PARKING TREES AND THE TORIC G-VECTOR OF NESTOHEDRA

RICHARD EHRENBORG, GÁBOR HETYEI AND MARGARET READDY

Abstract. We express the toric g-vector entries of any simple polytope as a nonnegative integer linear combination of its γ-vector entries. Using this expression we obtain that the toric g-vector of the associahedron is the ascent statistic of 123-avoiding parking functions. An analogous result holds for the cyclohedron and 123-avoiding functions. We prove that the toric g-vector of the permutahedron records the ascent statistics of parking trees representing 123-avoiding parking functions. We indicate how our approach extends to all chordal nestohedra.

1. Introduction

In his book, *Enumerative Combinatorics,* Richard Stanley wrote that the toric g-vector of an Eulerian poset is "an exceedingly subtle invariant"; see [34, Section 3.16]. As an example, he computed the toric g-vector of the permutahedron up to dimension 5; see [33, Page 195]. In the present paper we not only give a closed form formula for the toric g-vector of this polytope and of several other simple polytopes, but we also find a combinatorial interpretation in each case. The key result making such computations and interpretations possible is Theorem 3.4. It expresses the entries of the toric g-vector of a simple polytopes as nonnegative integer linear combinations of its γ-vector entries. This result naturally inspires us to revisit combinatorial models for the γ-vectors of classical simple polytopes.

The n-dimensional cube is the most basic example of a simple polytope, where the γ-vector is straightforward and the toric g-vector is well known; see Corollaries 4.9 and 4.10. Our approach offers new enumerative results. We give a combinatorial interpretation of the toric g-vector of the associahedron in Theorem 6.3, stating that the kth entry is given by the number of 123-avoiding parking functions $f : [n] \longrightarrow [n]$ that have exactly k ascents. The γ-vector of the associahedron is discussed alongside that of the permutahedron and of the cyclohedron in the work of Postnikov, Reiner and Williams [25]. As in their work on the γ-vector, our computation of the toric g-vector of the cyclohedron is analogous to that of the associahedron. For the cyclohedron we replace the set of 123-avoiding parking functions with the set of 123-avoiding functions $f : [n] \longrightarrow [n]$.

Postnikov, Reiner and Williams [25] expressed the γ-vector of a permutahedron in terms of the peak statistics of all permutations containing no double descents and no final descent. Our key observation is that their result may be translated into a result on the peak statistics of all plane 0-1-2 trees

Date: November 6, 2025.

2000 *Mathematics Subject Classification.* Primary 52B05, 52B12, 52B15, Secondary 05A15, 05E45.

Key words and phrases. Associahedron; Cyclohedron; Dyck paths; Nestohedron; Parking functions; Parking trees; Permutahedron.

having a descending labeling on the vertices. The equivalence of the two statistics may be shown using a restricted variant of the Foata–Strehl group action, first considered by Brändén [4]. Having a descending vertex labeling is also part of our definition of a *parking tree* that we introduce as a variant of the definitions given in [9, Section 8] and in [18], inspired by an exercise of Sagan [28, Ch 1. Exercise (32)(c)]. In Theorem 9.3 we express the toric g-vector of the permutahedron in terms of the peak statistics of parking trees representing 123-avoiding parking functions. This approach extends to all *chordal nestohedra*, studied by Postnikov, Reiner and Williams [25].

Our paper raises many open questions, the most important is the following. It is inspired by Theorem 3.4 and the Nevo–Petersen conjecture [23, Conjecture 1.4] stating that the γ-vector entries of a simplicial complex that is a flag homology sphere satisfy the Kruskal–Katona inequalities.

Conjecture 1.1. *When the γ-vector entries of a simple polytope satisfy the Kruskal–Katona inequalities, the same holds for its toric g-vector.*

The paper is organized as follows. In the preliminaries section we introduce two essential bijections. The first, due to Krattenthaler, is between 123-avoiding permutations and Dyck paths. The second, due to Garsia and Haiman, is between functions on the set $[n] = \{1, 2, \ldots, n\}$ and pairs of compatible permutations and lattice paths. Their important observation is that when the function is a parking function, the associated lattice path is a Dyck path. We also discuss the Foata–Strehl group action on the symmetric group.

