MONOTONICITY OF THE SET OF ZEROS OF THE LYAPUNOV EXPONENT WITH RESPECT TO SHIFT EMBEDDINGS

OLEG SAFRONOV

ABSTRACT. We consider the discrete Schrödinger operators with potentials whose values are read along the orbits of a shift of finite type. We study a certain subset of the collection of energies at which the Lyapunov exponent is zero and prove monotonicity of this set with respect to the shift embeddings. Then we introduce a certain function $\mathcal{J}(A,\mu)$ determined by the position of these zeros and prove monotonicity of $\mathcal{J}(A,\mu)$ with respect to embeddings.

In this short paper, we study the discrete Schrödinger operators H_ω defined on $\ell^2(\mathbb{Z})$ by

$$\big[H_\omega u\big](n) = u(n+1) + u(n-1) + V(T^n\omega)u(n), \qquad \omega \in \Omega.$$

Here Ω is a compact metric space whose elements are infinite sequences $\{\omega_n\}_{n\in\mathbb{Z}}$ such that $\omega_n \in \{1,\dots,\ell\} = \mathcal{A}$ for each n. There are sequences in $\mathcal{A}^{\mathbb{Z}}$ that are not allowed to be in Ω and we assume that forbidden words are of length 2. The metric $d(\cdot,\cdot)$ on Ω is defined by

$$d(\omega,\omega') = e^{-N(\omega,\omega')},$$

where $N(\omega,\omega')$ is the largest nonnegative integer such that $\omega_n = \omega'_n$ for all $|n| < N(\omega,\omega')$. The mapping $T:\ \Omega \to \Omega$ is assumed to be a subshift of finite type defined by

$$\big(T\omega\big)_n = \omega_{n+1}, \qquad \forall n \in \mathbb{Z}.$$

Finally, the function V is assumed to be locally constant on Ω in the sense of the following definition.

Definition. A function $V:\ \Omega \to \mathbb{R}$ is said to be locally constant, if there is an $\epsilon > 0$ such that

$$V(\omega') = V(\omega) \qquad \text{whenever} \qquad d(\omega',\omega) < \epsilon.$$

Spectral properties of H_ω are related to the behavior of solutions to the equation

$$u(n+1) + u(n-1) + V(T^n\omega)u(n) = Eu(n), \qquad n \in \mathbb{Z}, \tag{0.1}$$

for $E \in \mathbb{R}$.

On the other hand, all solutions to (0.1) can be described in terms of the Schrödinger cocycles (T, A^E) with $A = A^E:\ \Omega \to \mathrm{SL}(2,\mathbb{R})$ defined by

$$A^E(\omega) = \begin{pmatrix} E - V(\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

Namely, u is a solution of (0.1) if and only if

$$\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = A_n(\omega) \cdot \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}, \qquad \forall n \in \mathbb{Z},$$

where

$$A_n(\omega) = \begin{cases} A(T^{n-1}\omega) \cdots A(\omega) & \text{if} \quad n \geq 1; \\ [A_{-n}(T^n\omega)]^{-1} & \text{if} \quad n \leq -1; \\ \mathrm{Id} & \text{if} \quad n = 0. \end{cases}$$

Since Ω is a metric space, we can talk about the Borel σ-algebra of subsets of Ω and consider probability measures on Ω. Let μ be a T-ergodic probability measure on Ω. The Lyapunov exponent for A and μ is defined by

$$L(A, \mu) = \lim_{n \to \infty} \frac{1}{n} \int \ln(\|A_n(\omega)\|) d\mu(\omega).$$

By Kingman's subaddive ergodic theorem,

$$\frac{1}{n} \ln(\|A_n(\omega)\|) \qquad \text{converges to} \quad L(A, \mu) \qquad \text{as} \quad n \to \infty,$$

for μ-almost every $\omega \in \Omega$. For simplisity, we write $L(E) = L(A, \mu)$.

One of the main theorems of the paper [1] gives sufficient conditions guaranteeing that the set

$$\mathfrak{L}(A, \mu) = \big\{E \in \mathbb{R} : \ L(A, \mu) = 0\big\} \tag{0.2}$$

is finite. One of these conditions is that μ has a local product structure.

