ON THE STRUCTURE OF THE d-INDIVISIBLE NONCROSSING PARTITION POSETS

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Abstract. We study the poset of d-indivisible noncrossing partitions introduced by Mühle, Nadeau and Williams. These are noncrossing partitions such that each block has cardinality 1 modulo d and each block of the dual partition also has cardinality 1 modulo d. Generalizing the work of Speicher, we introduce a generating function approach to reach new enumerative results and recover some known formulas on the cardinality, the Möbius function and the rank numbers. We compute the antipode of the Hopf algebra of d-indivisible noncrossing partition posets. Generalizing work of Stanley, we give an edge labeling such that the labels of the maximal chains are exactly the d-parking functions. This edge labeling induces an EL-labeling. We also introduce d-parking trees which are in bijective correspondence with the maximal chains.

1. Introduction

There has been a lot of work on subposets of the partition lattice. Most notable is the d-divisible partition lattice, which has been studied by Stanley [23], Calderbank–Hanlon–Robinson [2] and Wachs [28]. See also the papers by [5, 6] which extend this lattice to more general partition posets. In the d-divisible partition lattice the size of each block of each partition is divisible by d. Note that that this condition is upward closed in the partition lattice. Another condition that is natural to impose on partitions is that each block size is congruent to 1 modulo d. See for instance the subposet where all the block sizes are odd in [2] and also Wachs [29, Subsection 4.5.2] and the references therein. Note that the cover relations in such posets may be described by listing $d+1$ blocks of a partition which have to be merged into a single block to obtain a partition covering the original partition.

The noncrossing partition lattice NC_n is a widely studied subposet of the partition lattice having many applications; see the references [1, 16, 20, 21, 22, 26]. One of its most striking features is that it is self-dual, this fact was observed by Kreweras [16] and Simion–Ullman [21]. For an example; see Figure 1. In fact, Stanley pointed out that every interval in the noncrossing partition lattice is self-dual [26]. Also note Armstrong studied the subposet of noncrossing partitions where each block size is divisible by d; see [1].

In this paper we will consider the subposet NC_n^d of d-indivisible noncrossing partitions of the noncrossing partition lattice, first defined by Mühle, Nadeau and Williams [17]. These noncrossing partitions are the ones in which each block size is congruent to 1 modulo d and each block in the dual partition also has cardinality congruent to 1 modulo d. That we require this cardinality condition on each partition and its dual, makes the subposet NC_n^d naturally self-dual. On the other hand, many enumerative results for the noncrossing partition lattice extend to the subposet NC_n^d. One less studied invariant is the antipode of a poset which is the natural Hopf algebra extension of the Möbius function.

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For the poset NC_n^d we obtain its antipode in terms of noncrossing hypertrees, generalizing the results in [4]. We also enumerate the maximal chains in the poset NC_n^d and show that via an edge labeling they are in bijection with d-parking functions, as defined by Stanley [25] and Yan [30]. Furthermore, these structures are also in bijective correspondence with a class of trees that we call d-parking trees.

The paper is organized as follows. In Section 2 we present basic results about the noncrossing partition lattice NC_n including the Kreweras dual and review basic generating function results. Our main object of study, the poset NC_n^d is introduced in Section 3. We review the equivalent characterizations given in [17, Theorem 1.1] and outline a proof of it that does not require introducing permutation factorizations. We prove that intervals in this poset are products of smaller posets of the same form and characterize all cover relations. In Section 4 we obtain enumerative results on NC_n^d. Our approach is using generating functions; see subsection 4.1. This method generalizes results of Speicher [22]; see Proposition 4.2. In Section 5 we review a labeled tree representation of noncrossing partitions which also appears in the work of Mühle, Nadeau and Williams [17], defined in terms of permutation factorizations. We consider this model as a variant of the "dual tree representation" discussed by Kortchemski and Marzouk [15, Section 2] relying on a bijection introduced by Janson and Stefánsson [14]. These trees may be used to visualize the approach to proving the enumerative results in the work of Mühle, Nadeau and Williams [17], as well as our approach in Section 4. In Section 6 we obtain an expression for the antipode of NC_n^d in terms of noncrossing hypertrees. In Section 7 we generalize Stanley's edge labeling of NC_n to NC_n^d and show that the maximal chains are in bijection with the set of all d-parking functions. We introduce the notion of a d-parking tree in Section 8. These labeled trees are in bijection with the d-parking functions and visually encode the block merging processes represented by the maximal chains in NC_n^d. Finally, further research and open problems are outlined in Section 9.

2. Preliminaries

2.1. **The noncrossing partition lattice.** The partition lattice consists of all partitions of the set $[n] = \{1, 2, \ldots, n\}$. Two different blocks B and C are noncrossing if there are no four elements $i < j < k < \ell$ such that $i, k \in B$ and $j, \ell \in C$. This condition is viewed geometrically as follows. Place the elements of the set $[n]$ in a circle in positive orientation. The two blocks B and C are noncrossing if the convex hull of the two blocks are disjoint. A partition is noncrossing if all pairs of blocks are noncrossing. For an example, the partition $\pi = \{\{1\}, \{2, 9, 10\}, \{3\}, \{4, 5, 6, 7, 8\}, \{11\}\}$ is displayed in the first drawing in Figure 1. For short hand we write this partition as $1|2, 9, 10|3|4, 5, 6, 7, 8|11$. We call these drawings of the noncrossing partition and its dual the *circle representation* of the partition.

The *Kreweras dual* of a noncrossing partition is defined as follows. First observe that the complement of the union of the convex hull of the blocks form regions. For instance, in the partition π in Figure 1 we have a region R with the vertices 2, 3, 4, 8 and 9. However, to obtain the dual partition place the vertex i' between vertices i and $i+1$. That is, we have the cyclic order $1 < 1' < 2 < 2' < 3 < 3' < \cdots < n-1 < (n-1)' < n < n' < 1$. The region R now correspond to the block $\{2', 3', 8'\}$. The blocks corresponding to the regions form the dual partition which is noncrossing. In our example, we obtain the dual partition π' given by $1', 10', 11'|2', 3', 8'|4'|5'|6'|7'|9'$.

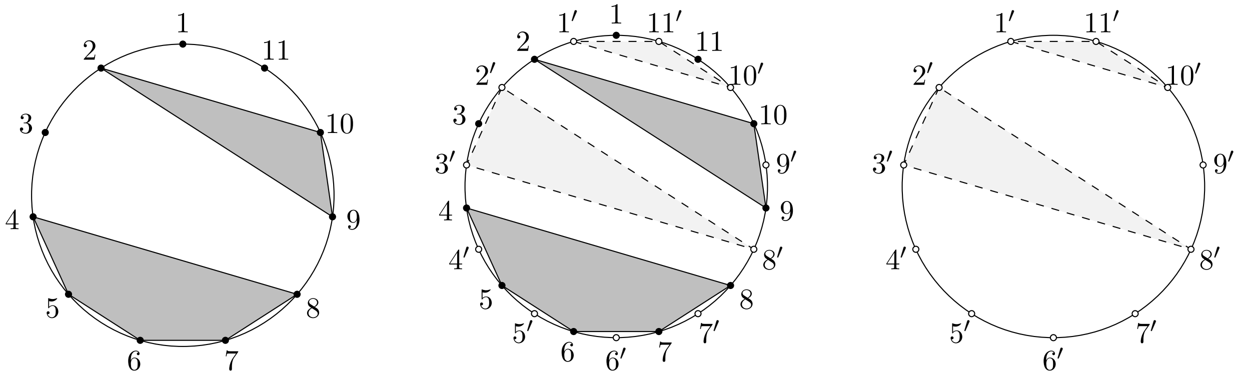

FIGURE 1. The noncrossing partition π given by $1|2,9,10|3|4,5,6,7,8|11$, its circle representation and the Kreweras dual partition $1',10',11'|2',3',8'|4'|5'|6'|7'|9'$.

Another natural dual to a noncrossing partition is the Simion–Ullman dual [21], that we denote by $\mathrm{SU}(\pi)$. It uses a different order on the set $1'$ through n': here we have $1 < n' < 2 < (n-1)' < \cdots < n-1 < 2' < n < 1' < 1$. The Simion–Ullman dual is an involution, that is, $\mathrm{SU}(\mathrm{SU}(\pi)) = \pi$. In this paper we will use the Kreweras dual by default as it yields an easier labeling of the dual partition.

We say that a block B of the noncrossing partition π and a block C' of the dual partition are *adjacent* if the block B contains the two elements i and j (they could be the same element) and the block C' contains the elements i' and $(j-1)'$. Note that the 4-gon with vertices i, i', $(j-1)'$ and j forms a channel between the block B and the dual block C'.

Following [4, Definitions 3.1 and 3.2] we make the following definition. For a block B in a noncrossing partition π and an adjacent block C' in the dual partition π' let i and j be the two vertices of the block B that are adjacent to the block C' and pick $\gamma(B, C')$ be the vertex of these two that is the most negative orientation from the block B's perspective.

A small note is in order at this point. The notion related to $\gamma(B, C')$ introduced in [4] is denoted $v_R(B)$. The definition of $\gamma(B, C')$ differs in two key points. First, here we work with the blocks of the dual partition, whereas [4] uses regions in the partition. Second, in [4] the authors pick the vertex in the most positive orientation. Here we pick the vertex in the most negative orientation. The reason for this difference is that we would like to be consistent with the Kreweras notation of the dual; see Section 5.

Let $|\pi|$ denote the number of blocks of the partition π. Let NC_n be the collection all noncrossing partitions on the set $[n]$ ordered by refinement. This poset is graded and is in fact a lattice. Observe that the map $\pi \longmapsto \mathrm{SU}(\pi)$ is an order-reversing involution. That is, the lattice NC_n is self-dual. Also note that rank of a noncrossing partition π is given by $\rho(\pi) = n - |\pi|$ and that the identity $|\pi| + |\pi'| = n + 1$ holds.

Consider a pair of noncrossing partitions π and σ in the noncrossing partition lattice NC_n such that $\pi \leq \sigma$. The interval $[\pi, \sigma]$ is given by $\{\tau \in \mathrm{NC}_n : \pi \leq \tau \leq \sigma\}$. It is well-known that intervals in the noncrossing partition lattice are isomorphic to Cartesian products of noncrossing partition lattices [22, Proposition 1].

2.2. **Generating functions.** We need the following classical results about generating functions. Assume that the generating function $G(z)$ satisfies the algebraic equation

$$G(z) = z \cdot G(z)^m + 1 \tag{2.1}$$

then the coefficient of z^n of $G(z)^\ell$ is given by

$$[z^n]G(z)^\ell = \frac{\ell}{mn+\ell} \cdot \binom{mn+\ell}{n} = \frac{\ell}{mn+\ell-n} \cdot \binom{mn+\ell-1}{n}, \tag{2.2}$$

for $\ell \geq 1$; see [12, Equations (7.68) and (7.69)]. See also [27, Example 6.2.6].

The Lagrange inversion formula allows one to obtain the coefficients of a power series $f(z)$ that satisfies the functional equation $f(z) = z \cdot g(f(z))$. Good stated a multivariate extension of the Lagrange inversion formula [10]. See also [7, Theorem 4]. We will only need this extension in two variables and hence we state it only in this case.

Theorem 2.1 (Good's Inversion Formula for two variables.). *Let $g_1(z, w)$ and $g_2(z, w)$ be formal power series with constant coefficients. Assume that the formal power series $f_1(z, w)$ and $f_2(z, w)$ satisfy the equations*

$$f_1(z, w) = z \cdot g_1(f_1(z, w), f_2(z, w)), \qquad f_2(z, w) = z \cdot g_2(f_1(z, w), f_2(z, w)).$$

Then the coefficient of $z^m w^n$ of $f_1(z, w)^k \cdot f_2(z, w)^\ell$ is given by

$$[z^m w^n] f_1(z, w)^k \cdot f_2(z, w)^\ell = [z^{m-k} w^{n-\ell}] \det \begin{pmatrix} g_1^m - z \cdot \frac{\partial g_1}{\partial z} \cdot g_1^{m-1} & -w \cdot \frac{\partial g_1}{\partial w} \cdot g_1^{m-1} \\ -z \cdot \frac{\partial g_2}{\partial z} \cdot g_2^{n-1} & g_2^n - w \cdot \frac{\partial g_2}{\partial w} \cdot g_2^{n-1} \end{pmatrix}.$$

3. The poset of d-indivisible noncrossing partitions

The following concept was introduced by Mühle, Nadeau and Williams [17].