In Section 3 we show how to compute the toric g-vector of a simple polytope in terms of its γ-vector. This computation involves the toric g-contribution polynomials $g_{n,j}(x)$ whose coefficients are described in terms of counting peaks in Dyck paths. In Section 4 we give two interpretations of the coefficients of these polynomials. One of these is in terms of ascent sets in 123-avoiding permutations.

In Section 5 we obtain generating functions for the toric g-contribution polynomials. We combine the bijections of Garsia–Haiman and Krattenthaler in Section 6 to prove our combinatorial interpretation of the toric g-vector of the associahedron. We modify our arguments to obtain a similar result for the cyclohedron in Section 7.

Parking trees are introduced in Section 8. They form the essential structure in order to obtain an explicit expression for the toric g-polynomial of the permutahedron, completing Stanley's quest for computing these polynomials; see Section 9. Finally in Section 10 we extend our results to nestohedra associated with chordal building sets. We end with open problems in Section 11.

2. Preliminaries

2.1. **Permutations.** For a non-negative integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Similarly for two integers $i \leq j$, let $[i, j]$ denote the interval $\{i, i+1, \ldots, j\}$.

Let $\mathfrak{S}_n$ denote the symmetric group on n elements. We choose the underlying set to be the set $[n]$. The index $i \in [n-1]$ is an *ascent* of a permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n)) \in \mathfrak{S}_n$ if $\pi(i) < \pi(i+1)$,

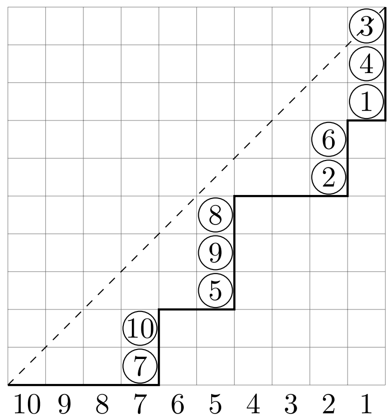

FIGURE 1. The Krattenthaler Dyck path $w = U^4D^2U^2D^3U^3D^2UD^3$ for the 123-avoiding permutation $\pi = (\underline{7}, 10, \underline{5}, 9, 8, \underline{2}, 6, \underline{1}, 4, 3)$ where the left-to-right minima are underlined.

otherwise i is a *descent*. The ascent set and descent set of a permutation π are given by

$$\mathrm{Asc}(\pi) = \{i \in [n-1] : \pi(i) < \pi(i+1)\}, \qquad \mathrm{Des}(\pi) = \{i \in [n-1] : \pi(i) > \pi(i+1)\}.$$

The number of ascents and descents are denoted by $\mathrm{asc}(\pi) = |\mathrm{Asc}(\pi)|$, respectively $\mathrm{des}(\pi) = |\mathrm{Des}(\pi)|$. Finally, a permutation $\pi \in \mathfrak{S}_n$ is 123-*avoiding* if there is no triple $1 \leq i_1 < i_2 < i_3 \leq n$ satisfying $\pi(i_1) < \pi(i_2) < \pi(i_3)$. For more on pattern avoidance, see [19].

2.2. **Lattice paths.** A *Dyck path* of semilength n is a lattice path from the origin $(0,0)$ to $(2n,0)$ composed of n up steps $(1,1)$ and n down steps $(1,-1)$ that does not go below the horizontal axis. We may encode such a Dyck path with the associated *Dyck word* by recording each up step by the letter U and each down step by the letter D. The number of Dyck paths, and hence also the number Dyck words, of semilength n is the nth Catalan number

$$C_n = \frac{1}{n+1} \cdot \binom{2n}{n}.$$

The generating function for the Catalan numbers is well-known:

$$C(t) = \sum_{n \geq 0} C_n \cdot t^n = \frac{1 - \sqrt{1-4t}}{2t}. \tag{2.1}$$

By rotating our perspective by $\pi/4$ radians counterclockwise, we may also consider paths from the origin to the point (n,n) taking North steps $(0,1)$ and East steps $(1,0)$ that never goes below the line $y = x$ as a Dyck path. Similarly, we also consider the mirror image of such a path that never goes above the line $y = x$ as a Dyck path. These types of Dyck paths will be useful in Sections 6 and 7.

We say that a word v is a *factor* of the word w if w can be factored as $w = u \cdot v \cdot z$. For instance, *peaks* in a Dyck path correspond to factors of the form UD, whereas *valleys* correspond to factors DU. We also say that a UD-word is *balanced* if it contains the same number of U letters as D letters.