Let us now give a formal definition of a measure having this property. We first define the spaces of semi-infinite sequences

$$\Omega_+ = \{\{\omega_n\}_{n \geq 0} : \ \omega \in \Omega\} \quad \text{and} \quad \Omega_- = \{\{\omega_n\}_{n \leq 0} : \ \omega \in \Omega\}.$$

Then using the natural projection $\pi_\pm$ from Ω onto $\Omega_\pm$, we define $\mu_\pm = (\pi_\pm)_* \mu$ on $\Omega_\pm$ to be the pushforward measures of μ. After that, for each $1 \leq j \leq \ell$, we introduce the cylinder sets

$$[0; j] = \{\omega \in \Omega : \ \omega_0 = j\} \quad \text{and} \quad [0; j]_\pm = \{\omega \in \Omega_\pm : \ \omega_0 = j\}.$$

A local product structure is a relation between the measures $\mu_j = \mu\big|_{[0;j]}$ and the measures $\mu_j^\pm = \mu_\pm\big|_{[0;j]}$. To describe this relation, we need to consider the natural homeomorphisms

$$P_j : [0; j] \to [0; j]_- \times [0; j]_+$$

defined by

$$P_j(\omega) = \big(\pi_-\omega, \pi_+\omega\big), \qquad \forall \omega \in \Omega.$$

Definition. We say that μ has a local product structure if there is a positive $\psi : \Omega \to (0, \infty)$ such that for each $1 \leq j \leq \ell$, the function $\psi \circ P_j^{-1}$ belongs to $L^1\big([0; j]_- \times [0; j]_+, \mu_j^- \times \mu_j^+\big)$ and

$$\big(P_j\big)_* d\mu_j = \psi \circ P_j^{-1} \, d(\mu_j^- \times \mu_j^+).$$

We will shortly divide points of the set (0.2) into two groups: removable and unremovable points. We will show that unremovable points in $\mathfrak{L}(A, \mu)$ do not disappear in the process of passing from Ω to a subshift $\tilde{\Omega} \subset \Omega$ with an ergodic measure $\tilde{\mu}$ on it.

A point $p \in \Omega$ is said to be periodic for T provided there is a positive integer n_p for which $T^{n_p}\, p = p$. If $p \in \Omega$ is periodic, then $V(T^n p)$ is a periodic function of n, because $V(T^{n_p+n}p) = V(T^n p)$ for every $n \in \mathbb{Z}$. For a periodic point p of period n_p, define $\Delta_p(E)$ to be the trace of the monodromy matrix $A_{n_p}(p)$

$$\Delta_p(E) = \mathrm{Tr}(A_{n_p}(p)).$$

By $\mathrm{Per}(T)$, we denoted the collection of all periodic points of T.

Definition. A point $E \in \mathfrak{L}(A, \mu)$ is said to be unremovable from $\mathfrak{L}(A, \mu)$ provided

either 1) there exists a T- periodic point $p \in \Omega$ for which $0 < |\Delta_p(E)| < 2$,

or 2) $|\Delta_p(E)| \in \{0, 2\}$ for all $p \in \mathrm{Per}(T)$.

The collection of unremovable from $\mathfrak{L}(A, \mu)$ points will be denoted by $\mathfrak{U}(A, \mu)$.

Theorem 1. *Let $T : \Omega \to \Omega$ be a subshift of finite type. Assume that μ is a T-ergodic measure on Ω that has a local product structure and the property* $\mathrm{supp}(\mu) = \Omega$. *Let V be a real-valued locally constant function on Ω. Then for any subshift $\tilde{T} : \tilde{\Omega} \to \tilde{\Omega}$ of T and any $\tilde{T}$-ergodic measure $\tilde{\mu}$ on $\tilde{\Omega} \subset \Omega$,*

$$\mathfrak{U}(A, \mu) \subseteq \mathfrak{U}(\tilde{A}, \tilde{\mu}), \tag{0.3}$$

where $\tilde{A}$ is the restriction of A to $\tilde{\Omega}$.

Corollary 2. *Let $T : \Omega \to \Omega$ be a subshift of finite type. Assume that μ and $\tilde{\mu}$ are T-ergodic measures on Ω that have a local product structure and have the property* $\mathrm{supp}(\mu) = \mathrm{supp}(\tilde{\mu}) = \Omega$. *Let V be a real-valued locally constant function on Ω. Then*

$$\mathfrak{U}(A, \mu) = \mathfrak{U}(A, \tilde{\mu}). \tag{0.4}$$

Remark. For any removable point E that belongs to the set $\mathfrak{L}(A, \mu) \backslash \mathfrak{U}(A, \mu)$, there is a subshift $\tilde{T} : \tilde{\Omega} \to \tilde{\Omega}$ of T and a $\tilde{T}$-ergodic measure $\tilde{\mu}$ on $\tilde{\Omega} \subset \Omega$ for which $E \notin \mathfrak{L}(\tilde{A}, \tilde{\mu})$. To see that, we find a periodic point $p \in \Omega$ for which $|\Delta_p(E)| > 2$ and then we define $\tilde{\mu}$ to be the ergodic probability measure supported on the union of the shifts $T^n p$ of the point p.