Definition 3.1. *Let d be a positive integer. A noncrossing partition $\pi \in \mathrm{NC}_n$ is* d-indivisible *if all the blocks of the partition π and the dual partition π' have cardinality congruent to 1 modulo d. We denote the subposet of d-indivisible noncrossing partitions of NC_n by NC_n^d.*

As an example, the partition in Figure 1 belongs the poset NC_{11}^2 since all the blocks in the partition and the dual partition have odd cardinality. We can replace the Kreweras dual π' with the Simion–Ullman dual $\mathrm{SU}(\pi)$ in the above definition, and it will not change the definition of the poset NC_n^d. Since taking the Simion–Ullman dual $\mathrm{SU}(\pi)$ is an order reversing involution, the poset NC_n^d is self-dual.

In [17] d-indivisible noncrossing partitions are only defined in the case when $n \not\equiv 1 \bmod d$ holds. The next lemma shows that a generalization to other values of n is not possible.

Lemma 3.2. *If the set NC_n^d is nonempty then $n \equiv 1 \bmod d$.*

Proof. Assuming that NC_n^d is nonempty, let π be a noncrossing partition in NC_n^d. Then we have the following string of congruences $n = \sum_{B\in\pi} |B| \equiv \sum_{B\in\pi} 1 = |\pi| \bmod d$. Similarly, we obtain $n \equiv |\pi'| \bmod d$. Adding these two congruences yields $2n \equiv |\pi| + |\pi'| = n+1 \bmod d$, that is, $n \equiv 1 \bmod d$. $\square$

Note that when $n \equiv 1 \bmod d$ then NC_n^d is nonempty since it contains the noncrossing partition $\{[n]\}$. Furthermore, this partition $\{[n]\}$ is the maximal element in NC_n^d and the noncrossing partition $\{\{1\},\{2\},\dots,\{n\}\}$ is the minimal element.

The following characterization of d-indivisible noncrossing partition is part of [17, Theorem 1.1].

Theorem 3.3 (Mühle–Nadeau–Williams)**.** *The noncrossing partition π belongs to NC_n^d if and only if the following two conditions are satisfied (i) $n \equiv 1 \bmod d$; (ii) for each pair of cyclically consecutive elements i and j of a nonsingleton block B in a noncrossing partition π, the number of elements in $[n]$ strictly between i and j in the cyclic order is divisible by d.*

In [17, Theorem 1.1] a third equivalent characterization was added in terms of permutation factorizations, and the cyclic proof relies on the use of permutation factorizations. In the rest of this section we will provide a direct proof of Theorem 3.3 that does not rely on permutation factorization, and characterize all covering relations. A key tool to reach our second goal will be Proposition 3.4 below.

Proposition 3.4. *Let π and σ be two noncrossing partitions in NC_n^d such that $\pi \leq \sigma$ holds. Then the interval $[\pi,\sigma]$ in NC_n^d is isomorphic to the Cartesian product of smaller posets of the form NC_k^d.*

By the same the same reasoning as in Lemma 3.2 we have the next lemma.

Lemma 3.5. *Let n be a positive integer such that $n \equiv 1 \bmod d$ and let π be a noncrossing partition in NC_n. Assume that all blocks B in π satisfy $|B| \equiv 1 \bmod d$. Assume furthermore that all blocks C' but one in the dual partition π' satisfy $|C'| \equiv 1 \bmod d$. Then the noncrossing partition π belongs to NC_n^d.*

Proof. Let D' be the block in the dual partition π' for which we do not have a congruence. We have the following string of congruences:

$$|D'| = n - \sum_{\substack{C'\in\pi' \\ C'\neq D'}} |C'| \equiv n - (|\pi'|-1) = n+1-|\pi'| \equiv |\pi| \equiv \sum_{B\in\pi} |B| \equiv n \equiv 1 \bmod d. \qquad \square$$

Lemma 3.6. *Assume that the noncrossing partition π belongs to the noncrossing partition poset NC_n^d. Then the two following equalities hold:*

$$\frac{|\pi'|-1}{d} = \sum_{B\in\pi} \frac{|B|-1}{d}, \qquad \frac{|\pi|-1}{d} = \sum_{C'\in\pi'} \frac{|C'|-1}{d},$$

where all the summands are integers.

Proof. By duality it is enough to prove the first identity. We have $|\pi'|-1 = n-|\pi| = \sum_{B\in\pi}(|B|-1)$, and the result follows by dividing by d. By the definition of NC^d_n, all summands in the statement are integers. □

The next Lemma is logically equivalent to [17, Corollary 2.3].

Lemma 3.7 (Mühle–Nadeau–Williams). *If $\sigma \in \mathrm{NC}^d_n$ is a coatom then it consists of $d+1$ blocks.*

Proof. We prove the contrapositive statement: a noncrossing partition π with $dr+1$ blocks, where $r \geq 2$, is not a coatom. By Lemma 3.6 we know that $r = \sum_{C'\in\pi'}(|C'|-1)/d$. We now have two cases. The first case is when this sum has at least two non-zero summands. Pick one of those nonzero terms corresponding to the block D' in the dual partition. Now form the noncrossing partition σ from π by joining all blocks adjacent to the block D'. The second case is when this sum $\sum_{C'\in\pi'}(|C'|-1)/d$ consists of one non-zero summand. Again, let this term correspond to the dual block D' which has cardinality $dr+1$. Say D' is given by $\{i'_1 < i'_2 < \cdots < i'_{dr+1}\}$. Now join the $d+1$ blocks of π that contain the elements i_1 through i_{d+1} to obtain a new noncrossing partition σ. In both cases, the partition σ belongs to NC^d_n and does not consist of one block. In the first case the block D' is replaced by all singleton blocks. In the second case D' is replaced by the singleton blocks $\{i'_1\}$ through $\{i'_d\}$ and the block $\{i'_{d+1}, i'_{d+2}, \ldots, i'_{rd+1}\}$. Hence we conclude that π is not a coatom. □

Lemma 3.8. *Let σ be a coatom in the noncrossing partition poset NC^d_n, and assume that $\sigma = \{B_1, B_2, \ldots, B_{d+1}\}$. Then the interval $[\widehat{0}, \sigma]$ in NC^d_n factors as*

$$[\widehat{0}, \sigma] \cong \prod_{i=1}^{d+1} \mathrm{NC}^d_{|B_i|}.$$

Proof. Let τ be a noncrossing partition in the interval $[\widehat{0}, \sigma]$, that is, $\tau \leq \sigma$. Let τ_i be the restriction of the partition τ to the block B_i. Note that $|B_i| \equiv 1 \bmod d$. Furthermore, each block B of τ_i is a block of τ and hence satisfies $|B| \equiv 1 \bmod d$. Note that all but one of the blocks of the dual partition τ'_i is a block of the dual partition τ' and thus satisfies $|C'| \equiv 1 \bmod d$. Hence by Lemma 3.5 we conclude that τ_i belongs to $\mathrm{NC}^d_{|B_i|}$.

Furthermore, the map from the interval $[\widehat{0}, \sigma]$ to the Cartesian product $\prod_{i=1}^{d+1} \mathrm{NC}^d_{|B_i|}$, sending τ to the $(d+1)$-tuple $(\tau_1, \tau_2, \ldots, \tau_{d+1})$ is bijective and order preserving. □

Lemma 3.9. *The noncrossing partition poset NC^d_n is graded of rank $(n-1)/d$ and the rank function is given by $\rho(\pi) = (n-|\pi|)/d$.*

Proof. It is enough to show that every maximal chain in the poset NC^d_{dk+1} has length k. We prove this by induction on k. When $k=0$ the statement is directly true. Assume now that it is true for noncrossing partition posets with smaller value of k and we prove it for k. Consider a maximal chain $\mathbf{m} = \{\widehat{0} \prec \cdots \prec \sigma \prec \widehat{1}\}$, where the coatom σ is given by $\{B_1, B_2, \ldots, B_{d+1}\}$. Since the interval $[\widehat{0}, \sigma]$ is given by a product of $\mathrm{NC}^d_{|B_i|}$, where each factor is graded, the poset $[\widehat{0}, \sigma]$ is graded. Its rank is given by the sum $\sum_{i=1}^{d+1}(|B_i|-1)/d = (n-d-1)/d = (n-1)/d-1$. Since the rank of $[\widehat{0},\sigma]$ is independent of the coatom σ, the poset NC_n^d is graded and its rank is one more than $(n-1)/d-1$.

Next observe that the suggested rank function satisfies $\rho(\widehat{0})=0$, $\rho(\widehat{1})=(n-1)/d$ and is strictly increasing. Hence it is the unique rank function. ☐

The next consequence is a generalization of [17, Corollary 2.3].

Corollary 3.10. *Let $\pi \prec \sigma$ be a cover relation in the noncrossing partition poset NC_n^d. Then σ is obtained by joining $d+1$ blocks of π.*

Proof. Since $\rho(\pi)+1=\rho(\sigma)$ we obtain that $|\pi|=|\sigma|+d$. But since every block size of π and σ is congruent to 1 modulo d, the only option is to join $d+1$ blocks of π. ☐

By Lemma 3.8 and the dual version of this lemma we now have a proof of Proposition 3.4.

Remark 3.11. For two sets $B \subseteq \{1,2,\ldots,n\}$ and $C' \subseteq \{1',2',\ldots,n'\}$ define the *intertwining number* $i(B,C')$ to be the number of times we move from the set B to set C' when reading the entries of $B\cup C'$ in the cyclic order $1<1'<2<2'<3<3'<\cdots<n-1<(n-1)'<n<n'<1$. Note that $i(B,C')=0$ if and only if B or C' is the empty set. Given two noncrossing partitions π and σ in NC_n^d such that $\pi\leq\sigma$. Then the interval $[\pi,\sigma]$ in NC_n^d is isomorphic to the Cartesian product

$$[\pi,\sigma]\cong\prod_{B\in\sigma}\prod_{C'\in\pi'}\mathrm{NC}^d_{i(B,C')}.$$

However, we do not need this statement and hence omit its proof.

Proof of Theorem 3.3. The forward implication that $n\equiv 1 \bmod d$ follows directly by Lemma 3.2. The other forward implication follows by constructing the noncrossing partition τ on the half open cyclic interval $[i,j)$ by letting

$$\tau=\{\{i\}\}\cup\{D\in\pi : D\subseteq(i,j)\}.$$

Note that this construction amounts to restricting π to the cyclic interval $[i,j]$ and identifying the two elements i and j. Note that the block B becomes the singleton block $\{i\}$. Every block $D\neq B$ of π is either contained in the open interval (i,j) or it is disjoint from it, yielding $|D|\equiv 1 \bmod d$ for the blocks in τ. Also observe that any block C' in the dual partition τ' is a dual block of π and hence satisfies $|C'|\equiv 1 \bmod d$. That is, the partition τ belongs $\mathrm{NC}^d_{|[i,j)|}$ and the previous lemma implies that $|[i,j)|\equiv 1 \bmod d$, proving the lemma.

Now consider the opposite implication. Given a block B in π of size k, that is, $B=\{i_1,i_2,\ldots,i_k\}$. Note that the set $[n]$ is the disjoint union of B and all the open cyclic intervals (i_j,i_{j+1}). Consider the cardinality modulo d, that is, $|B|\equiv|B|+\sum_{j=1}^{k}|(i_j,i_{j+1})|=n\equiv 1 \bmod d$. Hence every block has cardinality congruent to 1 modulo d.

