot,s)\|_{\vartheta-\beta_1,p} \le C\frac{\|\mathbf{u}\|_\Sigma}{\sqrt{s}^{\delta+\sigma\vartheta}}, \quad \|f_{\mathbf{v}}(\cdot,s)\|_{\vartheta-\beta_2,p} \le C\frac{\|\mathbf{u}\|_\Sigma}{\sqrt{s}^{\delta+\sigma\vartheta}}.$$

Moreover, provided $\alpha+\beta_1+\beta_2+\frac{n}{p'}-\vartheta \ge 0$, from P2 we also have

$$|\xi|^{\alpha+\beta_1+\beta_2+\frac{n}{p'}-\vartheta} e^{-g(\sqrt{\frac{t-s}{2}}^\sigma \xi)} \le \frac{C}{\sqrt{t-s}^{\sigma(\alpha+\beta_1+\beta_2+\frac{n}{p'}-\vartheta)}}.$$

Using these facts and applying (18) with $\theta_1=\vartheta-\beta_1, \theta_2=\vartheta-\beta_2$, from (27) we have

$$\|e^{-\frac{t-s}{2}\mathfrak{D}}\Lambda^\vartheta B[\mathbf{u}(s),\mathbf{v}(s)]\|_{G(s)} \le C\frac{\|\mathbf{u}\|_\Sigma\|\mathbf{v}\|_\Sigma}{\sqrt{s}^{2(\vartheta\sigma+\delta)}\sqrt{t-s}^{\sigma(\alpha+\beta_1+\beta_2+\frac{n}{p'}-\vartheta)}}. \tag{28}$$

The conditions on the parameters from applying P2 and (18) thus far are

$$\alpha+\beta_1+\beta_2+\frac{n}{p'}-\vartheta\ge 0, \max(\vartheta-\beta_1,\vartheta-\beta_2) < \frac{n}{p'},$$
$$2\vartheta-(\beta_1+\beta_2)\ge 0, 2\vartheta-(\beta_1+\beta_2) > n(\frac{1}{p'}-\frac{1}{p}).$$

Now inserting the estimate obtained in (28) in (26), we obtain

$$\begin{aligned}\sqrt{t}^{\delta+\sigma\vartheta}\|\Lambda^\vartheta b(\mathbf{u},\mathbf{v})\|_{G(t)} &\le C\|\mathbf{u}\|_\Sigma\|\mathbf{v}\|_\Sigma\sqrt{t}^{\delta+\sigma\vartheta}\int_0^t \frac{1}{\sqrt{s}^{2(\vartheta\sigma+\delta)}\sqrt{t-s}^{\sigma(\alpha+\beta_1+\beta_2+\frac{n}{p'}-\vartheta)}}\,ds \\ &\le C\|\mathbf{u}\|_\Sigma\|\mathbf{v}\|_\Sigma\sqrt{t}^{2-\delta-\sigma(\alpha+\beta_1+\beta_2+\frac{n}{p'})},\end{aligned}$$

provided

$$\vartheta\sigma+\delta<1, \sigma(\alpha+\beta_1+\beta_2+\frac{n}{p'}-\vartheta)<2.$$

We can estimate $\sqrt{t}^\delta\|b(\mathbf{u},\mathbf{v})\|_{G(t)}$ by proceeding exactly as above. Since the integrand in (27) contains the term $|\xi|^{p\alpha}$ (instead of $|\xi|^{p(\vartheta+\alpha)}$) we need to apply the estimate

$$|\xi|^{\alpha+\beta_1+\beta_2+\frac{n}{p'}-2\vartheta} e^{-g(\sqrt{\frac{t-s}{2}}^\sigma \xi)} \le \frac{C}{\sqrt{t-s}^{\sigma(\alpha+\beta_1+\beta_2+\frac{n}{p'}-2\vartheta)}},$$

for which we need the condition $\alpha+\beta_1+\beta_2+\frac{n}{p'}-2\vartheta\ge 0$. Since $\vartheta\ge 0$, this requirement is stronger than the previously obtained condition $\alpha+\beta_1+\beta_2+\frac{n}{p'}-\vartheta\ge 0$. The remaining conditions either remain unaltered or become weaker. These two estimates together yield

$$\|b(\mathbf{u},\mathbf{v})\|_\Sigma \le C\|\mathbf{u}\|_\Sigma\|\mathbf{v}\|_\Sigma\sqrt{T}^\mu,$$

where μ is as in (8). This is the desired estimate (25).