The following result is a consequence of our methods:

Theorem 3. *Let $T : \Omega \to \Omega$ be a subshift of finite type. Assume that μ is a T-ergodic measure on Ω that has a local product structure and the property* $\mathrm{supp}(\mu) = \Omega$. *Let V be a real-valued locally constant function on Ω. Then*

$$\mathfrak{U}(A,\mu) \;=\; \bigcap_{p\in \mathrm{Per}(T)} \sigma(p),$$

where $\sigma(p)$ denotes the spectrum of the Schrödinger operator H_p with the potential $V(T^n p)$.

Proof. Indeed, let $\tilde\Omega$ be the subshift consisting of the orbit of a periodic point p. Then $\sigma(p)$ coincides with the set $\mathfrak{U}(\tilde A,\tilde\mu)$. Thus, by Theorem 1, the spectrum $\sigma(p)$ contains $\mathfrak{U}(A,\mu)$. Therefore,

$$\mathfrak{U}(A,\mu) \;\subseteq\; \bigcap_{p\in \mathrm{Per}(T)} \sigma(p).$$

Conversely, let $E \in \bigcap_{p\in \mathrm{Per}(T)} \sigma(p)$. Then $L(A^E,\mu) = 0$ by Proposition 8 stated in Section 2. Consequently, E belongs to $\mathfrak{U}(A,\mu)$. $\square$

Several definitions below ivolve the set

$$\mathfrak{S}(T,\mu) = \bigcup_{p\in \mathrm{Per}(T)} \big\{E\in\mathbb{R}:\ \Delta_p(E)\in(-2,0)\cup(0,2)\big\}.$$

This set may only become smaller when one passes from T to $\tilde T$,

$$\mathfrak{S}(\tilde T,\tilde\mu) \subseteq \mathfrak{S}(T,\mu),$$

while $\mathfrak{U}(A,\mu)$ may only increase due to the property (0.3). This observation allows one to construct a real-valued function $\mathcal{J}(A,\mu)$ that decreases when either $\mathfrak{S}(T,\mu)$ becomes smaller, or $\mathfrak{U}(A,\mu)$ becomes larger. For this purpose, we recall that if T has a fixed point, then there are at most finitely many points in the set $\mathfrak{U}(A,\mu)$ (see Theorem 1.2 in [1]). Thus $\mathfrak{U}(A,\mu) = \{E_1, E_2, \dots, E_l\}$ where $E_1 < E_2 < \cdots < E_l$. We first enlarge the collection $\mathfrak{U}(A,\mu)$ by adding the two points $E_0 = -5/2 - \|V\|_\infty$ and $E_{l+1} = 5/2 + \|V\|_\infty$. Then, for each interval (E_j, E_{j+1}) whose intersection with $\mathfrak{S}(T,\mu)$ is not empty, we define N_j by

$$N_j = \text{ the integer part of } \Big[\frac{2|E_{l+1}-E_0|}{\big|(E_j,E_{j+1})\cap\mathfrak{S}(T,\mu)\big|}\Big].$$

Here, $|X|$ in the denominator denotes the Lebesgue measure of a Borel set $X \subset \mathbb{R}$. If $(E_j, E_{j+1}) \cap \mathfrak{S}(T,\mu) = \emptyset$, then we define N_j to be equal to 2. Finally, after setting

$$N(A,\mu) = \max_{0\le j\le l} N_j,$$

we define the function

$$\mathcal{J}(A,\mu) = \sum_{j=0}^{l} \mathcal{E}_j(A,\mu),$$

where

$$\mathcal{E}_j(A,\mu) = \frac{\big|(E_j,E_{j+1})\cap\mathfrak{S}(T,\mu)\big|}{\lambda}\ \ln\Big(\frac{\big|(E_j,E_{j+1})\cap\mathfrak{S}(T,\mu)\big|}{\lambda}\Big)+$$
$$+\frac{\big|(E_0,E_{l+1})\setminus\mathfrak{S}(T,\mu)\big|}{\lambda\cdot(l+1)}\ \ln\Big(\frac{\big|(E_0,E_{l+1})\setminus\mathfrak{S}(T,\mu)\big|}{N(A,\mu)\cdot\lambda}\Big)$$

and $\lambda = |E_{l+1}-E_0|$. The next result establishes monotonicity of the function $\mathcal{J}(A,\mu)$ with respect to embeddings of the subshifts.