Next, let i' and j' be two cyclically consecutive elements of a nonsingleton block C' in the dual partition π'. Note that the elements j and $i+1$ are two cyclically consecutive elements of a block B

of π. Hence by the second condition the open cyclic interval $(j, i+1) = \{j+1, j+2, \ldots, i\}$ has its size to be a multiple of d. Now the open interval $(i', j') = \{(i+1)', (i+2)', \ldots, (j-1)'\}$ has the same size as $\{i+1, i+2, \ldots, j-1\} = [n] - \{j, j+1, \ldots, i\}$. Again, by considering the cardinality modulo d we obtain $|(i', j')| \equiv n-1-|(j, i+1)| \equiv 0 \bmod d$. Hence the second condition of the proposition holds for the dual partition π'. Finally, by the argument in the previous paragraph of this proof, every block C' in the dual partition π' satisfies $|C'| \equiv 1 \bmod d$ and the proposition follows. $\square$

4. Enumerative results for the poset NC_n^d

4.1. **Generating function approach.** We now introduce generating functions in order to obtain enumerative results for the noncrossing partition poset NC_n^d. Let $a(0) = 1, a(1), a(2), \ldots$ and $a^*(0) = 1, a^*(1), a^*(2), \ldots$ be two given sequences. For a noncrossing partition π in the poset NC_{dk+1}^d define its weight by the product:

$$\mathrm{wt}(\pi) = \prod_{B \in \pi} a\left(\frac{|B|-1}{d}\right) \cdot \prod_{C' \in \pi'} a^*\left(\frac{|C'|-1}{d}\right) \cdot s^{(|\pi|-1)/d} \cdot t^{(|\pi'|-1)/d}.$$

Note that the power of the variable s is the corank of the partition π and that the power of t is the rank of π in the poset NC_{dk+1}^d. Hence every monomial $s^i t^j \cdot x^k$ that occurs satisfies $i + j = k$. We are interested in exploring the following sum and its ordinary generating function:

$$b(k) = \sum_{\pi \in \mathrm{NC}_{dk+1}^d} \mathrm{wt}(\pi), \qquad B(x) = \sum_{k \geq 0} b(k) \cdot x^k. \tag{4.1}$$

Let $A(x)$ and $A^*(x)$ be the two ordinary generating functions

$$A(x) = \sum_{k \geq 0} a(k) \cdot x^k, \qquad A^*(x) = \sum_{k \geq 0} a^*(k) \cdot x^k.$$

In order to obtain information about $B(x)$, we consider the two following partial sums and their generating functions

$$c(k) = \sum_{\substack{\pi \in \mathrm{NC}_{dk+1}^d \\ \{1\} \in \pi}} \mathrm{wt}(\pi), \qquad C(x) = \sum_{k \geq 0} c(k) \cdot x^k,$$

$$c^*(k) = \sum_{\substack{\pi \in \mathrm{NC}_{dk+1}^d \\ \{1'\} \in \pi'}} \mathrm{wt}(\pi), \qquad C^*(x) = \sum_{k \geq 0} c^*(k) \cdot x^k.$$

That is, $c(k)$ and its generating function $C(x)$, enumerate noncrossing partitions which has a given element as a singleton block. Similarly, $c^*(k)$ and $C^*(x)$ enumerate partitions with a given element as a singleton block in the dual.

Theorem 4.1. *The following three generating function identities hold:*

$$C(x) = A^*\left(x \cdot s \cdot C^*(x)^d\right), \qquad C^*(x) = A\left(x \cdot t \cdot C(x)^d\right), \tag{4.2}$$

$$B(x) = C(x) \cdot C^*(x). \tag{4.3}$$

Furthermore, the first two equations uniquely determine the generating functions $C(x)$ and $C^*(x)$.

Proof. Observe that the first two identities are dual to each other. Hence it is enough that we prove the second identity. We are considering the case where the singleton block $\{1'\}$ belongs to the dual partition. That is, the partition contains a block B such that

$$B = \{1 = i_1 < i_2 < \cdots < i_{dr+1} = n\}.$$

By Theorem 3.3 each difference $i_{j+1} - i_j$ is congruent to 1 modulo d. Let τ_j now be the restriction of the partition π to the union $[i_j, i_{j+1}] \cup B$ and identify all the elements of the block B. This construction is equivalent to letting τ_j be the restriction to the open interval (i_j, i_{j+1}) and adding a singleton block. Keeping track of the blocks in π we have that

$$|\pi| - 1 = (|\tau_1| - 1) + \cdots + (|\tau_{dr}| - 1).$$

Counting the blocks in the dual partition we obtain

$$|\pi'| - 1 = |\tau_1'| + \cdots + |\tau_{dr}'| = dr + (|\tau_1'| - 1) + \cdots + (|\tau_{dr}'| - 1).$$

Hence the weight $\mathrm{wt}(\pi)$ factors as

$$\mathrm{wt}(\pi) = a(r) \cdot t^r \cdot \mathrm{wt}(\tau_1) \cdots \mathrm{wt}(\tau_{dr}).$$

Now we express $c^*(k)$ as the sum

$$\begin{aligned} c^*(k) &= \sum_{r \geq 1} a(r) \cdot t^r \cdot \sum_{\substack{1 = i_1 < i_2 < \cdots < i_{dr+1} = n \\ i_{j+1} - i_j \equiv 1 \bmod d}} \sum_{\tau_1} \cdots \sum_{\tau_{dr}} \prod_{j=1}^{dr} \mathrm{wt}(\tau_j) \\ &= \sum_{r \geq 1} a(r) \cdot t^r \cdot \sum_{\substack{1 = i_1 < i_2 < \cdots < i_{dr+1} = n \\ i_{j+1} - i_j \equiv 1 \bmod d}} \prod_{j=1}^{dr} c\left(\frac{i_{j+1} - i_j - 1}{d}\right), \end{aligned} \tag{4.4}$$

where each τ_j ranges over partitions in $\mathrm{NC}^d_{i_{j+1} - i_j}$ with a given singleton block. Observe that k is given by the sum

$$k = r + \frac{i_2 - i_1 - 1}{d} + \cdots + \frac{i_{dr+1} - i_{dr} - 1}{d}.$$

Hence multiply with x^k and sum over all $k \geq 1$.

$$\begin{aligned} \sum_{k \geq 1} c^*(k) \cdot x^k &= \sum_{r \geq 1} a(r) \cdot t^r \cdot x^r \cdot \sum_{\substack{1 = i_1 < i_2 < \cdots < i_{dr+1} = n \\ i_{j+1} - i_j \equiv 1 \bmod d}} \prod_{j=1}^{dr} c\left(\frac{i_{j+1} - i_j - 1}{d}\right) \cdot x^{(i_{j+1} - i_j - 1)/d} \\ &= \sum_{r \geq 1} a(r) \cdot t^r \cdot x^r \cdot C(x)^{dr}. \end{aligned}$$

Lastly, adding the constant $c^*(0) = 1$ yields the second equation.

Equation (4.4) and its dual yield a pair of intertwined recursions for the sequences $(c(k))_{k \geq 0}$ and $(c^*(k))_{k \geq 0}$: equation (4.4) expresses $c(k)$ in terms of $c^*(0), \ldots, c^*(k-1)$, whereas the dual equation expresses $c^*(k)$ in terms of $c(0), \ldots, c(k-1)$. Together with the initial condition $c(0) = c^*(0) = 1$, the two functions $C(x)$ and $C^*(x)$ are uniquely determined.

Now we prove the third identity. Given a partition π in NC^d_{dk+1}, let j be the largest element in the block B containing 1. Theorem 3.3 implies that $j \equiv 1 \bmod d$, hence let $j = dr + 1$. Similarly, then j' is the smallest element in block of the dual partition containing n'. Let σ be the restriction π to the interval $[1, j]$, which has cardinality $dr + 1$. Let τ be the restriction of π to the union $[j + 1, n] \cup B$ and identify all elements of the block B. Note that τ is a partition on a set of size $d(k - r) + 1$. Note that σ has a given singleton block in the dual and τ has also a given singleton block. Enumerating the blocks in π and the blocks in the dual partition we have

$$|\pi| - 1 = |\sigma| - 1 + |\tau| - 1, \qquad |\pi'| - 1 = |\sigma'| - 1 + |\tau'| - 1.$$

Hence the weight of π factors as

$$\mathrm{wt}(\pi) = \mathrm{wt}(\tau) \cdot \mathrm{wt}(\sigma).$$

Summing over all partitions π in NC^d_{dk+1}, multiplying with $x^k = x^r \cdot x^{k-r}$ and summing over all $k \geq 0$ yield the desired identity. □

Proposition 4.2. *Set $s = t = 1$ and $a^*(k) = (-1)^k \cdot \binom{-1/d}{k}$ for all k. Define the two functions $\overline{A}(x) = 1 + x \cdot A(x)^d$ and $\overline{B}(x) = 1 + x \cdot B(x)^d$. Then the following identity holds*

$$\overline{B}(x) = \overline{A}(x \cdot \overline{B}(x)).$$

In the case when $d = 1$, this is Speicher's identity; see [22, Theorem on page 616].

Proof of Proposition 4.2. Note that $A^*(x) = 1/\sqrt[d]{1-x}$, that is, $A^*(x)^d = 1/(1 - x)$. Hence we have $C(x)^d = 1/(1 - x \cdot C^*(x)^d)$. Cross multiply to obtain $1 = C(x)^d - x \cdot C(x)^d \cdot C^*(x)^d = C(x)^d - x \cdot B(x)^d$. We obtain $\overline{B}(x) = 1 + x \cdot B(x)^d = C(x)^d$. Now we have

$$\begin{aligned}\overline{B}(x) &= 1 + x \cdot B(x)^d = 1 + x \cdot C(x)^d \cdot C^*(x)^d = 1 + x \cdot C(x)^d \cdot A(x \cdot C(x)^d)^d \\ &= 1 + x \cdot \overline{B}(x) \cdot A(x \cdot \overline{B}(x))^d = \overline{A}(x \cdot \overline{B}(x)).\end{aligned}$$

□

Example 4.3. The following results of Mühle, Nadeau and Williams [17] follow from Theorem 4.1.

(a) [17, Theorem 1.2] The cardinality of NC^d_{dk+1} is given by

$$\left|\mathrm{NC}^d_{dk+1}\right| = \frac{2}{dk+2} \cdot \binom{dk+k+1}{k}.$$

This follows by setting $a(k) = a^*(k) = s = t = 1$ and using the relations $A(x) = A^*(x) = 1/(1 - x)$ and $C(x) = C^*(x) = 1 + x \cdot C(x)^{d+1}$.

(b) [17, Corollary 4.6] The number of partitions of rank j in the noncrossing partition poset NC^d_{dk+1}, for $k = i + j$, is given by

$$\frac{dk+1}{(i+dj+1) \cdot (di+j+1)} \cdot \binom{i+dj+1}{i} \cdot \binom{di+j+1}{j}.$$

Set $a(k) = a^*(k) = 1$ such that $A(x) = A^*(x) = 1/(1-x)$. Let $D(s,t,x) = C(s,t,x) - 1$ and $D^*(s,t,x) = C^*(s,t,x) - 1$. Rewrite the two equations in (4.2) as

$$D = sx \cdot (1+D) \cdot (1+D^*)^d, \qquad D^* = tx \cdot (1+D)^d \cdot (1+D^*).$$

Note that every monomial that occurs has the form $s^i t^j \cdot x^{i+j}$. At this point we set $x = 1$ since all the information is carried by the coefficients of the monomials in the variables s and t. We can now determine the coefficients of $(1+D) \cdot (1+D^*)$ using Good's inversion formula, Theorem 2.1.

(c) [17, Corollary 4.9] The Möbius function of NC^d_{dk+1} is given by

$$\mu(\mathrm{NC}^d_{dk+1}) = (-1)^k \cdot \frac{1}{2dk-k+1} \cdot \binom{2dk}{k}.$$

Set $a^*(k) = s = t = 1$ and $a(k) = (-1)^k \cdot \frac{1}{2dk-k+1} \cdot \binom{2dk}{k}$. Then the four generating functions $A(x)$, $A^*(x)$, $C(x)$ and $C^*(x)$ satisfy the relations $A(x) + x \cdot A(x)^{2d} = 1$, $A^*(x) = 1/(1-x)$, $C(x)^d - C(x)^{d-1} = x$, $C(0) = 1$ and $C^*(x) = 1/C(x)$.

By considering the generating function $C(x)$ in Example 4.3 parts (a) and (b), we obtain the following results.

Example 4.4. (a) The number of noncrossing partitions in NC^d_{dk+1} that contain the singleton block $\{1\}$ is given by

$$\frac{1}{dk+1} \cdot \binom{dk+k}{k}.$$

(b) Furthermore, the number of partitions of rank j in the noncrossing partition poset NC^d_{dk+1} that contain $\{1\}$ as a singleton block is given by

$$\frac{1}{dj+1} \cdot \binom{i+dj}{i} \cdot \binom{di+j-1}{j}.$$

4.2. **Block sizes** 1 **and** $d+1$.