Recall that we now need to choose $\vartheta\ge 0$ and $\mu\ge 0$ subject to the previously prescribed constraints. In the resulting inequalities, all terms that occur except ϑ involve the given parameters in the problem. We rewrite these constraints as upper and lower bounds (in some cases, strict) for ϑ to obtain:
(i) $0\le\vartheta<\frac{1-\delta}{\sigma}$ (ii) $0\le 2\vartheta\le\alpha+\beta_1+\beta_2+\frac{n}{p'}$ (iii) $\vartheta\le\min\{\frac{n}{p'}+\beta_1,\frac{n}{p'}+\beta_2\}$ (iv) $\vartheta\ge\frac{\beta_1+\beta_2}{2}$

(v) $\vartheta > \frac{\beta_1+\beta_2}{2} + \frac{n}{2}\left(\frac{1}{p'} - \frac{1}{p}\right)$ (vi) $\vartheta > \alpha + \beta_1 + \beta_2 + \frac{n}{p'} - \frac{2}{\sigma}$.
In order to be able to make a choice of ϑ satisfying these constraints, we now require that all upper bounds exceed all lower bounds, with strict inequalities if the corresponding inequalities for ϑ are strict. This in turn leads to the conditions on the parameters specified in the theorem.

We will now prove part (c). From proofs of parts (a) and (b), for each t, we have

$$\int_0^t \|e^{\lambda(\sqrt{t}^\sigma|\xi|)^\gamma} e^{-(t-s)\mathfrak{D}} B(\mathbf{u},\mathbf{u})(\xi)\|_p \, ds < \infty. \tag{29}$$

We now note that $\left(\int e^{\lambda p'(\sqrt{t}^\sigma|\xi|)^\gamma}\, d\xi\right)^{1/p'} \leq \frac{C}{\sqrt{t}^{\sigma n/p'}}$. Using H older's inequality in (29) and this estimate, it follows that

$$\int_0^t |(e^{-(t-s)\mathfrak{D}} B(\mathbf{u},\mathbf{u}))(\xi)| \, d\xi \, ds < \infty$$

By Fubini's theorem, this in turn implies that

$$\mathcal{F}^{-1}(\int_0^t e^{-(t-s)\mathfrak{D}} B(\mathbf{u},\mathbf{u})\, ds) = \int_0^t \mathcal{F}^{-1}(e^{-(t-s)\mathfrak{D}} B(\mathbf{u},\mathbf{u}))\, ds.$$

This finishes the proof.

Proof of Theorem 3.2

We define the path space

$$\Sigma := \left\{ \mathbf{u} \in C_c((0,T); \mathbb{V}_{\theta_0,p}) : \|\mathbf{u}\|_\Sigma := \sup_{0<t<T} \max\{\|\mathbf{u}\|_{G_{\theta_0}(t)},\ \sqrt{t}^{\sigma\vartheta}\|\mathbf{u}\|_{G_{\theta_0+\vartheta}(t)}\} < \infty \right\}$$

for adequate $\vartheta \geq 0$. The proof of the contraction mapping argument is similar to Theorem 3.1 and is thus omitted. Concerning the proof of the fact that $\check{\mathbf{u}} = \mathcal{F}^{-1}(\mathbf{u})$ is a solution of (2), we proceed as in the part (c) of Theorem 3.1. Using the definition of the Gevrey class norm on the path space Σ and the fact that

$$\left(\int |\xi|^{-\theta_0 p'} e^{\lambda p'(\sqrt{t}^\sigma|\xi|)^\gamma}\, d\xi\right)^{1/p'} \leq \frac{C}{\sqrt{t}^{\frac{\sigma n}{p'}-\theta_0}} \text{ (provided } \theta_0 p' < n),$$

we conclude that $\int_0^t |(e^{-(t-s)\mathfrak{D}} B(\mathbf{u},\mathbf{u}))(\xi)| \, d\xi \, ds < \infty$.

5.2. **Borderline Case $p = 1$.** We will now provide the proofs for the 3D Navier-Stokes equations and the critical quasi-geostrophic equation.

5.2.1. *3D Navier-Stokes equations.* Here, $f(\xi) = g(\xi) = |\xi|^2$ in Assumption P1, $\sigma = 1$ in Assumption P2 and Assumption P3 holds for any $\lambda_0 > 0$ with $\gamma = 1$.

For $\mathbf{u} : [0,T] \to \mathbb{V}$, we define

$$\|\mathbf{u}\|_{\Sigma'} := \int \left(\int_0^T |e^{\lambda\sqrt{t}|\xi|}\mathbf{u}(\xi,t)|^2\, dt\right)^{1/2} d\xi.$$

We will need the following path space with the corresponding path space norm

$$\Sigma := \left\{ \mathbf{u} \in C_c((0,T);\mathbb{V}_{-1,1}) : \|\mathbf{u}\|_\Sigma := \max\{ \sup_{0<t<T} \|\mathbf{u}\|_{G_{-1}(t)}, \|\mathbf{u}\|_{\Sigma'}\} < \infty \right\}. \tag{30}$$