Theorem 4. *Let $T:\Omega\to\Omega$ be a subshift of finite type. Assume that μ is a T-ergodic measure on Ω that has a local product structure and the property $\operatorname{supp}(\mu)=\Omega$. Let V be a real-valued locally constant function on Ω. Suppose $\tilde T:\tilde\Omega\to\tilde\Omega$ is a further subshift of T and $\tilde\mu$ is a $\tilde T$-ergodic measure on $\tilde\Omega\subset\Omega$ for which the set $\mathfrak{U}(\tilde A,\tilde\mu)$ is finite (by $\tilde A$, we denote the restriction of A to $\tilde\Omega$). Then*

$$\mathcal{J}(A,\mu)\ \ge \mathcal{J}(\tilde A,\tilde\mu).$$

1. Begining of the proof of Theorem 1. Main ingredients

Note that a Schrödinger cocycle $A=A^E$ with a locally constant potential $V:\Omega\to\mathbb{R}$ is also locally constant. Put differently, there is an $\epsilon>0$ such that

$$A(\omega')=A(\omega) \qquad \text{whenever} \qquad d(\omega',\omega)<\epsilon.$$

Definition . Let $T:\Omega\to\Omega$ be a subshift of finite type. The local stable set of a point $\omega\in\Omega$ is defined by

$$W^s(\omega)=\{\omega'\in\Omega:\ \omega'_n=\omega_n \quad \text{for} \quad n\ge 0\}$$

and the local unstable set of ω is defined by

$$W^u(\omega)=\{\omega'\in\Omega:\ \omega'_n=\omega_n \quad \text{for} \quad n\le 0\}.$$

For $\omega'\in W^s(\omega)$, define $H^{s,n}_{\omega',\omega}$ to be

$$H^{s,n}_{\omega,\omega'}=\big[A_n(\omega')\big]^{-1}A_n(\omega).$$

Since $d(T^j\omega',T^j\omega)\le e^{-j}$ tends to 0 as $j\to\infty$, there is an index n_0 for which

$$H^{s,n}_{\omega,\omega'}=H^{s,n_0}_{\omega,\omega'} \qquad \text{for} \quad n\ge n_0.$$

In this case, we define the stable holonomy $H^s_{\omega,\omega'}$ by

$$H^s_{\omega,\omega'}=H^{s,n_0}_{\omega,\omega'}.$$

The unstable holonomy $H^u_{\omega,\omega'}$ for $\omega'\in W^u(\omega)$ is defined similarly by

$$H^u_{\omega,\omega'}=\big[A_n(\omega')\big]^{-1}A_n(\omega) \qquad \text{for all} \quad n\le -n_0.$$

The general theory of dynamical systems tells us that the cocycle

$$(T, A) : \Omega \times \mathbb{RP}^1 \to \mathbb{RP}^1$$

defined by

$$(T, A)(\omega, \xi) = (T\omega, A(\omega)\xi)$$

has an invariant probability measure m on $\Omega \times \mathbb{RP}^1$. We say that such a measure m projects to μ if $m(\Delta \times \mathbb{RP}^1) = \mu(\Delta)$ for all Borel subsets Δ of Ω. Given any T-invariant measure μ on Ω, one can find a (T, A)-ivariant measure m that projects to μ by applying the Krylov-Bogolyubov trick.

Definition. Suppose m is a (T, A)-invariant probability measure on $\Omega \times \mathbb{RP}^1$ that projects to μ. A disintegration of m is a measurable family $\{m_\omega : \quad \omega \in \Omega\}$ of probability measures on $\mathbb{RP}^1$ having the property

$$m(D) = \int_\Omega m_\omega(\{\xi \in \mathbb{RP}^1 : \ (\omega, \xi) \in D\}) d\mu(\omega)$$

for each measurable set $D \subset \Omega \times \mathbb{RP}^1$.

Existence of such a disintegration is guaranteed by Rokhlin's theorem. Moreover, $\{\tilde{m}_\omega : \omega \in \Omega\}$ is another disintegration of m then $m_\omega = \tilde{m}_\omega$ for μ-almost every $\omega \in \Omega$. It is easy to see that m is (T, A)-invariant if and only if $A(\omega)_* m_\omega = m_{T\omega}$ for μ-almost every $\omega \in \Omega$.

Definition. A (T, A)-invariant measure m on $\Omega \times \mathbb{RP}^1$ that projects to μ is said to be an su-state for A provided it has a disintegration $\{m_\omega : \quad \omega \in \Omega\}$ such that for μ-almost every $\omega \in \Omega$,

1)
$$A(\omega)_* m_\omega = m_{T\omega},$$

2)
$$\left(H^s_{\omega,\omega'}\right)_* m_\omega = m_{\omega'} \qquad \text{for every} \quad \omega' \in W^s(\omega).$$

3)
$$\left(H^u_{\omega,\omega'}\right)_* m_\omega = m_{\omega'} \qquad \text{for every} \quad \omega' \in W^u(\omega)$$

The following statement was proved in [1] (Proposition 4.7) for a significantly larger class of functions A.