A key tool for our proofs will be a variant of the bijection between 123-avoiding permutations of the set $[n]$ and Dyck paths of semilength n introduced by Krattenthaler [21]. An example of our bijection is shown in Figure 1. Note the lattice path never goes above the line $y = x$. Given a 123-avoiding permutation π of $[n]$, we rephrase Krattenthaler's bijection as follows.

(1) We label the U-steps (these are the East steps in Figure 1) in decreasing order from n to 1.
(2) For each left-to-right minimum $\pi(i)$, we insert a D step (North step) right after the U-step labeled $\pi(i)$ and we label this D-step also with $\pi(i)$. By abuse of terminology we say that the peak is labeled $\pi(i)$.
(3) From each peak labeled $\pi(i)$ we continue with D steps labeled $\pi(i+1), \pi(i+2), \ldots, \pi(j-1)$ where $\pi(j)$ is the next left-to-right minimum.
(4) After the D step labeled $\pi(j-1)$, we continue with U steps until we reach the next peak labeled $\pi(j)$.

We note that the complement of the left-to-right minima of a 123-avoiding permutation forms a decreasing sequence. This observation is essential in the argument that the inverse of Krattenthaler's bijection is well-defined.

We also consider lattice paths taking up steps $(1, 1)$ and down steps $(1, -1)$ that start at the origin and end at the lattice point $(n, n-2k)$ that do not go below the x-axis. Any such path must contain $n-k$ up steps and k down steps. They are enumerated by

$$C(n, k) = \binom{n}{k} - \binom{n}{k-1} \quad \text{for } 0 \leq k \leq n/2. \tag{2.2}$$

The numbers $C(n, k)$ are the entries of the *Catalan triangle*, listed as sequence A008315 in OEIS [24].

Finally we also consider the set of Dyck words of semilength n that do not contain UUU as a factor. The number of such words is known to be the *Motzkin number*

$$M_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \cdot C_j. \tag{2.3}$$

By definition, the Motzkin number M_n is the number of *Motzkin paths of length* n from $(0, 0)$ to $(n, 0)$ composed of up steps $(1, 1)$, down steps $(1, -1)$ and horizontal steps $(1, 0)$ that never go below the horizontal axis. The parameter j in (2.3) is the number of up steps (and down steps) in the path. A bijection between Motzkin paths of length n and UUU-avoiding Dyck paths of semilength n is outlined in the entry of sequence A001006 in OEIS [24]: replace each UUD factor in the Dyck word with an up step, then replace each remaining UD factor with a horizontal step, and finally the remaining D steps are unchanged. From this discussion we have part (a) of the following lemma.

Lemma 2.1. (a) *The number of UUU-avoiding Dyck paths of semilength n having exactly j factors of UU is given by $\binom{n}{2j} \cdot C_j$.*
(b) *The number of balanced UD-words with n U's and n D's, having no UUU factors, ending with a D and having exactly j UU factors is $\binom{n}{2j} \cdot \binom{2j}{j}$.*

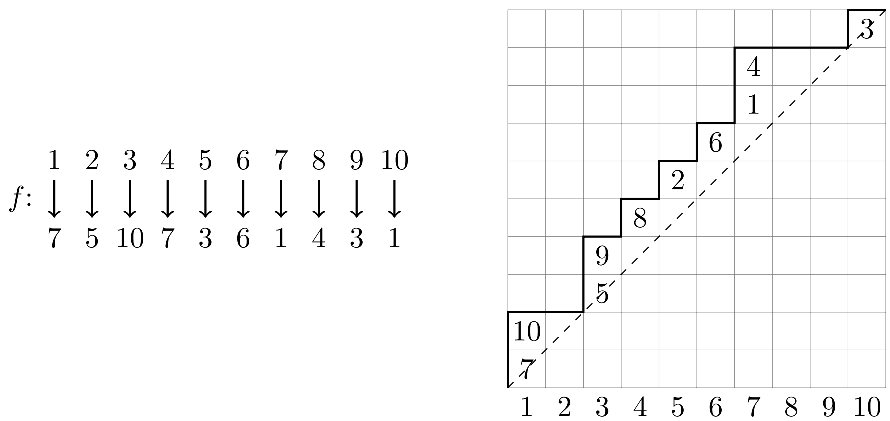

FIGURE 2. The Dyck path of Garsia and Haiman representing the parking function $f = (7, 5, 10, 7, 3, 6, 1, 4, 3, 1)$. The associated permutation is $\pi = (\underline{7}, 10, \underline{5}, 9, 8, \underline{2}, 6, \underline{1}, 4, 3)$ and the Dyck word is $v = U^2D^2U^2DUDUDUDU^2D^3UD$.