Theorem 4.5. *The number of noncrossing partitions on $dk+1$ elements where each block size is either 1 or $d+1$ and each block size of the dual partition is also either 1 or $d+1$ is given by*

$$\frac{2}{dk+2} \cdot \binom{dk+2}{k}.$$

If we further assume that the partition contains the singleton block $\{1\}$, the number reduces to

$$\frac{1}{dk+1} \cdot \binom{dk+1}{k}.$$

Proof. In this case we use $A(x) = A^*(x) = 1 + x$ and $s = t = 1$. By symmetry we have $C(x) = C^*(x)$ and this generating function satisfies $C(x) = 1 + x \cdot C(x)^d$. Again equation (2.2) yields these two results. $\square$

Theorem 4.6. *Let $i+j=k$. The number of noncrossing partitions in NC^d_{dk+1} such that the partition only has blocks of sizes 1 and $d+1$ and the dual partition also only has blocks of sizes 1 and $d+1$ of rank j is given by*

$$\frac{dk+1}{(dj+1)\cdot(di+1)}\cdot\binom{dj+1}{i}\cdot\binom{di+1}{j}.$$

Proof. This proof is similar to the proof of Example 4.3(c) and hence we highlight the differences. Let $D(s,t,x)=C(s,t,x)-1$ and $D^*(s,t,x)=C^*(s,t,x)-1$. Rewrite the two equations in (4.2) recalling that in this case we have $A(x)=A^*(x)=1+x$, as

$$D = sx\cdot(1+D^*)^d, \qquad D^* = tx\cdot(1+D)^d.$$

Set $x=1$ and apply Good's inversion formula using $g_1(s,t)=(1+t)^d$ and $g_2(s,t)=(1+s)^d$. Let M be the matrix given by

$$M_{1,1} = g_1(s,t)^i - s\cdot\frac{\partial g_1}{\partial s}\cdot g_1(s,t)^{i-1} = (1+t)^{di}, \qquad M_{1,2} = -d\cdot t\cdot(1+t)^{di-1},$$
$$M_{2,1} = -s\cdot\frac{\partial g_2}{\partial s}\cdot g_2(s,t)^{j-1} = -d\cdot s\cdot(1+s)^{dj-1}, \qquad M_{2,2} = (1+s)^{dj}.$$

Its determinant is given by

$$\det(M) = (1+s)^{dj}\cdot(1+t)^{di} - d^2\cdot st\cdot(1+s)^{dj-1}\cdot(1+t)^{di-1}.$$

Good's inversion formula now implies

$$[s^it^j]D = [s^{i-1}t^j]\det(M) = \binom{dj}{i-1}\cdot\binom{di}{j} - d^2\cdot\binom{dj-1}{i-2}\cdot\binom{di-1}{j-1}$$

and

$$[s^it^j]D\cdot D^* = [s^{i-1}t^{j-1}]\det(M) = \binom{dj}{i-1}\cdot\binom{di}{j-1} - d^2\cdot\binom{dj-1}{i-2}\cdot\binom{di-1}{j-2}.$$

We have that

$$[s^it^j]B(s,t) = [s^it^j]D\cdot D^* + [s^it^j]D + [s^it^j]D^* + [s^it^j]1.$$

Hence the result follows by adding the two previous identities and the symmetric version for $[s^it^j]D^*$. $\square$

5. A tree representation behind the enumerative results

In this section we review a two-colored labeled tree representation of a noncrossing partition which simultaneously represents the blocks of a noncrossing partition and the blocks of its Kreweras dual. Essentially the same representation was defined in [17, Figure 3] in terms of permutation factorizations. We consider these labeled trees as a variant of the "dual tree representation" discussed by Kortchemski and Marzouk [15, Section 2] relying on a bijection introduced by Janson and Stefánsson [14]. The topological structure and the two-coloring are the same in all of the above mentioned sources, the labeling and the choice of the root vary. The structure of these trees may be used to visualize the approach to proving the enumerative results in the work of Mühle, Nadeau and Williams [17], as well as our approach in Section 4. At the end of this section we visualize the difference between the two approaches in terms of decomposing these trees.

We begin with the description of a *labeled topological tree* encoding a noncrossing partition. Recall that a *topological graph* is a graph enriched with a fixed cyclic order of the incident edges around each vertex. Consider the circle representation of a noncrossing partition π of $\{1, 2, \ldots, n\}$. On the left hand side of Figure 2 we see the circle representation of the noncrossing partition $\pi = 1|2, 9, 10|3|4, 5, 6, 7, 8|11$ that was introduced in Figure 1. The non-singleton blocks of its Kreweras dual partition $\pi' = 1', 10', 11'|2', 3', 8'|4'|5'|6'|7'|9'$ are polygons with dashed line boundaries.

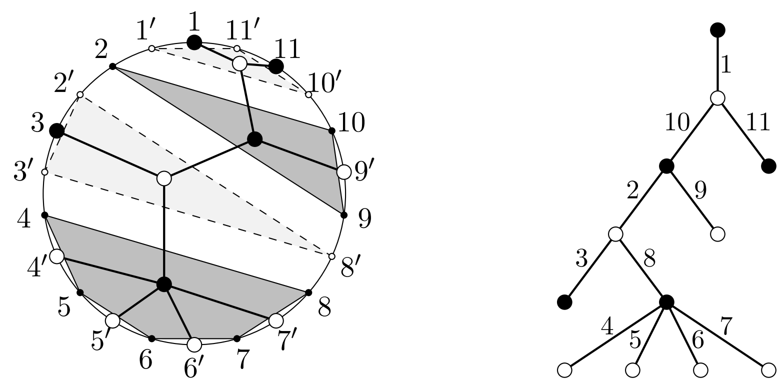

FIGURE 2. A noncrossing partition and its plane tree representation

Recall that a block B of noncrossing partition π and a block C' in the dual partition π' are adjacent if the block B contains the two elements i and j and the dual block C' contains the two elements i' and $(j-1)'$. Here $i = j$ and $i' = (j-1)'$ are possible. Furthermore, in subsection 2.1 we defined $\gamma(B, C')$ to be the element in the most in the negative orientation of i and j from the block B's perspective. Observe that this is the element i since it has i in the block B and i' in the dual block C'.

Definition 5.1. *The* labeled topological tree representing the noncrossing partition $\pi \in \mathrm{NC}_n$ *is a two vertex-colored tree with n labeled edges, defined as follows:*

(1) *The black nodes represent the blocks of π, the white nodes represent the blocks of π'.*
(2) *If a block B of π and a block C' in the dual partition are adjacent then the associated nodes are connected with an edge.*
(3) *Label the edge between the two nodes representing the two adjacent blocks B and C' with the canonical element $\gamma(B, C')$.*

The labeled topological tree associated to the noncrossing partition π is shown on the left hand side of Figure 2 in such a way that the vertex representing a non-singleton (dual) block is inside the polygon representing the same block, whereas the vertices representing the singleton (dual) blocks appear at correct place representing the respective point i (i') on the circle. So far our representation is essentially the same as the dual tree representation in [15, Section 2], the only difference being that they draw the edge labels near the respective black vertex. We place the same labels in the middle of each vertex, and we think of the label i as the representative of i in the block of π associated to the black end of the edge and as the representative of i' in the block of π' associated to the white end of the same edge.

Remark 5.2. The careful reader should notice that the labeled topological tree represented in [15, Figure 2] is the same after taking the mirror image of the figure. The topological graph structure there is defined in terms of the permutations associated to the noncrossing partition and to its Kreweras dual.

As in [15, Section 2], the following statement is a direct consequence of this construction.

Proposition 5.3. *The operation associating to each noncrossing partition its labeled topological tree is injective.*

Next we turn our labeled topological tree into a *(rooted) plane tree*. This is where our choices differ from [15, Section 2] substantially, as we define the root and its leftmost child (called sometimes "corner" in [15, Section 2]) in a different fashion. The way of turning a topological tree into a plane tree is implicit in [17].

Definition 5.4. *Given a noncrossing partition $\pi \in \mathrm{NC}_n$ we define its* labeled plane tree *as the drawing of its labeled topological tree in the plane subject to the following rules.*

1. *The root of the tree represents the block B_1 of π containing the element* 1*.*
2. *The leftmost child of each vertex is connected to it by the edge whose label is the least among the labels of all edges connecting the vertex to its children.*

The labeled plane tree representation of the noncrossing partition shown in Figure 2 is on the right hand side of the figure. Note that we may omit the mention of the two-coloring of the vertices, because assigning the color black to the root determines the two-coloring uniquely. As a consequence of Proposition 5.3 we obtain the following corollary.

Corollary 5.5. *The operation associating to each noncrossing partition its labeled plane tree is injective.*

Next we show that keeping track of the labels is not necessary either, as it may be uniquely reconstructed from the plane tree structure. The following statement plays a key role in seeing this. Since the vertices represent (dual) blocks of a noncrossing partition, by a slight abuse of notation we will denote the label on the edge connecting the vertices u and v also by $\gamma(u, v)$.

Proposition 5.6. *The labeling of a labeled plane tree representing a noncrossing partition $\pi \in \mathrm{NC}_n$ has the following properties:*

1. *The labels on the edges are the elements of the set $\{1, 2, \ldots, n\}$ each appearing exactly once.*
2. *For each vertex in the tree, the labels on the edges connecting it to its children increase left to right.*

(3) *For each nonroot black (white) vertex u in the tree the largest (least) label appearing on an edge connecting u to another vertex is on the edge connecting u to its parent in the tree.*
(4) *For each vertex in the tree the set of all labels appearing in its subtree form an interval $[i,j] = \{i, i+1, \ldots, j\}$ of consecutive integers.*
(5) *For each black (white) vertex in the tree, the least (largest) label in its subtree labels the edge connecting the root of the subtree to its leftmost (rightmost) child.*
(6) *If v_1 and v_2 are children of the same vertex v and v_1 is to the left from v_2 then $\gamma(v,v_1)$ is smaller than the label on any edge in the subtree of v_2, and $\gamma(v,v_2)$ is greater than the label on any edge in the subtree of v_1.*

Proof. In our proof we use the fact that each black vertex represents a block $\{i_1, i_2, \ldots, i_r\}$ of a noncrossing partition, here we assume $i_1 < i_2 < \cdots < i_r$. Similarly, each white vertex represents a dual block $\{j_1', j_2', \ldots, j_s'\}$, here we assume $j_1 < j_2 < \cdots < j_s$.

Property (1) is a consequence of Definition 5.1. It is also a consequence of this definition that for each vertex in the tree, the labels on the edges connecting it to its children follow the counterclockwise order. By part (2) of Definition 5.4 each such cyclic list of labels begins with its least element, this implies property (2). Next we prove part (3). There is nothing to show for the root vertex, which represents a block $\{i_1, i_2, \ldots, i_r\}$ satisfying $i_1 = 1$. Any other black vertex represents a block $\{i_1, i_2, \ldots, i_r\}$ where $1 < i_1 < \cdots < i_r$. The edge connecting this vertex to its parent crosses the side $\{i_1, i_r\}$ because this line separates the polygon from the point 1. The label of this edge is i_r. The proof of property (3) for white vertices is similar. (Note that a white vertex can never be the root of the entire tree). A white vertex represents a dual block $\{j_1', j_2', \ldots, j_s'\}$, where $j_1 < j_2 < \cdots < j_s$. Once again the edge connecting this vertex to its parent crosses the side $\{j_1', i_s'\}$ because this line separates the polygon from the point 1.

We prove the remaining parts (4), (5) and (6) at once. Consider first a nonroot black vertex u representing the block $\{i_1, i_2, \ldots, i_r\}$ where $1 < i_1 < i_2 < \cdots < i_r$. The set of labels appearing in its subtree is $\{i_1, i_1+1, \ldots, i_r - 1\}$, because the points appearing on the counterclockwise (half open) arc $[i_1, i_r)$ in our circle representation are $i_1, i_1', (i_1+1), (i_1+1)', \ldots, i_r - 1, (i_r-1)'$. (Keep in mind that i_r is the label of the edge connecting our vertex to its parent.) Furthermore, the set $\{i_1, i_2, \ldots, i_{r-1}\}$ is used to label the edges connecting u to its children. The labels strictly between i_t and i_{t+1} (for $1 \leq t \leq r-1$) are the ones appearing in the subtree of the child v of u satisfying $\gamma(u,v) = i_t$.