Proposition 5. *Let A be locally constant. Suppose μ has a local product structure and $L(A, \mu) = 0$. If the support of the measure μ coincides with all of Ω, then there exists an su-state for A.*

We apply the following method to extend m_ω to a continuous function of ω on all of Ω. For each $1 \le j \le \ell$, we select a point $\omega^{(j)}$ in $[0; j] \cap \Omega_0$ for which the measure $m_{\omega^{(j)}}$ is well defined. Then we set

$$m_\omega = \left(H^u_{\omega \wedge \omega^{(\omega_0)}, \omega} H^s_{\omega^{(\omega_0)}, \omega \wedge \omega^{(\omega_0)}}\right)_* m_{\omega^{(\omega_0)}}. \tag{1.5}$$

Obviously m_ω depends continuously on ω.

Observe that $\mathbb{RP}^1$ may be aslo viewed as $\mathbb{R} \cup \{\infty\}$, because any vector of the form $(\xi, 1) \in \mathbb{RP}^1$ is uniquily characterized by $\xi \in \mathbb{R} \cup \{\infty\}$. Aslo, $\mathbb{CP}^1$ may be aslo viewed as $\mathbb{C} \cup \{\infty\}$ because there is a 1:1 mapping of one set onto another. The part of $\mathbb{CP}^1$ that is mapped onto the extended upper half-plane $\mathbb{C}_+ \cup \{\infty\}$ will be denoted by $\mathbb{C}_+\mathbb{P}^1$.

Now we will state Proposition 4.9 from [1] in the following more convenient form:

Proposition 6. *For each probability measure ν on $\mathbb{RP}^1$ containing no atom of mass $\geq 1/2$, there is an unique point $B(\nu) \in \mathbb{C}_+\mathbb{P}$, called the conformal barycenter of ν, such that*

$$B(P_*\nu) = P \cdot B(\nu)$$

for each $P \in \mathrm{SL}(2, \mathbb{R})$.

Let m be an su-state with a continuous disintegration m_ω. If m_ω does not have an atom of mass $\geq 1/2$, then we set $Z(\omega) \subset \mathbb{C}_+\mathbb{P}$ to be $\{B(m_\omega)\}$. Otherwise $Z(\omega)$ is defined to be the collection of points ξ at which $m_\omega(\{\xi\}) \geq 1/2$. Since m_ω is a probability measure, the set $Z(\omega)$ can contain at most two points. The following theorem is a consequence of Proposition 6.

Theorem 7. *Let A be locally constant. Suppose μ has a local product structure and $L(A, \mu) = 0$. Then*

$$A(\omega)Z(\omega) = Z(T\omega) \qquad \textit{for each} \quad \omega \in \Omega.$$

If ω', ω are two points in Ω such that $\omega'_0 = \omega_0$, then

$$Z(\omega) = \left(H^u_{\omega\wedge\omega',\omega} H^s_{\omega',\omega\wedge\omega'}\right) Z(\omega'). \tag{1.6}$$

In particular, the number of the points in $Z(\omega)$ does not depend on ω. Moreover, if $Z(\omega)$ is real for one ω, then it is real for all $\omega \in \Omega$.

The last two lines of the theorem follow from the fact that for any two points ω and ω' in Ω, there is a real matrix $P \in \mathrm{SL}(2, \mathbb{R})$ for which $Z(\omega) = P \cdot Z(\omega')$. Indeed, if $\omega'_0 = \omega_0$, then this property is guaranteed by (1.6). On the other hand, since T is transitive, for any two points ω' and ω, there is an index n and a point $\tilde{\omega}$ such that $\left(T^n\tilde{\omega}\right)_0 = \omega'_0$ while $\tilde{\omega}_0 = \omega_0$. Therefore

$$Z(T^n\tilde{\omega}) = A_n(\tilde{\omega})Z(\tilde{\omega}) = \left(H^u_{T^n\tilde{\omega}\wedge\omega',T^n\tilde{\omega}} H^s_{\omega',T^n\tilde{\omega}\wedge\omega'}\right) Z(\omega'),$$

which implies that

$$Z(\tilde{\omega}) = [A_n(\tilde{\omega})]^{-1} \left(H^u_{T^n\tilde{\omega}\wedge\omega',T^n\tilde{\omega}} H^s_{\omega',T^n\tilde{\omega}\wedge\omega'}\right) Z(\omega').$$

It remains to note that

$$Z(\omega) = \left(H^u_{\omega\wedge\tilde{\omega},\omega} H^s_{\tilde{\omega},\omega\wedge\tilde{\omega}}\right) Z(\tilde{\omega}).$$

2. END OF THE PROOF OF THEOREM 1

Let $E \in \mathfrak{U}(A, \mu)$. We must show that $E \in \mathfrak{U}(\tilde{A}, \tilde{\mu})$.