Part (b) of this lemma follows by the same bijection. Instead of Motzkin paths we consider all possible paths from $(0, 0)$ to $(n, 0)$ consisting of up, down and horizontal steps with no restriction.

2.3. Parking functions.

Definition 2.2. *For a function $f : [n] \longrightarrow [n]$ we have the following three notions:*

(i) *The function f is a* parking function *if the fiber $f^{-1}([k])$ has at least k elements for $1 \leq k \leq n$.*
(ii) *The function f is* 123-avoiding *if there is no triple $1 \leq i_1 < i_2 < i_3 \leq n$ such that $f(i_1) \leq f(i_2) \leq f(i_3)$.*
(iii) *The function f has an* ascent *at position i if $1 \leq i \leq n-1$ and $f(i) \leq f(i+1)$.*

The study of parking functions originated with Konheim and Weiss [20]. Note that parts (ii) and (iii) of the above definition are extensions of permutation notions to functions.

We say that a balanced word v of semilength n and a permutation $\pi \in \mathfrak{S}_n$ are *compatible* if

$$v = U^{q_1}DU^{q_2}D \cdots U^{q_n}D \qquad \mathrm{Des}(\pi) \subseteq \{q_1, q_1 + q_2, \ldots, q_1 + \cdots + q_{n-1}\},$$

where $q_1, q_2, \ldots, q_n$ are nonnegative integers with sum n. Garsia and Haiman [15, pages 226–227] introduced a bijection between functions $f : [n] \longrightarrow [n]$ and the set

$$\{(\pi, v) : \pi \in \mathfrak{S}_n, v \text{ balanced } UD\text{-word}, \pi \text{ and } v \text{ compatible}\}.$$

Given a function f, the balanced word v is given by $U^{q_1}DU^{q_2}D \cdots U^{q_r}D$ where q_i is the cardinality of the fiber $f^{-1}(i)$, that is, $q_i = |f^{-1}(i)|$. Furthermore, let τ_i be the list of the elements in the fiber $f^{-1}(i)$ in increasing order. The permutation π is then the concatenation of the n lists τ_1 through τ_n.

We visualize this bijection as follows; see Figure 2. We will denote the vertical steps $(0, 1)$ by the letter U and the horizontal steps $(1, 0)$ by the letter D.

(1) We associate the element $i \in [n]$ in the range of f to the interval $(i-1, i)$ on the x-axis.
(2) We write the elements of the fiber $f^{-1}(i)$ in upward increasing order as labels of the U steps whose horizontal coordinate is $i-1$. These steps are consecutive and we set the vertical coordinate of the first such U step to be $|f^{-1}([i-1])|$.
(3) We connect the resulting runs of U steps with D steps.
(4) Finally, we add enough D steps after the last run of U steps so that the path ends at (n, n).

Garsia and Haiman showed that the function f is a parking function if and only if the balanced word v is a Dyck word. In our visualization in Figure 2 this is the lattice path occurring weakly above the line $y = x$.

Definition 2.3. *Let $F = \{f_0, f_1, \ldots\}$ be an infinite alphabet and let the weight of the letter f_q be $w(f_q) = q - 1$ for $q \geq 0$. Furthermore, define the weight of a word $v = v_1 \cdots v_k$ with letters from this alphabet to be the sum $w(v) = w(v_1) + \cdots + w(v_k)$. A word $v = v_1 \cdots v_m$ is a* Łukasziewicz word *if the weight of each initial factor satisfies $w(v_1 \cdots v_k) \geq 0$ for $k < m$ and $w(v) = -1$.*

Lemma 2.4. *The map sending the word $U^{q_1} D U^{q_2} D \cdots U^{q_n} D$ of length $2n$ to the word $f_{q_1} f_{q_2} \cdots f_{q_n} f_0$ is a bijection between Dyck words of length $2n$ and Łukasziewicz words of length $n + 1$.*

The straightforward verification is left to the reader.