For the root vertex representing the block $\{i_1, i_2, \ldots, i_r\}$ where $1 = i_1 < i_2 < \cdots < i_r$, we need to amend slightly the above reasoning. The subtree of the root is the entire tree, containing all labels. The set $\{i_1, i_2, \ldots, i_r\}$ is used to label the edges connecting the root to its children. The labels strictly between i_t and i_{t+1} (for $1 \leq t \leq r-1$) are the ones appearing in the subtree of the child v of u satisfying $\gamma(u,v) = i_t$, and the labels appearing in the subtree of the last child of the root are the larger than all other labels.

For white vertices parts (4), (5) and (6) may be shown in a similar fashion. A white vertex represents a dual block $\{j_1', j_2', \ldots, j_s'\}$ where $j_1 < j_2 < \cdots < j_s$ holds. The label of the edge connecting our white vertex to its parent is j_1', and the labels appearing in the subtree of our vertex are $\{j_1+1, \ldots, j_s\}$, because the points appearing on the counterclockwise (half open) arc $(j_1', j_s']$ in our circle representation are $j_1+1,(j_1+1)',(j_1+2),(j_1+2)',\ldots,j_s,j_s'$. The set $\{j_2,j_3\ldots,j_s\}$ is used to label the edges connecting u to its children. The labels strictly between j_{t-1} and j_t for $2\leq t\leq s$ are the ones appearing in the subtree of of the child v of u satisfying $\gamma(u,v)=j_s$. □

Proposition 5.7. *Given a plane tree with $n+1$ vertices, there is a unique way to label its edges in such a way that the conditions stated in Proposition 5.6 are satisfied.*

Proof. We prove by induction on the number of labels the following statement: given a vertex, and the set of labels on the edges of its subtree, there is at most one way to place the labels on the edges. The statement is obviously true for the empty set, and we will prove the induction step for the subtree of a black vertex u: the case of the subtree of a white vertex is completely analogous.

Assume that the set of children of u is $\{v_1,\ldots,v_k\}$, listed in the left to right order, and the label of the edge $\{v,v_t\}$ is i_t for $t=1,2,\ldots,k$. By property (2) we have $i_1<i_2<\cdots<i_k$ and by property (5) the label i_1 is the least element of the set of all labels appearing in the subtree. By property (4) the set of all labels in the subtree is a set of consecutive integers, the same holds for the subtrees of v_1, v_2, …and v_k. Combining properties (3) through (5) we obtain that the labels in the subtree of u must be listed in the following increasing order: i_1, labels in the subtree of v_1, i_2, labels in the subtree of v_2, …, i_k, labels in the subtree of v_k. (Keep in mind that each v_t is a white vertex.) There is exactly one way to partition the set of labels and assign them to the subtrees of each v_t and to the edges connecting u to its children. □

Theorem 5.8. *The operation assigning to each noncrossing partition $\pi\in\mathrm{NC}_n$ its labeled plane tree is a bijection. Furthermore the range of this operation is the set of all labeled plane trees on $n+1$ vertices that satisfy the conditions stated in Proposition 5.6.*

Proof. The operation is an injection by Proposition 5.3 and Corollary 5.5. By Proposition 5.7, no information is lost either if we remove the labels of the edges. The surjectivity now follows from the well known fact that the noncrossing partitions of $\{1,2,\ldots,n\}$ and the plane trees on $n+1$ vertices are enumerated by the same Catalan number. □

Since the degree of each vertex in our tree representations equals the number of elements of the block it represents, we obtain the following corollaries to Theorem 5.8. Following [17, Remark 4.2],we call a rooted plane tree *d-divisible* if the number of children of each vertex is divisible by d.

Corollary 5.9. *The cardinality of the poset NC_n^d is the number of rooted plane trees on $n+1$ vertices such that the degree of each vertex is congruent to 1 modulo d.*

Corollary 5.10. *The number of noncrossing partitions in NC^d_{dk+1} that contains the singleton block $\{1\}$ is the number of d-divisible rooted plane trees on $dk+1$ vertices.*

Indeed, an element of NC^d_{dk+1} contains the singleton block $\{1\}$ if and only if the root of the corresponding plane tree on $dk+2$ vertices has a single child. Such plane trees correspond bijectively to the plane trees on $dk+1$ vertices obtained by removing the root and designating its only child as the new root. Using the same idea we also obtain the following corollary.

Corollary 5.11. *The number of noncrossing partitions in* NC^d_{dk+1} *having only has blocks and dual blocks of size* 1 *and* $d+1$ *and containing the singleton block* $\{1\}$ *is the number of rooted* d*-ary trees on* $dk+1$ *vertices.*

Hence the second statement in Theorem 4.5 is a direct consequence of the well-known fact that the number of rooted d-ary trees is a Fuss–Catalan number. We may also derive the second statement of Example 4.3(a) from Corollary 5.10 by observing that the number of d-divisible rooted plane trees on $dk+1$ vertices is the same as the number of $(d+1)$-ary trees with $d+1$ nonleaf vertices. A bijective proof of this statement is outlined in [17, Remark 4.2].

Remark 5.12. Proposition 6.2.1 in [27] presents six classes of objects that are in bijection with each other. See also the historical remarks in the Notes in [27, Chapter 6]. We add a seventh class to this collection. Let S be a subset of the positive integers $\mathbb{P}$ and let n and m be two positive integers. The set of noncrossing partitions π in the set NC_n such that for all blocks B of π and the blocks C' of π' the block sizes belongs to the set $\{1\} \cup (S+1)$, the singleton block $\{1\}$ belongs to the partition π and there are $m+1$ singleton blocks total among the two partitions π and the dual π'. For instance in the case $S = \{2,3\}$, $n = 6$ and $m = 4$, the noncrossing partition $1|2|3,4,5,6$ corresponds to the tree in [27, Proposition 6.2.1, part (i)].

We conclude this section with visually highlighting the main difference between the main idea used to obtain enumerative results in [17] and in our work. Each labeled two-colored topological tree is decomposed into a pair of d-divisible plane trees in [17] by removing the *root edge* labeled 1. The roots of the two trees have opposite color. Our approach corresponds to splitting the *root vertex* v, that is, the black vertex incident to the edge labeled 1 into $\deg(v)$ copies. Thus we obtain several labeled topological trees of the same structure, whose root vertex is has degree 1. After identifying each new root vertex with its only child, we obtain a collection of $\deg(v)$ d-divisible rooted plane trees. The use of the Good inversion formula seems unavoidable with either approach: in the paper of Mühle, Nadeau and Williams [17] this work is done in a cited paper of Goulden and Jackson [11].

6. The antipode of the poset NC^d_n

Let $\mathcal{P}$ denote the linear span of all the isomorphism classes of all finite posets with that has a minimal element $\widehat{0}$ and a maximal element $\widehat{1}$. The vector space $\mathcal{P}$ forms a Hopf algebra. That is, there is a product, a coproduct, a unit, a counit and an antipode. For more details; see [3, 19]. The antipode is described by the following expression due to Schmitt [19, Theorem 6.1]:

$$S(P) = \sum_{k \geq 0} \sum_{c} (-1)^k \cdot [x_0, x_1] \cdot [x_1, x_2] \cdots [x_{k-1}, x_k], \tag{6.1}$$

where the sum is over all chains $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$ taking k steps. Observe that this is a generalization of Philip Hall's formula for the Möbius function. The disadvantage with the above sum is that it yields an expression that may contain a lot of cancellations.

A *hypergraph* G on a vertex set $\mathcal{V}$ is a collection $\mathcal{E}$ of subsets of $\mathcal{V}$ such that the subsets $E \in \mathcal{E}$ have cardinality at least 2. The subsets in $\mathcal{E}$ are called edges. A hypergraph is connected if for every pair of vertices x and y there is a sequence of edges $E_1, E_2, \ldots, E_k \in \mathcal{E}$ such that $x \in E_1$, $y \in E_k$ and the intersection $E_i \cap E_{i+1}$ is nonempty for all $1 \leq i \leq k-1$. A connected hypergraph is a hypertree if $\sum_{E \in \mathcal{E}} (|E| - 1) = |\mathcal{V}| - 1$. Note that if all the edges have cardinality 2 we have a tree.

Einziger [8, 9] proved the following result for the antipode of the noncrossing partition lattice. A proof using a sign reversing involution was given in [4].

Theorem 6.1 (Einziger)**.** *The antipode of the noncrossing partition lattice* NC_n *is given by the sum*

$$S(\mathrm{NC}_n) = \sum_{T} (-1)^{|T|} \cdot \prod_{E \in T} \mathrm{NC}_{|E|},$$

where T ranges over all noncrossing hypertrees on the set $[n]$.

We now extend this result to the noncrossing partition poset NC^d_n.

Theorem 6.2. *Let n and d be positive integers such that $n \equiv 1 \bmod d$. Then the antipode of the noncrossing partition poset* NC^d_n *is given by the sum*

$$S(\mathrm{NC}^d_n) = \sum_{T} (-1)^{|T|} \cdot \prod_{E \in T} \mathrm{NC}^d_{|E|},$$

where T ranges over all noncrossing hypertrees on the set $[n]$ such that each edge E of T satisfies $|E| \equiv 1 \bmod d$.

Recall that in subsection 2.1 for a block B of a partition π and an adjacent block C' in the dual partition π', we defined the notion $\gamma(B, C')$ to be the vertex of B adjacent to the block C' that is the most negative orientation from the block B's perspective.

Next, for two noncrossing partitions π and σ in in NC^d_n such that $\pi < \sigma$ define the hypergraph $\varphi(\pi, \sigma)$ as follows. For each block C' in the dual partition and a maximal collection of blocks $B_1, B_2, \ldots, B_j$ of π, where $j \geq 2$, which are adjacent to the block C' and are all contained in one block of the partition σ, add the edge $\{\gamma(B_1, C'), \gamma(B_2, C'), \ldots, \gamma(B_j, C')\}$ to the hypergraph $\varphi(\pi, \sigma)$. It is important to note that j satisfies the congruence $j \equiv 1 \bmod d$.

For a chain $c = \{\widehat{0} = \pi_0 < \pi_1 < \cdots < \pi_r = \widehat{1}\}$ in the noncrossing partition poset NC^d_n define the hypergraph $\varphi(c)$ to be the union $\cup_{i=1}^{r} \varphi(\pi_{i-1}, \pi_i)$.

Similar to [4, Lemma 3.3 and Corollary 3.5] we have the next result whose proof we omit.

Lemma 6.3. *For a chain c in the poset* NC^d_n *the isomorphism holds*

$$\prod_{i=1}^{r} [\pi_{i-1}, \pi_i] \cong \prod_{E \in \varphi(c)} \mathrm{NC}^d_{|E|}.$$

Finally, their Lemma 3.6 becomes:

Lemma 6.4. *For a chain c in the poset NC_n^d the hypergraph $\varphi(c)$ is a noncrossing hypertree where each edge E satisfies $|E| \equiv 1 \bmod d$.*

The only difference in the proof of this lemma is the factor of d in the following chain of equalities:

$$\sum_{E \in \varphi(c)} (|E| - 1) = d \cdot \sum_{E \in \varphi(c)} \rho\left(\mathrm{NC}_{|E|}^d\right) = d \cdot \sum_{i=1}^{r} \rho([\pi_{i-1}, \pi_i]) = dk = n - 1.$$

Finally, Proposition 4.3 in [4] states:

Proposition 6.5 (Ehrenborg–Happ)**.** *For a noncrossing hypertree H on n elements with r edges, the following alternating sum holds:*

$$\sum_{c \in \varphi^{-1}(H)} (-1)^{\ell(c)} = (-1)^r.$$

Proof of Theorem 6.2. By combining the Schmitt expression for the antipode (6.1) and Proposition 6.5 the result follows. $\square$

By combining Example 4.3(b) and 6.2 using Philip Hall's formula for the Möbius function, we obtain:

Corollary 6.6. *Let k and d be nonnegative integers such that $d \geq 1$. Then the following holds*

$$(-1)^k \cdot \frac{1}{2dk - k + 1} \cdot \binom{2dk}{k} = \sum_T (-1)^{|T|},$$

where T ranges over all noncrossing hypertrees on the set $[dk+1]$ such that each edge E of T satisfies $|E| \equiv 1 \bmod d$.