Proposition 5.4. *For* $\mathbf{u}_0 \in \mathbb{V}_{-1,1}$ *we have*

$$\|e^{-t\mathfrak{D}}\mathbf{u}_0\|_\Sigma \le e^{-\lambda^2}\|\mathbf{u}_0\|_{-1,1} \text{ and } \lim_{T\to 0^+} \|e^{-t\mathfrak{D}}\mathbf{u}_0\|_\Sigma = 0. \tag{31}$$

Proof. First note that with $f(\xi) = |\xi|^2$, we have

$$\int_0^T e^{-tf(\xi)}\, dt = \frac{1}{|\xi|^2}\left(1 - e^{-T|\xi|^2}\right) \le \frac{1}{|\xi|^2}. \tag{32}$$

Now as in the proof of Proposition 5.3, we have

$$\int_0^T |e^{\lambda\sqrt{t}|\xi|}\mathbf{u}(\xi,t)|^2\, dt = \int_0^T e^{2\lambda\sqrt{t}|\xi| - t|\xi|^2}|e^{-\frac{t}{2}|\xi|^2}\mathbf{u}_0(\xi)|^2\, dt$$
$$\le e^{\lambda^2}|\mathbf{u}_0|^2 \int_0^T e^{-t|\xi|^2}\, dt \le |\mathbf{u}_0(\xi)|^2\frac{e^{\lambda^2}}{|\xi|^2},$$

where to derive the first inequality of the second line above, we have used the fact that

$$2\lambda\sqrt{t}|\xi| - t|\xi|^2 = 2\lambda|\sqrt{t}\xi| - |\sqrt{t}\xi|^2 \le \lambda^2 \text{ for all } \xi \in \mathbb{R}^3.$$

Using this estimate, it immediately follows that

$$\int \left(\int_0^T |e^{\lambda(\sqrt{t^\sigma}|\xi|)^\gamma}\mathbf{u}(\xi,t)|^2\, dt\right)^{1/2} d\xi \le e^{\lambda^2}\|\mathbf{u}_0\|_{-1,1}.$$

The term $\sup_{0<t<T}\|\mathbf{u}\|_{G_{-1}(t)}$ can be estimated in a similar manner. this proves the first relation in (31).

For the second, let $\epsilon > 0$ be given. Define $\mathbf{v}_0(\xi) = \mathbf{1}_{|\xi|\le N}\mathbf{u}_0(\xi)$ and $\mathbf{w}_0 = \mathbf{u}_0 - \mathbf{v}_0$ and note that $\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{w}_0$. Here N is chosen large enough such that $\|\mathbf{w}_0\| < \dfrac{\epsilon}{2e^{\lambda^2}}$. Thus, from the first part of (31),

$$\|e^{-t\mathfrak{D}}\mathbf{u}_0\|_\Sigma \le \|e^{-t\mathfrak{D}}\mathbf{v}_0\|_\Sigma + \|e^{-t\mathfrak{D}}\mathbf{w}_0\|_\Sigma \le \|e^{-t\mathfrak{D}}\mathbf{v}_0\|_\Sigma + \frac{\epsilon}{2}.$$

It will now be enough to show that $\lim_{T\to 0^+}\|e^{-t\mathfrak{D}}\mathbf{w}_0\|_\Sigma = 0$. To that end, proceeding exactly as in the proof of the first part of (31) and recalling the definition of $\mathbf{v}_0$, we obtain

$$\|e^{-t\mathfrak{D}}\mathbf{v}_0\|_\Sigma \le \int_{|\xi|\le N} |\mathbf{u}_0(\xi)|\frac{e^{\frac{\lambda^2}{2}}}{|\xi|}\sqrt{(1 - e^{-T|\xi|^2})}d\xi$$
$$\le \sqrt{(1 - e^{-TN^2})}\, e^{\frac{\lambda^2}{2}} \int \frac{|\mathbf{u}_0(\xi)|}{|\xi|} = \sqrt{(1 - e^{-TN^2})}\, e^{\frac{\lambda^2}{2}}\|\mathbf{u}_0\|_{-1,1}.$$

Letting $T \to 0^+$ in the right hand side of the above inequality concludes the proof. $\square$