Assume first that $0 < |\Delta_p(E)| < 2$ for some T- periodic point $p \in \Omega$. By the symbol n_p, we denote the period of p. We also set

$$L(A, p) = \lim_{n\to\infty} \frac{1}{n} \ln\bigl(\|A_n(p)\|\bigr).$$

It is easy to see that $Z(p)$, viewed as a set of complex numbers, is not real. In fact, $Z(p)$ consists of one point $(a + i\sqrt{4 - (\Delta_p(E))^2})/b$, where a and $b \neq 0$ are the two elements of the first row of the matrix $A_{n_p}(p)$. Since, for any periodic point $q \in \Omega$, the set $Z(q)$ is the image of $Z(p)$ under an $\mathrm{SL}(2, \mathbb{R})$ transformation, $Z(q)$ is not real and consists of one point. Therefore, for any periodic point $q \in \Omega$, the matrix $A_{n_q}(q)$ has two complex eigenvalues that belong to the unit circle. The latter observation leads to the conclusion that $E \in \sigma(H_q)$ and

$$L(A, q) = 0 \qquad \text{for all periodic points} \quad q \in \Omega. \tag{2.7}$$

In particular, $L(A, q) = L(\tilde{A}, q) = 0$ for all periodic points that belong to $\tilde{\Omega}$.

Now we use the following result proved in a much more general setting by Kalinin (see Theorem 1.4 in [12]).

Proposition 8. *Let A be locally constant on $\tilde{\Omega}$. Then for each $\delta > 0$ there is a periodic point $q \in \tilde{\Omega}$ such that $|L(\tilde{A}, q) - L(\tilde{A}, \tilde{\mu})| < \delta$.*

Combining Proposition 8 with the equality (2.7), we obtain that

$$L(\tilde{A}, \tilde{\mu}) = 0.$$

Thus, $E \in \mathfrak{U}(\tilde{A}, \tilde{\mu})$.

Now assume that $|\Delta_p(E)| \in \{0, 2\}$ for all $p \in \mathrm{Per}(T)$. Then $|\Delta_p(E)| \in \{0, 2\}$ for all $p \in \mathrm{Per}(\tilde{T})$. In particular, this implies that all eigenvalues of A_{n_p} belong to the unit circle and, hence, $L(\tilde{A}, p) = 0$ for any periodic $p \in \tilde{\Omega}$. Thus, we infer from Proposition 8 that $L(\tilde{A}, \tilde{\mu}) = 0$.

The proof is complete. □

Corollary 9. *A point $E \in \mathfrak{L}(A, \mu)$ is unremovable from $\mathfrak{L}(A, \mu)$ if and only if the point E belongs to the spectrum of H_p for each $p \in \mathrm{Per}(T)$.*

The zeros of the Lyapunov exponent that belong to the set

$$\mathfrak{S}(T, \mu) = \bigcup_{p\in\mathrm{Per}(T)} \bigl\{E \in \mathbb{R} : \ \Delta_p(E) \in (-2, 0) \cup (0, 2)\bigr\}$$

have simple and interesting properties described in the following statement.

Corollary 10. *Let V be a real-valued locally constant function on Ω. Let $T : \Omega \to \Omega$ be a subshift of finite type. Assume that μ is a T-ergodic measure on Ω that has a local product structure and the property* $\mathrm{supp}(\mu) = \Omega$. *Then for any subshift $\tilde{T} : \tilde{\Omega} \to \tilde{\Omega}$ of T and any $\tilde{T}$-ergodic measure $\tilde{\mu}$ on $\tilde{\Omega} \subset \Omega$,*

$$\mathfrak{S}(T,\mu) \cap \mathfrak{L}(A,\mu) \subseteq \mathfrak{S}(T,\mu) \cap \mathfrak{L}(\tilde{A},\tilde{\mu}),$$

where $\tilde{A}$ is the restriction of A to $\tilde{\Omega}$.