7. Edge labeling of the poset NC_n^d

We define an edge labeling of the noncrossing partition poset NC_n^d as follows. Let $\pi \prec \sigma$ be a cover relation is the poset NC_n^d. By Corollary 3.10 there are $d+1$ blocks of π that are joined to form a block of σ. Let us denote the joined blocks by $B_1, B_2, \ldots, B_{d+1}$ and assume that the inequalities $\min(B_1) < \min(B_2) < \cdots < \min(B_{d+1})$ hold. Then let the edge label $\lambda(\pi, \sigma)$ be given by

$$\lambda(\pi, \sigma) = \max(\{i \in B_1 : i < \min(B_2)\}).$$

This edge labeling generalizes the edge labeling of NC_n introduced by Stanley [26]. For a maximal chain $\mathbf{m} = \{\widehat{0} = \pi_0 \prec \pi_1 \prec \cdots \prec \pi_k = \widehat{1}\}$ in the poset NC_n^d define the labeling of the chain $\mathbf{m}$ to be

$$\lambda(\mathbf{m}) = (\lambda(\pi_0, \pi_1), \lambda(\pi_1, \pi_2), \ldots, \lambda(\pi_{k-1}, \pi_k)).$$

Define a *d-parking function* to be a list $(a_1, a_2, \ldots, a_k)$ of positive integers such that when the list is ordered $(a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(k)})$, it satisfies the inequality $a_{(i)} \leq d \cdot (i-1) + 1$. This definition is a shift from the definitions occurring in the two papers [25, 30], where the entries are nonnegative integers.

The purpose of this section is to prove the following theorem.

Theorem 7.1. *The map of sending a maximal chain* $\mathbf{m}$ *of* NC^d_{dk+1} *to its list of labels* $\lambda(\mathbf{m})$ *is a bijection between all maximal chains of* NC^d_{dk+1} *and all* d*-parking functions of length* k*.*

The proof of this theorem is by Lemmas 7.2 through 7.6.

Lemma 7.2. *The list of labels* $\lambda(\mathbf{m})$ *of a maximal chain* $\mathbf{m}$ *in* NC^d_n *is a* d*-parking function.*

Proof. Let R be a subset of $[n] \times [n]$ defined by $(\lambda_i, h) \in R$ if and only if $\lambda_i = \lambda(\pi_i, \pi_{i+1})$ and $h \in \{\min(B_2), \min(B_3), \ldots, \min(B_{d+1})\}$ and B_1 through B_{d+1} are the blocks that are joined in the cover relation $\pi_i \prec \pi_{i+1}$ ordered by $\min(B_1) < \min(B_2) < \cdots < \min(B_{d+1})$. Observe that if $(\lambda_i, h), (\lambda_j, h) \in R$ then $i = j$ since h can only once be the minimal element of a block being joined to another block with an even smaller minimal element. Also note that $(\lambda_i, h) \in R$ implies the inequality $\lambda_i < h$.

Assume now that the list $\lambda(\mathbf{m})$ has r elements ℓ_1 through ℓ_r that are greater than or equal to s. Then we have the following block of size $d \cdot r$:

$$\{h \ : \ \exists i \, (\ell_i, h) \in R\} \subseteq \{s+1, s+2, \ldots, n\} = [n] - [s].$$

Hence we have the inequality $d \cdot r \leq n - s = dk + 1 - s$. In other words, r is bounded above by $k - \lceil (s-1)/d \rceil$. This bound is equivalent to the definition of a d-parking function. $\square$

Lemma 7.3. *Let* $\mathbf{m} = \{\widehat{0} = \pi_0 \prec \pi_1 \prec \cdots \prec \pi_k = \widehat{1}\}$ *be a maximal chain in* NC^d_{dk+1} *and let* $\lambda(\mathbf{m}) = (a_1, a_2, \ldots, a_k)$*. Let* r *be the largest label occurring among these labels, that is,* $r = \max(a_1, a_2, \ldots, a_k)$*. Furthermore, let* s *be the last position this label occurs, that is,* $s = \max(\{i : a_i = r\})$*. Then the partition* π_{s-1} *contains the singleton blocks* $\{r+1\}, \{r+2\}, \ldots, \{r+d\}$*.*

Proof. We claim that for $0 \leq e \leq d$ that the partition π_{s-1} contains the singleton blocks $\{r+1\}, \{r+2\}, \ldots, \{r+e\}$. We prove this by induction on e. The basis case is $e = 0$ which is directly true.

Assume now that the statement is true for $0 \leq e \leq d-1$ and we prove it for $e+1$. Let B be the block of π_{s-1} that contains the element $r+e+1$. We claim that the block B does not contain the element r. If $e = 0$ note that $\{r, r+1\} \subseteq B$ contradicts that $\lambda(\pi_{s-1}, \pi_s) = r$. If $e \geq 1$ then we have a block in the dual partition whose cardinality lies strictly between 1 and $d+1$, which leads to a contradiction. Hence we may assume that r and $r+e+1$ lie in different blocks, proving the claim. If $r+e+1$ is the smallest element in the block B, then this contradicts that the largest label of the maximal chain is r. Hence assume there is an element t such that $1 \leq t < r$ in the block B. Let B_1 be the block of π_{s-1} that contains the element r. Since $\lambda(\pi_{s-1}, \pi_s) = r$ we know that B_1 is joined with d other blocks B_2 through B_{d+1} to obtain the partition π_s. These other blocks must have their minimal elements greater than r. By the noncrossing property, the only such blocks are the singletons $\{r+1\}$ through $\{r+e\}$. But there are fewer than d of these blocks. Hence there is no such element t and we conclude that the element $r+e+1$ forms a singleton block, proving the induction step. $\square$

Lemma 7.4. *With the same notation as in Lemma 7.3, the partition π_s is obtained from the partition π_{s-1} by joining the block B_1 containing the element r with the d singleton blocks $\{r+1\}$ through $\{r+d\}$.*

Proof. Since $\lambda(\pi_{s-1}, \pi_s) = r$, the block B_1 is joined by d blocks whose elements are all greater than r. Assume that h of these blocks are from the singleton blocks $\{r+1\}$ through $\{r+d\}$. We would to establish that $h = d$. If $1 \leq h \leq d-1$ then the dual partition of π_s has a block of size strictly between 2 and d, yielding a contradiction. Hence the case $h = 0$ remains. Let j be the next smallest element in the the block of π_s that contains r. That is, j is strictly greater than $r + d$. By the noncrossing property, what happens to the elements $r+1$ through $r+j-1$ is independent of what happens outside this interval. At some point further up the maximal chain the block containing the element $r+1$ must join the block containing the element r. This yields a label of r in the maximal chain, contradicting that s was chosen to be maximal. Hence $h = d$ and the lemma follows. □

Lemma 7.5. *Given a d-parking function $\vec{a} = (a_1, a_2, \ldots, a_k)$ we can reconstruct a maximal chain $\mathbf{m}$ in NC^d_{dk+1} such that $\lambda(\mathbf{m}) = \vec{a}$.*

Proof. We proceed by induction on k. The base case is $k = 0$ which is direct. As in Lemma 7.3 let $r = \max(a_1, a_2, \ldots, a_k)$ and $s = \max(\{i : a_i = r\})$. Observe that

$$\vec{b} = (b_1, b_2, \ldots, b_{k-1}) = (a_1, a_2, \ldots, a_{s-1}, a_{s+1}, \ldots, a_k)$$

is a d-parking function. By induction we have a maximal chain $\mathbf{n} = \{\widehat{0} = \sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_{k-1} = \widehat{1}\}$ in $\mathrm{NC}^d_{d(k-1)+1}$ such that $\lambda(\mathbf{n}) = \vec{b}$. Let $f : [d(k-1)+1] \longrightarrow [dk+1]$ be the following relabeling function: we set

$$f(j) = \begin{cases} j & \text{if } 1 \leq j \leq r, \\ j + d & \text{if } r+1 \leq j \leq d(k-1)+1. \end{cases} \tag{7.1}$$

Define the maximal chain $\mathbf{m} = \{\widehat{0} = \pi_0 \prec \pi_1 \prec \cdots \prec \pi_k = \widehat{1}\}$ by

$$\pi_i = \{f(B) \ : \ B \in \sigma_i\} \cup \{\{r+1\}, \ldots, \{r+d\}\}$$

for $0 \leq i \leq s-1$ and

$$\pi_i = \{f(B) \ : \ r \notin B \in \sigma_{i-1}\} \cup \{f(B) \cup \{r+1, \ldots, r+d\} \ : \ r \in B \in \sigma_{i-1}\}$$

for $s \leq i \leq k$. Note that the relabeling function f opens up for an interval of d new elements, that is, $r+1$ through $r+d$. Let a and b be two elements in the cyclic order on $[d(k-1)+1]$. Then the number elements between a and b in the cyclic order is congruent to the number of elements between $f(a)$ and $f(b)$ in the cyclic order on $[dk+1]$ modulo d. Hence by Theorem 3.3 we know that the partition π_i belongs to the poset NC^d_{dk+1} for all i. We also obtain $\lambda(\mathbf{m}) = \vec{a}$. □

Combining Lemmas 7.3 through 7.5 we have proven the next result.

Lemma 7.6. *For two maximal chains $\mathbf{m}$ and $\mathbf{m}'$ in the noncrossing partition poset NC^d_{dk+1} the equality $\lambda(\mathbf{m}) = \lambda(\mathbf{m}')$ implies $\mathbf{m} = \mathbf{m}'$.*

Observe now that Lemmas 7.2 through 7.6 prove Theorem 7.1. Next we show that the order complex of the poset NC_n^d is shellable.

Proposition 7.7. *The labeling $\lambda^*(\pi,\sigma) = |\pi| - \lambda(\pi,\sigma)$ is an EL-labeling of the noncrossing partition poset NC_n^d.*

Proof. We first show that the poset NC_n^d has a unique rising chain. Consider a rising maximal chain

$$\mathbf{m} = \{\widehat{0} = \pi_0 \prec \pi_1 \prec \cdots \prec \pi_k = \widehat{1}\}.$$

Since $\mathbf{m}$ is a maximal chain we have $|\pi_i| = |\pi_{i+1}| + d$ for $0 \leq i \leq k-1$ using Lemma 3.9. Hence the rising condition $\lambda^*(\pi_{i-1},\pi_i) \leq \lambda^*(\pi_i,\pi_{i+1})$ implies $\lambda(\pi_{i-1},\pi_i) \geq \lambda(\pi_i,\pi_{i+1}) + d$. As a consequence the d-parking function $\vec{a} = (a_1, a_2, \ldots, a_k)$ associated to $\mathbf{m}$ must satisfy

$$(a_1, a_2, \ldots, a_k) \geq ((k-1)d+1, (k-2)d+1, \ldots, d+1, 1)$$

coordinate-wise. Rearranging the entries of $\vec{a}$ into increasing order, the resulting vector satisfies

$$(a_{(1)}, a_{(2)}, \ldots, a_{(k)}) \geq (1, d+1, \ldots, (k-1)d+1)$$

coordinate-wise. By the definition of a d-parking function the above lower bound for $\vec{a}$ is also an upper bound. That is, equality holds. Hence the only rising chain $\mathbf{m}$ is the one associated to the d-parking function $((k-1)d+1, (k-2)d+1, \ldots, d+1, 1)$.

Next we show that the unique rising chain of NC_n^d described above is also lexicographically first. Consider any other maximal chain $\mathbf{n} = \{\widehat{0} = \sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_k = \widehat{1}\}$ associated to the d-parking function $\vec{b} = (b_1, b_2, \ldots, b_k)$. Let i be the least index such that $a_i \neq b_i$ holds. As a consequence of $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_{i-1} = b_{i-1}$, the first $i-1$ coordinates of $\vec{b}$ are also its $i-1$ largest coordinates, and b_i must satisfy $b_i \leq (k-i+1)d$. Hence $\lambda^*(\sigma_i,\sigma_{i+1}) > \lambda^*(\pi_i,\pi_{i+1})$ must be satisfied, while $\lambda^*(\sigma_j,\sigma_{j+1}) = \lambda^*(\pi_j,\pi_{j+1})$ holds for all $j < i$. Therefore $\mathbf{m}$ precedes $\mathbf{n}$ in the lexicographic order.