Proof of Theorem 3.3

As before, it will be enough to obtain an estimate of the form

$$\|b(\mathbf{u},\mathbf{v})\|_{\Sigma} \le C_T \|\mathbf{u}\|_{\Sigma'} \|\mathbf{v}\|_{\Sigma'}, \tag{33}$$

where $\|\cdot\|_{\Sigma}$ is as in (30). Using P1–P3 and the second inequality in (19), we have

$$\begin{aligned} &e^{\lambda\sqrt{t}|\xi|} e^{-(t-s)|\xi|^2} |B[\mathbf{u},\mathbf{v}](\xi)| \\ &\le C e^{\lambda\sqrt{s}|\xi|} |\xi|^{\alpha} (|\mathbf{u}| * |\mathbf{v}|)(\xi) \le C e^{-\frac{(t-s)}{2}|\xi|^2} |\xi|^{\alpha} (\tilde{\mathbf{u}} * \tilde{\mathbf{v}})(\xi), \end{aligned} \tag{34}$$

where $\tilde{\mathbf{u}}(\xi, s) = e^{\lambda\sqrt{s}|\xi|}|\mathbf{u}(\xi, s)|$ and $\tilde{\mathbf{v}}(\xi, s) = e^{\lambda\sqrt{s}|\xi|}|\mathbf{v}(\xi, s)|$. In order to obtain the last inequality in (34), we used the first inequality in (19) with $\gamma = 1$. Using (34) we obtain

$$\begin{aligned} &\|b(\mathbf{u},\mathbf{v})\|_{G_{-1}(t)} \le \int |\xi|^{-1} \int_0^t e^{\lambda\sqrt{t}|\xi|} e^{-(t-s)|\xi|^2} |B[\mathbf{u},\mathbf{v}](\xi)|\, ds\, d\xi \\ &\le C \int \int_0^t (\tilde{\mathbf{u}} * \tilde{\mathbf{v}})(\xi)\, ds\, d\xi \\ &\le C \int \left(\int_0^T \tilde{\mathbf{u}}^2(\cdot, s)\, ds \right)^{1/2} * \left(\int_0^T \tilde{\mathbf{v}}^2(\cdot, s)\, ds \right)^{1/2} \le C \|\mathbf{u}\|_{\Sigma'} \|\mathbf{v}\|_{\Sigma'}, \end{aligned}$$

where the inequalities in the last line follow from Cauchy-Schwartz and the fact that L^1 is a Banach algebra under convolution.

We will now estimate the second term in (30). To that end, using (34) and Minkowski's inequality, we have

$$\begin{aligned} &\int \left(\int_0^T \left(\int_0^t e^{\lambda\sqrt{t}|\xi|} e^{-(t-s)|\xi|^2} |B[\mathbf{u},\mathbf{v}]\, ds \right)^2 dt \right)^{1/2} d\xi \\ &= \int \left(\int_0^T \left(\int_0^T \mathbf{1}_{\{s\le t\}} e^{\lambda\sqrt{t}|\xi|} e^{-(t-s)|\xi|^2} |B[\mathbf{u},\mathbf{v}]|\, ds \right)^2 dt \right)^{1/2} d\xi \\ &\le C \int \left(\int_0^T \left(\int_0^T \mathbf{1}_{\{s\le t\}} |\xi| e^{-\frac{t-s}{2}|\xi|^2} (\tilde{\mathbf{u}} * \tilde{\mathbf{v}})(\xi, s)\, ds \right)^2 dt \right)^{1/2} d\xi \\ &\le C \int \int_0^T |\xi| (\tilde{\mathbf{u}} * \tilde{\mathbf{v}})(\xi, s) \left(\int_0^T \mathbf{1}_{\{s\le t\}} e^{-(t-s)|\xi|^2}\, dt \right)^{1/2} ds\, d\xi \qquad (35) \\ &\le C \int \left(\int_0^T (\tilde{\mathbf{u}} * \tilde{\mathbf{v}})(\xi, s)\, ds \right) d\xi \qquad (36) \\ &\le C \int \left(\int_0^T |\tilde{\mathbf{u}}(\cdot, s)|^2\, ds \right)^{1/2} * \left(\int_0^T |\tilde{\mathbf{v}}(\cdot, s)|^2\, ds \right)^{1/2} d\xi \le C_T \|\mathbf{u}\|_{\Sigma'} \|\mathbf{v}\|_{\Sigma'}, \end{aligned}$$

where to obtain (35) we used Minkowski while to obtain (36) we used (32). The last inequality follows from the fact that L^1 is a Banach algebra under convolution and the definition of the $\|\cdot\|_{\Sigma'}$ norm. We have thus established (33) which, in view of Proposition (5.4), is enough to prove the global result for small initial data.

5.2.2. *Critical Quasi-geostrophic equation.* In this case, in Assumptions P1–P3, $f(\xi) = g(\xi) = |\xi|, \gamma = 1, \sigma = 2$ with $\lambda_0 = \frac{1}{2}$. For $\mathbf{u} : [0,T] \to \mathbb{V}$ and $\lambda \le \frac{1}{2}$, we define

$$\|\mathbf{u}\|_\Sigma := \int \left(\sup_{0<t\le T} |e^{\lambda t|\xi|}\mathbf{u}(\xi,t)| \right) d\xi.$$

Note that $\sup_{0<t<T} \|\mathbf{u}\|_{G(t)} \le \|\mathbf{u}\|_\Sigma$, where recall that $\|\mathbf{u}\|_{G(t)} = \int |e^{\lambda t|\xi|}\mathbf{u}(\xi,t)|\, d\xi$. We consider the path space

$$\Sigma := \{\mathbf{u} \in C_c((0,T);\mathbb{V}_1) : \|\mathbf{u}\|_\Sigma < \infty\}. \tag{37}$$

We will need the following proposition.