3. Proof of Theorem 4

Observe that under the assumptions of Theorem 4,

$$N(\tilde{A},\tilde{\mu}) \geq N(A,\mu).$$

First consider the case where $\mathfrak{U}(A,\mu) = \mathfrak{U}(\tilde{A},\tilde{\mu})$ while $\mathfrak{S}(\tilde{T},\tilde{\mu}) \subset \mathfrak{S}(T,\mu)$ is a proper inclusion. Then the inequality

$$\mathcal{J}(\tilde{A},\tilde{\mu}) \leq \mathcal{J}(A,\mu) \tag{3.8}$$

may be established by the means of Calculus. Indeed, since $N(\tilde{A},\tilde{\mu}) \geq N(A,\mu)$, we only need to show that the derivative of $\mathcal{J}(A,\mu)$ with respect to $x = |(E_{j_0}, E_{j_0+1}) \cap \mathfrak{S}(T,\mu)|$ is positive, provided $|(E_0, E_{l+1}) \cap \mathfrak{S}(T,\mu)|$ is viewed as the linear function $\tau - x$, where $\tau = E_{j_0+1} - E_{j_0} + \sum_{j \neq j_0} |(E_j, E_{j+1}) \cap \mathfrak{S}(T,\mu)|$. Put differently, we must show that the derivative of

$$\psi(x) = \frac{x}{\lambda} \ln\Big(\frac{x}{\lambda}\Big) + \frac{\tau - x}{\lambda} \ln\Big(\frac{\tau - x}{N(A,\mu)\cdot\lambda}\Big)$$

is positive.

The direct computation shows that

$$\psi'(x) = \frac{1}{\lambda} \ln\Big(\frac{N(A,\mu)\cdot x}{\tau - x}\Big).$$

Thus, we infer that $\psi'(x) > 0$ from the finequality $(N(A,\mu)+1)x > \tau$.

It remains to prove (3.8) in the case where $\mathfrak{U}(A,\mu) \subset \mathfrak{U}(\tilde{A},\tilde{\mu})$ is a proper inclusion. In the case $\mathfrak{U}(A,\mu) = \mathfrak{U}(\tilde{A},\tilde{\mu})$, the quantity $\mathcal{J}(A,\mu)$ was an increasing function of $\mathfrak{S}(T,\mu)$. Therefore, it is enough to consider the case where $\mathfrak{S}(T,\mu) = \mathfrak{S}(\tilde{T},\tilde{\mu})$.

Suppose $\mathfrak{U}(\tilde{A},\tilde{\mu})$ consists of the points $\tilde{E}_1 < \tilde{E}_2 < \cdots < \tilde{E}_r$. Define $\tilde{E}_0 = E_0$ and $\tilde{E}_{r+1} = E_{l+1}$. Then each interval $[E_j, E_{j+1})$ is the union of a finite collection of intervals $[\tilde{E}_k, \tilde{E}_{k+1})$:

$$[E_j, E_{j+1}) = \bigcup_{k=k_0(j)}^{\tilde{k}(j)} [\tilde{E}_k, \tilde{E}_{k+1}).$$

Therefore,

$$|[E_j, E_{j+1}) \cap \mathfrak{S}(T,\mu)| \ln\Big(|[E_j, E_{j+1}) \cap \mathfrak{S}(T,\mu)|\Big) \geq$$
$$\sum_{k=k_0(j)}^{\tilde{k}(j)} |[E_k, E_{k+1}) \cap \mathfrak{S}(T,\mu)| \ln\Big(|[E_k, E_{k+1}) \cap \mathfrak{S}(T,\mu)|\Big),$$

which implies (3.8).

4. THE MONOTONICITY IS NOT STRICT

Here we give an example of a subshift for which Ω is a proper subset of $\mathcal{A}^{\mathbb{Z}}$, and yet $\mathfrak{U}(A,\mu) = \emptyset$.

Assume that V depends only on the zero-coordinate ω_0 of ω. Then all stable and unstable holonomies are identity operators. Consequently, if m_ω is a continuous disintegration of an su-state, then

$$m_\omega = m_{\omega'} \quad \text{whenever} \quad \omega_0 = \omega_0'.$$

But then the equality

$$A(\omega) m_\omega = m_{T\omega}$$

implies that $m_{T\omega'} = m_{T\omega}$ whenever $\omega_0 = \omega_0'$.