As noted by Stanley [26] about the $d = 1$ case, the statement is easily generalized to an arbitrary interval of NC_n^d using the fact that any such interval is a direct product of smaller copies of $\mathrm{NC}_{n_i}^d$ □

Corollary 7.8. *The number of d-parking functions $(a_1, a_2, \ldots, a_k)$ such that $a_i \leq a_{i+1} + d - 1$ for all indices i is given by $\binom{2dk}{k}/(2dk-k+1)$.*

Proof. This result follows from the formula for the Möbius function of the poset NC_{dk+1}^d stated in Example 4.3(b) and Proposition 7.7. Recall that the Möbius function times the sign $(-1)^k$ enumerates the number of falling chains. □

8. A tree representation of the maximal chains in NC_n^d

Parsing a maximal chain of NC_{dk+1}^d amounts to describing a process each step of which consists of merging $d+1$ blocks of a partition. In this section we describe a tree representation which offers a visual record of this process.

Definition 8.1. *A d-*parking tree *is a labeled rooted plane tree on $n = dk + 1$ vertices, such that the number of children of each vertex is a multiple of d and the labeling satisfies the following conditions:*

(1) *The label of the root is ∞.*
(2) *The label of any other vertex is of the form i_j where $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, d\}$.*
(3) *For each fixed i, the vertices labeled $i_1, \dots, i_d$ are consecutive labels in the left-to-right order of the same parent.*
(4) *Each $i \in \{1, 2, \dots, k\}$ is only used to label one d-element set of siblings.*
(5) *If i_j and $i'_{j'}$ are children of the same vertex and $i < i'$ holds then the vertex labeled i_j is to the* right *of the vertex labeled $i'_{j'}$.*

Note that for $d = 1$ our definition yields a labeled rooted plane tree with no restriction on the degrees of the vertices. A 2-parking tree is shown on the left hand side of Figure 3. We consider d-parking trees as labeled trees with *distinct vertices.* To identify the vertices in the tree, in this paper we rely on the *depth-first search ordering using the interval* $[1, n]$. For a tree T and a vertex u in the tree T, let $T(u)$ be the subtree that has the node u as a root, that is, $T(u)$ consists of all vertices that are descendants of u.

Definition 8.2. *Given a plane tree T on n vertices and a positive integer a, the* depth-first search ordering of T using the interval $[a, a + n - 1]$ *is defined by the following procedure:*

(1) *We label the root r of the tree T with the number a, that is, we set $\omega(r) = a$.*
(2) *Suppose $r_1, r_2, \dots, r_s$ are the children of the root r in the left-to-right order and let T_i be the subtree $T(r_i)$ for $i = 1, 2, \dots, s$. We label each subtree T_i recursively with depth-first search ordering using the interval $[a + |T_1| + \cdots + |T_{i-1}| + 1, a + |T_1| + \cdots + |T_i|]$.*

The right hand side of Figure 3 shows the *depth-first search ordering* of the vertices.

Remark 8.3. Essentially the same d-parking trees (the mirror images of the present ones) were also defined in [13, Section 6] as part of a different model that was used to encode *Athanasiadis-Linusson diagrams.* The key difference between the two models is that a breadth-first search ordering was used to identify the vertices in [13, Section 6].

Recall there is a bijection between d-parking functions of length k and maximal chains in the noncrossing partition poset NC^d_{dk+1}; see Theorem 7.1. We now include d-parking trees in this bijective family. We begin with a key observation, that is applicable in all situations when we identify the vertices of a plane tree using an *increasing ordering.*

Definition 8.4. *Given a plane tree T on n vertices an* increasing ordering *of the vertices of T is a bijection ω from the vertex set of T to the set $\{1, 2, \dots, n\}$ such that for every cover relation $u \prec v$ in the tree, that is, u is the parent of v, the inequality $\omega(u) < \omega(v)$ holds.*

Clearly, the depth-first search ordering and any breadth-first search are increasing orderings.

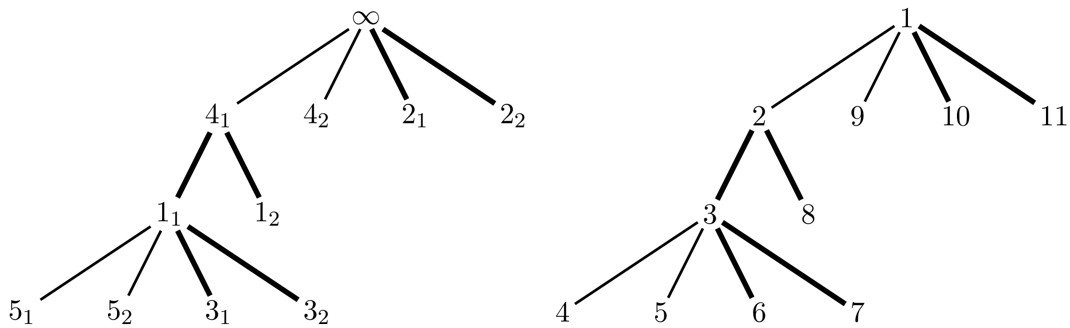

Figure 3. A 2-parking tree and the associated depth-first search ordering. This example corresponds to the 2-parking function $(2, 1, 3, 1, 3)$.

Lemma 8.5. *Let T be a d-parking tree on $n = dk + 1$ vertices and let ω be an increasing ordering of its vertices. For all $1 \leq i \leq k$ we define a_i as the label $\omega(p)$ of the common parent of the vertices $\omega(i_1), \omega(i_2), \ldots, \omega(i_d)$. Then the resulting vector $\vec{a}_\omega(T) = \vec{a} = (a_1, a_2, \ldots, a_k)$ is a d-parking function.*

Proof. Note that the number j appears in the list $\vec{a}$ if and only if the vertex $\omega^{-1}(j)$ is not a leaf. If $\omega^{-1}(j)$ has $d \cdot c(j)$ children then j appears in the list $\vec{a}$ exactly $c(j)$ times. Consider the ordered list $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(k)}$ obtained from $\vec{a}$. Let us relabel the vertices of the d-parking tree in such a way that the order $(1_1, 1_2, \ldots, 1_d, \ldots, k_1, k_2, \ldots, k_d)$ corresponds to listing all children of the vertex $\omega^{-1}(a_{(1)})$, then of $\omega^{-1}(a_{(2)})$, and so on, finally of $\omega^{-1}(a_{(k)})$ in the left-to-right order. In the case when $a_{(i)} = a_{(i+1)}$ we list all children only at once, but we list them in increasing order of the value of ω, and we will consider the first d children associated to the first copy, the next d children associated to the second copy, and so on.

Let us call the resulting d-parking tree the *straightened d-parking tree with respect to ω*. The straightened 2-parking tree with respect to the depth-first search order, obtained from the 2-parking tree shown in Figure 3 is represented in Figure 4. For $1 \leq i \leq k$ consider the list of children associated

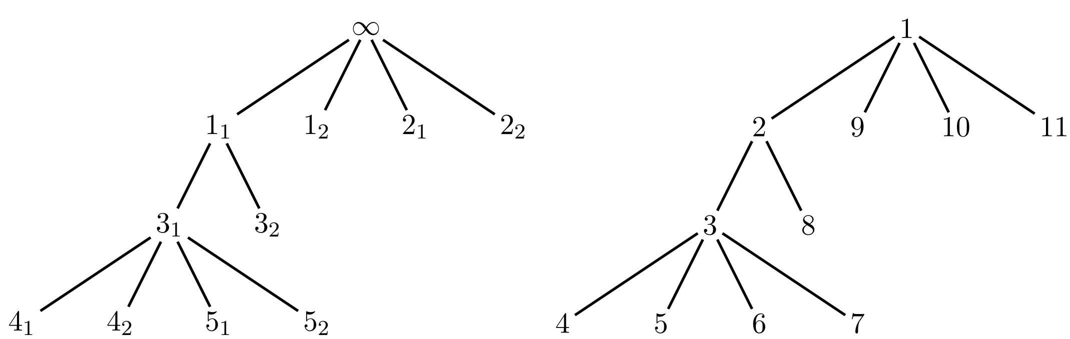

Figure 4. The straightened of the 2-parking tree in Figure 3 with respect to the depth-first search. This example corresponds to the 2-parking function $(1, 1, 2, 3, 3)$.

to the parents in the initial segment $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(i-1)}$. In the straightened d-parking tree this is exactly the list of the first $d \cdot (i-1)$ entries of the list $(1_1, 1_2, \ldots, 1_d, \ldots, k_1, k_2, \ldots, k_d)$. Observe that $a_{(i)}$ is exactly the value of $\omega(p)$ for the common parent of $i_1, i_2, \ldots, i_d$ in the straightened parking tree. Consider any nonroot-vertex v that is not yet listed. The parent $p(v)$ of v satisfies $\omega(p(v)) \geq a_{(i)}$. Since the ordering ω is increasing on subtrees we must also have $\omega(v) > a_{(i)}$. We obtained that the label of any non-yet listed nonroot vertex is greater than the label $a_{(i)}$. There are $n - (d(i-1)+1)$ such vertices which forces $a_{(i)} \leq d(i-1)+1$. □

Every d-parking tree may be turned into a 1-parking tree, using the following simple relabeling.

Definition 8.6. *Let T be a d-parking tree on $n = dk+1$ vertices. We call the* expansion of T *the 1-parking tree obtained by replacing each label i_j with $(i-1)d+j$ for $1 \leq i \leq k$ and $1 \leq j \leq d$.*

The following statement is a direct consequence of Lemma 8.5 and Definition 8.6.

Corollary 8.7. *Let T be a d-parking tree on $n = dk+1$ vertices, let ω be an increasing ordering of its vertices and let $\vec{a}_\omega(T) = \vec{a} = (a_1, \ldots, a_k)$ be the d-parking function associated to T ordered by ω in Lemma 8.5. Let U be the expansion of the tree T. Then the parking function $\vec{b} = \vec{a}_\omega(U)$ is the vector obtained by replacing each entry a_i in $\vec{a}$ by a list of d consecutive copies of a_i.*

Example 8.8. The 2-parking tree shown in Figure 3 corresponds to the 2-parking function $\vec{a} = (2,1,3,1,3)$ with respect to he depth-first search ordering. Its expansion corresponds to the parking function $\vec{b} = (2,2,1,1,3,3,1,1,3,3)$. The fourth coordinate of $\vec{a}$ is 1 because the common parent ∞ of 4_1 and 4_2 satisfies $\omega(\infty) = 1$. In the expansion 4_1 becomes $3 \cdot 2 + 1 = 7$ and 4_2 becomes $3 \cdot 2 + 2 = 8$. The seventh and eighth coordinates of $\vec{b}$ are both 1.

In the remainder of the section we assume that the increasing ordering ω is the depth-first search ordering. In this case we will write $\vec{a}(T)$ for $\vec{a}_\omega(T)$ and call $\vec{a}(T)$ *the d-parking function associated to the d-parking tree T.*

Theorem 8.9. *The map $T \longmapsto \vec{a}(T)$ from d-parking trees on $n = dk+1$ vertices to the d-parking functions of length k is a bijection.*

Proof. It suffices to show that given a d-parking function $\vec{a} = (a_1, a_2, \ldots, a_k)$ there is exactly one d-parking tree T on $n = dk+1$ whose associated d-parking function is $\vec{a}$.

Observe first that we may restrict our attention to straightened d-parking trees (defined in the proof of Lemma 8.5) and d-parking functions $\vec{a} = (a_1, a_2, \ldots, a_k)$ satisfying $a_1 \leq a_2 \leq \cdots \leq a_k$. Indeed, given any permutation π on the set $\{1, 2, \ldots, k\}$, let T_π be the d-parking tree obtained from the d-parking tree T by replacing the labels $i_1, i_2, \ldots, i_d$ with $\pi(i)_1, \pi(i)_2, \ldots, \pi(i)_d$ for each $i \in \{1, 2, \ldots, k\}$. If $\vec{a} = (a_1, a_2, \ldots, a_k)$ is the d-parking function associated to T then $\vec{a}_\pi = (a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(k)})$ is the d-parking function associated to T_π. In other words, once we prove the statement for straightened d-parking trees and d-parking functions $\vec{a} = (a_1, a_2, \ldots, a_k)$ satisfying $a_1 \leq a_2 \leq \cdots \leq a_k$, the general statement may be obtained by permuting the labels.