Proposition 5.5. *For $\mathbf{u}_0 \in \mathbb{V}_1$ we have*

$$\|e^{-t\mathfrak{D}}\mathbf{u}_0\|_\Sigma \le \|\mathbf{u}_0\|_{-1,1}. \tag{38}$$

Proof. The proof of this proposition simply follows by noting that since $\lambda \le \frac{1}{2}$, we have

$$\sup_{0<t<T} \left(e^{\lambda t|\xi|}e^{-t|\xi|}|\mathbf{u}_0(\xi)|\right) \le \sup_{0<t<T} \left(e^{-\frac{1}{2}t|\xi|}|\mathbf{u}_0(\xi)|\right) \le |\mathbf{u}_0(\xi)|. \tag{39}$$

□

Proof of Theorem 3.4

For any $\mathbf{w} \in C([0,T];\mathbb{V}_1)$, denote

$$\tilde{\mathbf{w}}(\xi,s) = e^{\lambda\sqrt{s}|\xi|}|\mathbf{v}(\xi,s)| \text{ and } \tilde{W}(\xi) = \sup_{0<s\le T} e^{\lambda\sqrt{s}|\xi|}|\mathbf{w}(\xi,s)| = \sup_{0<s\le T} \tilde{\mathbf{w}}(\xi,s). \tag{40}$$

Since $\lambda \le \frac{1}{2}$, we have

$$\begin{aligned} &|e^{\lambda t|\xi|}e^{-(t-s)|\xi|}B[\mathbf{u},\mathbf{v}](\xi,s)| = e^{-(\frac{1}{2}-\lambda)(t-s)|\xi|}e^{-\frac{t-s}{2}|\xi|}e^{\lambda s|\xi|}|B[\mathbf{u},\mathbf{v}](\xi,s)| \\ &\le e^{-\frac{t-s}{2}|\xi|}e^{\lambda s|\xi|}|\xi|(|\mathbf{u}| * |\mathbf{v}|)(\xi,s) \le e^{-\frac{t-s}{2}|\xi|}|\xi|(\tilde{\mathbf{u}} * \tilde{\mathbf{v}})(\xi,s), \end{aligned} \tag{41}$$

where to obtain the last inequality, we used the first inequality in (19). It is easy to see that for each ξ and s,

$$(\tilde{\mathbf{u}} * \tilde{\mathbf{v}})(\xi,s) \le (\tilde{U} * \tilde{V})(\xi) \text{ where } \tilde{U}(\xi) = \sup_{0<s<T} \tilde{\mathbf{u}}(\xi,s), \tilde{V}(\xi) = \sup_{0<s<T} \tilde{\mathbf{v}}(\xi,s).$$

Using this fact and (41), we obtain

$$\begin{aligned} &\sup_{0<t<T} \int_0^t |e^{\lambda\sqrt{t}|\xi|}e^{-(t-s)|\xi|^2}B[\mathbf{u},\mathbf{v}](\xi)|\, ds \\ &\le \sup_{0<t<T} \int_0^t e^{-\frac{t-s}{2}|\xi|}|\xi|(\tilde{\mathbf{u}} * \tilde{\mathbf{v}})(\xi)\, ds \\ &\le (\tilde{U} * \tilde{V})(\xi) \sup_{0<t<T} \int_0^t e^{-\frac{t-s}{2}|\xi|}|\xi| \le 2(\tilde{U} * \tilde{V})(\xi)\left(1 - e^{-\frac{T}{2}|\xi|}\right). \end{aligned} \tag{42}$$

From (42) we obtain

$$\|b(\mathbf{u},\mathbf{v})\|_\Sigma \le 2\int (\tilde{U} * \tilde{V})(\xi)d\xi \le 2\|\tilde{U}\|_1\|\tilde{V}\|_1 = 2\|\mathbf{u}\|_\Sigma\|\mathbf{v}\|_\Sigma,$$

where the last equality follows from the definition of $\tilde{U}, \tilde{V}$ and $\|\cdot\|_\Sigma$. As before, this is enough to establish global well-posedness for small data.

5.3. **Proof of Applications.** Proofs of Theorems 4.3 and 4.4 follow directly from Theorem 3.1 and 3.2. We focus on the proofs of Theorems 4.1 and 4.2.