Let us now give a condition that makes the latter equality impossible. For each $j_0 \in \mathcal{A}$, define the set

$$D_{j_0} = \{ j \in \mathcal{A} : \ \exists \omega \in \Omega \, \text{such that} \, \omega_0 = j_0, \, \omega_1 = j \}. \tag{4.9}$$

Suppose that for each pair of symbols j and j' in $\mathcal{A}$, there is an ordered collection of letters $j_1, j_2, \dots, j_k$, such that $j \in D_{j_1}$ and $j' \in D_{j_k}$, while $D_{j_n} \cap D_{j_{n+1}} \neq \emptyset$ for all $1 \leq n \leq k-1$. Then m_ω is constant on Ω. This would imply that $Z(\omega)$ is constant on Ω, which would turn the relation $A^E(\omega) \cdot Z(\omega) = Z(T\omega)$ into the equality

$$A^E(\omega) \cdot Z(\omega) = Z(\omega).$$

This cannot be true in the case where $V(\omega)$ takes at least two different values. The obtained contradiction shows that there is no su-state for the cocycle A^E, which implies that the Lyapunov exponent is positive for every $E \in \mathbb{R}$.

Theorem 11. *Assume that V depends only on the zero-coordinate ω_0 of ω and takes at least two different values on Ω. Let the sets D_{j_0} be defined by* (4.9)*. Suppose that for each pair of symbols j and j' in $\mathcal{A}$, there is an ordered collection of letters $j_1, j_2, \dots, j_k$, such that $j \in D_{j_1}$ and $j' \in D_{j_k}$, while $D_{j_n} \cap D_{j_{n+1}} \neq \emptyset$ for all $1 \leq n \leq k-1$. Finally, assume that μ is a T-ergodic measure on Ω that has a local product structure and the property* $\operatorname{supp}(\mu) = \Omega$*.Then the Lyapunov exponent is positive for each $E \in \mathbb{R}$:*

$$L(A^E, \mu) > 0.$$

REFERENCES

[1] A. Avila, D. Damanik, and Z. Zhang: *Schrödinger operators with potentials generated by hyperbolic transformations: I–positivity of the Lyapunov exponent*, Invent. Math. **231**, (2023) 851-927.

[2] A. Avila, M. Viana: *Extremal Lyapunov exponents: an invariance principle and applications*, Invent. Math. **181** (2010), 115-189.

[3] L. Backes, A. Brown, C. Butler: *Continuity of Lyapunov exponents for cocycles with invariant holonomies*, J. Mod. Dyn. **12** (2018), 223-260.

[4] K. Bjerklöv: *Positive Lyapunov exponent for some Schrödinger cocycles over strongly expanding circle endomorphisms,* Commun. Math. Phys. **379** (2020), 353-360.

[5] C. Bonatti, M. Viana: *Lyapunov exponents with multiplicity 1 for deterministic products of matrices*, Ergodic Theory Dynam. Systems **24** (2004), 1295-1330.

[6] J. Bourgain, E. Bourgain-Chang: *A note on Lyapunov exponents of deterministic strongly mixing potentials*, J. Spectr. Theory **5** (2015), 1-15.

[7] J. Bourgain, M. Goldstein, W. Schlag: *Anderson localization for Schrödinger operators on Z with potentials given by the skew-shift*, Commun. Math. Phys. **220** (2001), 583-621.

[8] J. Bourgain, W. Schlag: *Anderson localization for Schrödinger operators on Z with strongly mixing potentials*, Commun. Math. Phys. **215** (2000), 143-175.

[9] D. Damanik: *Schrödinger operators with dynamically defined potentials*, Ergodic Theory Dynam. Systems **37** (2017), 1681-1764.

[10] D. Damanik, R. Killip: *Almost everywhere positivity of the Lyapunov exponent for the doubling map,* Commun. Math. Phys. **257** (2005), 287-290.

[11] D. Damanik, S. Tcheremchantsev: *Power-law bounds on transfer matrices and quantum dynamics in one dimension*, Commun. Math. Phys. **236** (2003), 513-534.

[12] B. Kalinin: *Livšic theorem for matrix cocycles*, Ann. of Math. **173** (2011), 1025-1042.

[13] R. Leplaideur: *Local product structure for equilibrium states*, Trans. Amer. Math. Soc. **352** (2000), 1889-1912.

[14] C. Sadel, H. Schulz-Baldes: *Positive Lyapunov exponents and localization bounds for strongly mixing potentials*, Adv. Theo. Math. Phys. **12** (2008), 1377-1400.

[15] M. Viana: *Almost all cocycles over any hyperbolic system have non-vanishing Lyapunov exponents*, Ann. of Math. **167** (2008), 643-680.

[16] M. Viana, J. Yang: *Continuity of Lyapunov exponents in the C^0 topology,* Israel J. Math. **229** (2019), 461-485.

[17] Z. Zhang: *Uniform hyperbolicity and its relation with spectral analysis of 1D discrete Schrödinger operators*, J. Spectr. Theory **10** (2020) no. 4, 1471-1517.

Email address: osafrono@uncc.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNCC, CHARLOTTE, NC