Consider now a d-parking function $\vec{a} = (a_1, a_2, \ldots, a_k)$ satisfying $a_1 \leq a_2 \leq \cdots \leq a_k$. As in the proof of Lemma 8.5 let $c(j)$ be the number of times the number j occurs in $\vec{a}$. Let $\{j_1 < j_2 < \cdots < j_m\}$ be the set of integers satisfying $c(j) \geq 1$. Since $a_1 = 1$, we must have $c(1) \geq 1$, $j_1 = 1$, and the straightened d-parking tree T giving rise to $\vec{a}$ must have exactly $d \cdot c(1)$ children: $1_1, 1_2, \ldots, 1_d, \ldots, c(1)_1, c(1)_2, \ldots, c(1)_d$. The numbers $2, 3, \ldots, j_2 - 1$ do not appear in $\vec{a}$, they must be numbers of leaves in the depth-first search ordering, and these $j_2 - 2$ leaves must be the leftmost $j_2 - 2$ children of the root. The vertex j_2 in the depth-first search order is then child number $j_2 - 1$ of the root. This assignment is possible if and only if

$$j_2 \leq d \cdot c(1) + 1 \tag{8.1}$$

holds, as this is the number of children of the root. Observe also that we have $a_1 = a_2 = \cdots = a_{c(1)} = 1$ and j_2 is the common value of $a_{c(1)+1} = a_{c(1)+2} = \cdots = a_{c(1)+c(j_2)}$. The inequality (8.1) is equivalent to $a_{c(1)+1} \leq d \cdot c(1) + 1$ which is the d-parking condition for $a_{c(1)+1}$. (It is worth noting that the parking conditions for $a_2 = \cdots = a_{c(1)} = 1$ are automatically satisfied, and they follow from (8.1) for $a_{c(1)+2} = \cdots = a_{c(1)+c(j_2)}$.)

We proceed now by induction of m, the number of distinct values listed in $\vec{a}$. Assume that we have already found a unique straightened parking tree T' on $d \cdot (c(j_1) + c(j_2) + \cdots + c(j_{m-1})) + 1$ vertices whose associated d-parking function is $(a_1, a_2, \ldots, a_{c(j_1)+c(j_2)+\cdots+c(j_{m-1})})$. By our construction, v is a nonleaf vertex of T' if and only if $\omega(v)$ belongs to the set $\{j_1, j_2, \ldots, j_{m-1}\}$. Now we want to add $d \cdot c(j_m)$ children to the vertex labeled j_m in T', to obtain the tree T. In the depth-first search ordering this will increase the value of ω by $d \cdot c(j_m)$ for the vertices v of T that satisfy $\omega(v) > j_m$, but by $j_{m-1} < j_m$ all these vertices are leaves, and the d-parking function associated to T will agree with the one associated to T' in the first $c(j_1) + c(j_2) + \cdots + c(j_{m-1})$ coordinates. The insertion of children is possible if and only if

$$j_{m-1} < j_m \leq d \cdot (c(j_1) + c(j_2) + \cdots + c(j_{m-1})) + 1 \tag{8.2}$$

holds. The upper bound is equivalent to

$$a_{c(j_1)+c(j_2)+\cdots+c(j_{m-1})+1} \leq d \cdot (c(j_1) + c(j_2) + \cdots + c(j_{m-1})) + 1,$$

the d-parking condition for $a_{c(j_1)+c(j_2)+\cdots+c(j_{m-1})+1}$. If this condition is satisfied, the weaker conditions for $a_{c(j_1)+c(j_2)+\cdots+c(j_{m-1})+1}, \ldots, a_{c(j_1)+c(j_2)+\cdots+c(j_m)}$ are also satisfied. The inequality

$$j_{m-1} < d \cdot (c(j_1) + c(j_2) + \cdots + c(j_{m-1})) + 1$$

needed to have at least one choice for j_m is also a direct consequence of the d-parking condition for $a_{c(j_1)+c(j_2)+\cdots+c(j_{m-2})+1}$. □

Theorem 7.1 establishes a bijection between the set of all maximal chains in NC^d_{dk+1} and the set of all d-parking functions of length k. Combining this bijection with Theorem 8.9 above we obtain a bijection between the set of all maximal chains in NC^d_{dk+1} and the set of all d-parking trees on $n = dk + 1$ vertices. The correspondence is visually straightforward, as exhibited in the following example.

Example 8.10. Consider the 2-parking function $(a_1, \ldots, a_5) = (2, 1, 3, 1, 3)$, corresponding to the 2-parking tree in Figure 3. Note that 5_1 and 5_2 has the node 1_1 as their parent and the depth-first search label of 1_1 is $\omega(1_1) = 3 = a_5$. Furthermore, this 2-parking function corresponds to the following maximal chain in the noncrossing partition poset NC^2_{11}:

$$\begin{aligned} \widehat{0} &\prec 1|2,3,8|4|5|6|7|9|10|11 \prec 1,10,11|2,3,8|4|5|6|7|9 \\ &\prec 1,10,11|2,3,6,7,8|4|5|9 \prec 1,2,3,6,7,8,9,10,11|4|5 \prec \widehat{1}. \end{aligned}$$

In the 3rd step we join together the blocks containing the elements $\omega(1_1) = 3$, $\omega(3_1) = 6$ and $\omega(3_2) = 7$. Note that the node 1_1 is the parent of 3_1 and 3_2. This relation is explained in the next theorem.

Theorem 8.11. *Let* $\mathbf{m} = \{\widehat{0} = \pi_0 \prec \pi_1 \prec \cdots \prec \pi_k = \widehat{1}\}$ *be a maximal chain in* NC^d_{dk+1} *and let* T *be the* d*-parking tree such that* $\lambda(\mathbf{m}) = \vec{a} = \vec{a}(T)$*. Then* π_i *is the noncrossing partition whose blocks consist of the connected components of the graph containing all edges that connect a vertex* j_s *to its parent for some* $1 \leq j \leq i$ *and some* $1 \leq s \leq d$*. In particular, the cover relation from the partition* π_{i-1} *to the partition* π_i *is obtained by joining the blocks containing the* $d+1$ *elements* $\omega(p_i), \omega(i_1), \omega(i_2), \ldots, \omega(i_d)$*, where* p_i *is the common parent of* i_1 *through* i_d*.*

Proof. We proceed by induction on k. When $k = 0$ there is is nothing to prove completing the induction basis. Assume now that the result is true for $k-1$ and we prove it for k.

Let r and s be defined as in Lemma 7.3. That is, r is the largest entry in the k-parking function $\vec{a} = (a_1, a_2, \ldots, a_k)$ and s is the last entry such that $a_s = r$. Lemma 7.3 implies that the partition π_{s-1} contains the singleton blocks $\{r+1\}, \{r+2\}, \ldots, \{r+d\}$. As in Lemma 7.5 let $\vec{b} = (b_1, b_2, \ldots, b_{k-1}) = (a_1, a_2, \ldots, a_{s-1}, a_{s+1}, \ldots, a_k)$ be the d-parking function where we remove the a_s entry. Let U be the d-parking tree corresponding to the d-parking function $\vec{b}$, that is, we define U by $\vec{a}(U) = \vec{b}$.

Construct a new tree T^* by the following two steps:

(1) Relabel the nodes i_j in the tree U where $i \geq s$ to be $(i+1)_j$. Note that after the relabeling, there are no nodes with the labels s_1 through s_d.
(2) Let p be the node in the tree U such that $\omega_U(p) = r$. Attach the new leaves s_1 through s_d to the node p such that they are the d right-most children of the node p. Note that these nodes in the depth-first search labeling receive the labels $r+1$ through $r+d$, that is, $\omega_{T^*}(s_j) = r+j$ for $1 \leq j \leq d$.

The node x in the tree U satisfies $f(\omega_U(x)) = \omega_{T^*}(x)$ where f is the relabeling function defined in (7.1) Hence $\vec{a}(T^*) = \vec{a}$ holds. Since the correspondence between d-parking trees and d-parking functions is a bijection, we conclude that T^* is the tree T.

Consider the cover relation $\pi_{i-1} \prec \pi_i$ in the chain $\mathbf{m}$. By the induction hypothesis when $i \neq s$ the partition π_i is obtained from π_{i-1} by joining the blocks containing the elements $f(\omega_U(p_i))$, $f(\omega_U(i_1))$, $f(\omega_U(i_2))$, …, $f(\omega_U(i_d))$. But these are exactly the elements $\omega_T(p_i), \omega_T(i_1), \omega_T(i_2), \ldots, \omega_T(i_d)$. Finally, when i is s by the above construction we are joining $\omega_T(p), \omega_T(s_1), \omega_T(s_2), \ldots, \omega_T(s_d)$ which are the elements r through $r+d$, completing the induction. ☐

Remark 8.12. In [26] Stanley mentions (about the case $d = 1$ of all noncrossing partitions) that “The above proof of the injectivity of the map Λ from maximal chains to parking functions is reminiscent of the proof [20, p. 5] that the Prüfer code of a labelled tree determines the tree." It is a direct consequence of the proof of [13, Theorem 6.10] (which uses a homogenized variant of the Prüfer code algorithm) that the number of d-parking trees is n^k.

It is an immediate consequence of the definitions that for a d-parking function $\vec{a} = (a_1, a_2, \ldots, a_k)$ and a permutation τ of the set $\{1, 2, \ldots, k\}$ the vector $\tau(\vec{a}) = (a_{\tau(1)}, a_{\tau(2)}, \ldots, a_{\tau(k)})$ is also a d-parking function. Since d-parking functions bijectively label the maximal chains of NC^d_{dk+1}, the above action of the symmetric group $\mathfrak{S}_k$ on the set of d-parking functions of length k induces an action of the same group on the maximal chains of NC^d_{dk+1}. As a direct consequence of Theorem 8.11 we obtain the following statement.

Corollary 8.13. *Two maximal chains of NC^d_{dk+1} are in the same orbit of the above described action of the symmetric group if and only if the removal of the labels on the d-parking trees associated to them yields the same plane tree.*

Theorem 8.11 and Corollary 8.13 have the following consequence.

Corollary 8.14. *Let $n = dk+1$. If a partition π of $[n]$ belongs to the noncrossing partition poset NC^d_n then there is a d-parking tree T and an $s \in \{0, 1, \ldots, k\}$ such that the blocks of π are the connected components of the graph obtained from the tree T by deleting all the edges connecting each vertex i_j to its parent for all $i > s$ and $j \in \{1, 2, \ldots, d\}$. Conversely, if there is a d-parking tree T and a subset S of $\{1, 2, \ldots, k\}$, such that the connected components of the graph obtained from the tree T by deleting all the edges connecting each vertex i_j to its parent for all $i \in S$ and $j \in \{1, 2, \ldots, d\}$ are the blocks of π, then π belongs to NC^d_n.*

For $d = 1$ Corollary 8.14 yields the following statement.

Corollary 8.15. *A partition π of $[n]$ is a noncrossing partition if and only if its blocks are the connected components of a graph obtained from a plane tree ordered by the depth-first search order after deleting an arbitrary subset of its edges.*

9. Concluding remarks

As it was first pointed out in [17], many of the enumerative results are related to the Raney numbers. Some of these results already have a combinatorial proof, but to count the elements of a fixed rank seems impossible without some use of the Good inversion formula at this time. Is there a combinatorial proof of this result? Also, is there a combinatorial proof for Corollary 7.8?

The quasisymmetric function of a graded poset encodes all of the flag f-vector information of the poset; see [3]. Stanley observed that if every interval of a poset P is self-dual then the quasisymmetric function of P is a symmetric function; see [24, Theorem 1.4]. Hence the noncrossing partition poset NC^d_n has a symmetric quasisymmetric function. In the paper [26] Stanley explores the quasisymmetric function of the noncrossing partition lattice NC_n. Are there similar results for the d-indivisible noncrossing partition poset NC^d_n?

The set of d-parking functions are known to be in bijection with the regions of the extended Shi-arrangement via the Pak-Stanley labeling [24, 2.1 Theorem]. The same d-parking functions also label the maximal chains of the poset NC_n^d. Is there a geometric way to directly connect the poset NC_n^d with the extended Shi arrangement?

Einziger explored the Hopf algebra structure of the Hopf subalgebra of $\mathcal{P}$ generated by the noncrossing partition lattice NC_n; see [8, 9]. Would the Hopf subalgebra generated by NC_n^d, for a fixed d, have a similar structure?

Noncrossing partitions have connections with free probability. Does the subposet NC_n^d of the lattice NC_n have a similar connection?

There are noncrossing partitions for other Coxeter systems; see [1, 18], a type B analogue of the d-indivisible noncrossing partitions has been proposed by Mühle, Nadeau and Williams [17]. The authors are currently developing the analogous results for noncrossing partitions of these types.

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