Proof of Theorem 4.1: In this case, we define $\mathbb{V} = \{\mathbf{v} \in \mathbb{V}' : \xi \cdot \mathbf{v}(\xi) = 0\}$. The first two parts follow directly from Theorems 3.1 and 3.2. For part (iii), we first remark that due the results in [19] (see, for instance, Theorem 15.2 in [35]), there exists $\epsilon > 0$ such that if $\|\check{\mathbf{u}}(t_0)\|_{\dot{\mathbb{H}}^{1/2}} < \epsilon$ for some $t_0 > 0$, then there exists an unique weak (and mild) solution of the 3D NSE on $(t_0, \infty) \times \mathbb{R}^3$, belonging to $C([t_0, \infty); \dot{\mathbb{H}}^{1/2}(\mathbb{R}^3)) \cap L^2_{loc}((t_0, \infty); \mathbb{H}^{3/2}(\mathbb{R}^3))$. This solution is also smooth on $(t_0, \infty) \times \mathbb{R}^3$. By Plancheral theorem, denoting $\mathbf{u}_0 = \mathcal{F}(\check{\mathbf{u}}_0)$, it follows that $\|\mathbf{u}_0\|_{\frac{1}{2},2} < \epsilon$. We can now apply part (ii) of our theorem with $p = 2, \theta_0 = \frac{1}{2}$, to obtain a solution $\mathbf{u}(t), t \geq t_0$ satisfying

$$\int e^{2\lambda\sqrt{t-t_0}|\xi|}|\xi||\mathbf{u}(t))(\xi)|^2\, d\xi \leq 2\|\check{\mathbf{u}}(t_0)\|^2_{\dot{\mathbb{H}}^{1/2}} < 2\epsilon \text{ for all } t \geq t_0.$$

By Remark 4, the unique mild solution to the 3D NSE is given by $\check{\mathbf{u}}(t) = \mathcal{F}^{-1}(\mathbf{u}(t))$. The desired L^2 decay estimate (11) now follows from Lemma 5.6.

The fact that $\|\check{\mathbf{u}}(t_0)\|_{\dot{\mathbb{H}}^{1/2}} < \epsilon$ for adequately large t_0 can be proven as follows. From the energy inequality, we have

$$\frac{1}{T}\|\check{\mathbf{u}}(T)\|^2_{L^2} + \frac{1}{T}\int_0^T \|(\Delta)^{1/2}\check{\mathbf{u}}(s)\|^2_{L^2}\, ds \leq \frac{1}{T}\|\check{\mathbf{u}}_0\|^2_{L^2}. \tag{43}$$

Let $T = \frac{2\|\check{\mathbf{u}}_0\|^2_{L^2}}{\delta}$. It follows that there exists $t_0 \in [0, T]$ such that $\|(-\Delta)^{1/2}\mathbf{u}(t_0)\|^2 < \delta$. By interpolation inequality, we have

$$\|\check{\mathbf{u}}(t_0)\|_{\dot{\mathbb{H}}^{\frac{1}{2}}} \leq \|\check{\mathbf{u}}(t_0)\|^{1/2}_{L^2}\|(-\Delta)^{1/2}\mathbf{u}(t_0)\|^{1/2} \leq \|\check{\mathbf{u}}_0\|^{1/2}_{L^2}\delta^{1/4}.$$

If δ is chosen suitably small, the claim follows.

We will now prove (13) in part (iv). For $n = 2$, the critical borderline space is L^2 (i.e., $\theta_0 = 0, p = 2$). The proof is similar to part (iii). Let $\check{\mathbf{u}}(t)$ be the classical solution of the NSE on $\mathbb{R}^2 \times (0, \infty)$ with initial data $\check{\mathbf{u}}_0$. For the requisite L^2 decay estimates on the higher 9homogeneous) Sobolev norms, it will be enough to show that there exists $t_0 >$ with $\|\mathbf{u}(t_0)\|_{L^2} < \epsilon$, where $\mathbf{u}(t) = \mathcal{F}(\check{\mathbf{u}}(t))$. In case $\check{\mathbf{u}}_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, it follows from the Schonbek's result that $\|\check{\mathbf{u}}(t)\|^2_{L^2} = O(\frac{1}{t})$ and the requisite estimate follows.

We will now show that the L^2 norm of the solution is small for large time if $\check{\mathbf{u}}_0 \in \dot{\mathbb{H}}^{-\frac{1}{2}}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Note that by interpolation $\|\mathbf{u}(t_0)\|_{L^2} \leq \|\mathbf{u}(t_0)\|^{2/3}_{\dot{\mathbb{H}}^{-\frac{1}{2}}}\|\mathbf{u}(t_0)\|^{1/3}_{\dot{\mathbb{H}}^1}$. As before, for any $\delta > 0$, we can choose t_0 large enough so that $\|\mathbf{u}(t_0)\|_{\dot{\mathbb{H}}^1} < \delta$. It will thus be enough to prove that $\sup_{[0,\infty)} \|\check{\mathbf{u}}(t)\|_{\dot{\mathbb{H}}^{-\frac{1}{2}}} < \infty$ provided $\check{\mathbf{u}}_0 \in \dot{\mathbb{H}}^{-\frac{1}{2}}$.

For two functions f and g defined on $\mathbb{R}^n$, we will now need the following inequality for the homogeneous Sobolev norm of their product, namely,

$$\|fg\|_{\dot{\mathbb{H}}^{\theta_1+\theta_2-\frac{n}{2}}} \leq C\|f\|_{\dot{\mathbb{H}}^{\theta_1}}\|g\|_{\dot{\mathbb{H}}^{\theta}_2}, \theta_1 + \theta_2 > 0, \theta_1, \theta_2 < \frac{n}{2}, \theta_1, \theta_2 \in \mathbb{R}. \tag{44}$$

This is a direct consequence of the weighted convolution inequality (18) with $p = 2$, once one expresses the homogeneous Sobolev norm in the Fourier space and thereby converting the product to a convolution. Let $A = (-\Delta)$. We will need the following estimate on the nonlinear term. Let $0 < \epsilon < \frac{1}{2}$ be fixed. We have

$$\begin{aligned} &|(B(\check{\mathbf{u}}, \check{\mathbf{u}}), A^{-\frac{1}{2}}\check{\mathbf{u}})| = |(A^{-\frac{3}{4}-\frac{\epsilon}{2}}B(\check{\mathbf{u}}, \check{\mathbf{u}}), A^{\frac{1}{4}+\frac{\epsilon}{2}}\check{\mathbf{u}})| \\ &\leq C|A^{-\frac{1}{4}}\check{\mathbf{u}}||A^{\frac{1}{2}-\frac{\epsilon}{2}}\check{\mathbf{u}}||A^{\frac{1}{4}+\frac{\epsilon}{2}}\check{\mathbf{u}}| \leq C|A^{-\frac{1}{4}}\check{\mathbf{u}}||A^{\frac{1}{2}}\check{\mathbf{u}}||A^{\frac{1}{4}}\check{\mathbf{u}}|, \end{aligned} \tag{45}$$

where to obtain the first inequality in (45), we first note that $B(\check{\mathbf{u}}, \check{\mathbf{u}}) = \nabla \cdot (\check{\mathbf{u}} \otimes \check{\mathbf{u}})$ satisfies (5) with $\alpha = 1$. We then use (44) with $n = 2$ and $\theta_1 = -\frac{1}{2}, \theta_2 = 1 - \epsilon$ to obtain the requisite inequality. The last inequality in the second line is obtained using interpolation.

Multiplying the NSE by $A^{-\frac{1}{2}}\check{\mathbf{u}}$, integrating and using (45), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|A^{-\frac{1}{4}}\check{\mathbf{u}}|^2 + |A^{\frac{1}{4}}\check{\mathbf{u}}|^2 \leq |(B(\check{\mathbf{u}}, \check{\mathbf{u}}), A^{-\frac{1}{2}}\check{\mathbf{u}})| \\ &\qquad \leq |A^{-\frac{1}{4}}\check{\mathbf{u}}||A^{\frac{1}{4}}\check{\mathbf{u}}||A^{\frac{1}{2}}\check{\mathbf{u}}| \leq \frac{1}{2}|A^{\frac{1}{4}}\check{\mathbf{u}}|^2 + \frac{C}{2}|A^{-\frac{1}{4}}\check{\mathbf{u}}|^2|A^{\frac{1}{2}}\check{\mathbf{u}}|^2, \end{aligned} \tag{46}$$

where, the first inequality in (46) follows from (44) and the second from Young's inequality. From (46), it follows that

$$\frac{d}{dt}|A^{-\frac{1}{4}}\check{\mathbf{u}}|^2 - C|A^{-\frac{1}{4}}\check{\mathbf{u}}|^2|A^{\frac{1}{2}}\check{\mathbf{u}}|^2 \leq 0.$$

Applying Gronwall's inequality and recalling that $|A^{-\frac{1}{4}}\check{\mathbf{u}}|^2 = \|\check{\mathbf{u}}\|^2_{\dot{\mathbb{H}}^{-\frac{1}{2}}}$, we immediately obtain

$$\|\check{\mathbf{u}}(t)\|^2_{\dot{\mathbb{H}}^{-\frac{1}{2}}} \leq e^{C\int_0^t |A^{1/2}\check{\mathbf{u}}(s)|^2\,ds}\|\check{\mathbf{u}}_0\|^2_{\dot{\mathbb{H}}^{-\frac{1}{2}}} \leq e^{C|\check{\mathbf{u}}_0|^2}\|\check{\mathbf{u}}_0\|^2_{\dot{\mathbb{H}}^{-\frac{1}{2}}}, \tag{47}$$

where the last inequality follows from (10). Thus,

$$\sup_{t\in[0,\infty)} \|\check{\mathbf{u}}(t)\|^2_{\dot{\mathbb{H}}^{-\frac{1}{2}}} \leq e^{|\check{\mathbf{u}}_0|^2}\|\check{\mathbf{u}}_0\|^2_{\dot{\mathbb{H}}^{-\frac{1}{2}}} < \infty.$$

Finally, we will prove the requisite decay estimates (12) and (14) for $\|\check{\mathbf{u}}\|_{L^q}$. Note that by Proposition 2.1, $\|\mathbf{u}\|_{\mathbb{B}_p^{1-\frac{n}{p'},\infty}} \leq C\|\mathbf{u}\|_{\mathbb{V}_{\frac{n}{2}-1,2}} = C\|\check{\mathbf{u}}\|_{\dot{\mathbb{H}}^{\frac{n}{2}-1}}$, provided $p < \frac{n}{n-1} = n'$. Since $\|\check{\mathbf{u}}\|_{\dot{\mathbb{H}}^{\frac{n}{2}-1}}$ can be made arbitrarily small by choosing t_0 large enough, the same is true for $\|\mathbf{u}\|_{\mathbb{B}_p^{1-\frac{n}{p'},\infty}}$. We can thus apply part (i) of the Theorem for critical spaces in conjunction with Lemma 5.6 to conclude

$$\|\mathbf{u}(t)\|_{\mathbb{V}_p} \leq \frac{\epsilon}{\sqrt{(t-t_0)}^{\zeta+\delta}}\frac{\zeta^\zeta}{e^\zeta} \text{ where } \delta = 1 - \frac{n}{p'}.$$

By the Hausdorff-Young inequality, provided $p < 2$, we have $\|\check{\mathbf{u}}(t)\|_{L^q} \leq \|\mathbf{u}(t)\|_{\mathbb{V}_p}$ where $q = p'$. Since the condition $p < n'$ implies $q = p' > n$ as well as $p < 2$ for both $n = 2, 3$, the claim follows.

Proof of Theorem 4.2: The proof is very similar to the previous case. We only need to note that the "energy inequality" here yields smallness of $\|\check{\mathbf{u}}\|_{\dot{\mathbb{H}}^{\frac{\kappa}{2}}}$ for large time. The critical space for the global result is $\dot{\mathbb{H}}^{2-\kappa}$. By interpolation, we have $\|\check{\mathbf{u}}\|_{\dot{\mathbb{H}}^{2-\kappa}} \le \|\check{\mathbf{u}}\|_{L^2}^{\theta}\|\check{\mathbf{u}}\|_{\dot{\mathbb{H}}^{\frac{\kappa}{2}}}^{1-\theta}$ for adequate $0 \le \theta \le 1$ provided $\frac{\kappa}{2} \ge 2-\kappa$ or equivalently, $\kappa \ge 4/3$.

Lemma 5.6. *For $\lambda, p, \sigma, \gamma$ and $\theta_0 \ge 0$ fixed, let $\|\mathbf{v}\|_{G_{\theta_0}(\tau)} < \infty$. We then have the estimate*

$$\|\mathbf{v}\|_{\theta,p} \le \frac{1}{(e\lambda)^{\frac{\theta-\theta_0}{\gamma}}} \frac{1}{\tau^{\frac{\sigma(\theta-\theta_0)}{2}}} \left(\frac{\theta-\theta_0}{\gamma}\right)^{\frac{\theta-\theta_0}{\gamma}} \|\mathbf{v}\|_{G_{\theta_0}(\tau)} \text{ for all } \theta > \theta_0.$$

Proof. This follows by writing

$$\begin{aligned}\|\mathbf{v}\|_{\theta,p}^p &= \int_{\mathbb{R}^n} |\xi|^{\theta p} |\mathbf{v}(\xi)|^p \, d\xi \\ &= \int_{\mathbb{R}^n} |\xi|^{(\theta-\theta_0)p} e^{-\lambda p(\sqrt{\tau}^\sigma |\xi|)^\gamma} e^{\lambda p(\sqrt{\tau}^\sigma |\xi|)^\gamma} |\xi|^{\theta_0 p} |\mathbf{v}(\xi)|^p \, d\xi\end{aligned}$$

and then by recalling the definition of the Gevrey norm in (6) and noting that

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^{(\theta-\theta_0)} e^{-\lambda(\sqrt{\tau}^\sigma |\xi|)^\gamma} \le \frac{1}{(e\lambda)^{\frac{\theta-\theta_0}{\gamma}}} \frac{1}{\tau^{\frac{\sigma(\theta-\theta_0)}{2}}} \left(\frac{\theta-\theta_0}{\gamma}\right)^{\frac{\theta-\theta_0}{\gamma}}.$$

□

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