2.4. **The restricted Foata–Strehl action.** Some of our key results in Sections 9 and 10 make use of a restriction of the Foata–Strehl action [13] on the set of permutations of $[n]$ that was first considered by Brändén [4], albeit in a slightly modified form. We review these actions in terms of the *Foata–Strehl tree* representation of the permutations of $[n]$.

Definition 2.5. *A rooted tree on n vertices has an* increasing vertex labeling *if its vertices are bijectively labeled with the elements of an n-element totally ordered set in such a way that the label on each child is greater than the label on its parent. A* Foata–Strehl tree on n vertices *is a special rooted tree with an increasing vertex labeling where every vertex has at most two children: at most one left child and at most one right child.*

Figure 3 displays three Foata–Strehl trees. These trees are called *increasing binary trees* in [34, Section 1.5]. We prefer the above terminology, because some sources insist that every non-leaf vertex should have two children in a binary tree, and they would call rooted trees of the above type 0-1-2 *trees*.

Definition 2.6. *A* 0-1-2 *tree is a rooted tree in which each vertex has at most* 2 *children.*

Even after disregarding the vertex labeling, a Foata–Strehl tree is not immediately identifiable with a plane 0-1-2 tree, because there is no distinction between a right child and a left child in a plane tree if the child is the only child of its parent. In this paper we will use the following identification.

Definition 2.7. *We say that a Foata–Strehl tree is* right-adjusted *if no vertex has an only child that is a left child. Given a plane* 0-1-2 *tree with an increasing vertex labeling, we identify it with the right-adjusted Foata–Strehl tree obtained by designating each only child to be a right child.*

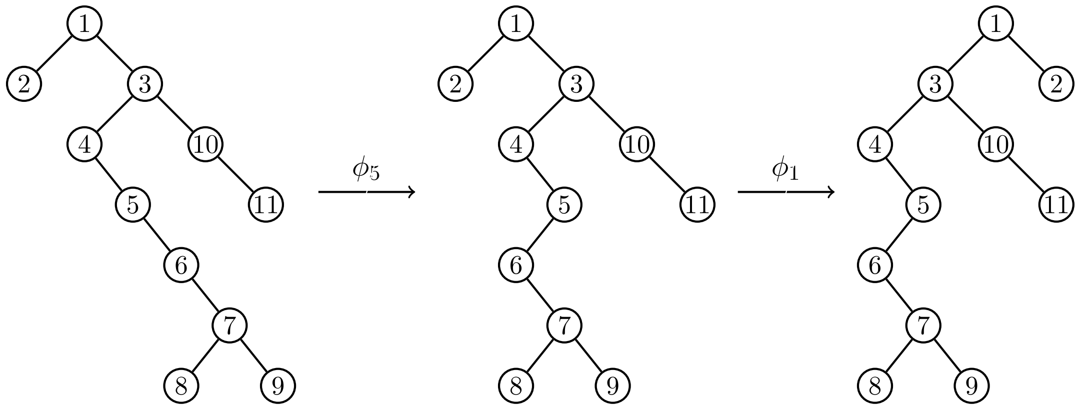

FIGURE 3. The Foata–Strehl trees for the permutations $\tau = (2,\ 1,\ 4,\ 5,\ 6,\ 8,\ 7,\ 9,\ 3,\ 10,\ 11)$, $\phi_5\tau$ and $\phi_1\phi_5\tau$.

Clearly there is a bijection between labeled plane 0-1-2 trees with an increasing vertex labeling and right-adjusted Foata–Strehl trees. As explained in [34, Section 1.5], there is a bijection between Foata–Strehl trees with n-vertices and permutations of $[n]$, defined as follows.

Definition 2.8. *Given a permutation $\pi = \pi(1)\cdots\pi(n)$ of letters of an n-element totally ordered set, we define its Foata–Strehl tree recursively as follows:*

(1) We label the root of our tree with $\pi(m) = \min\{\pi(1),\ldots,\pi(n)\}$.
(2) We define the left, respectively right subtree of the root as the Foata–Strehl tree of the permutation $\pi(1)\cdots\pi(m-1)$, respectively $\pi(m+1)\cdots\pi(n)$. (Either or both subtrees may be empty.) The root of the left (respectively right) subtree is the left (respectively right) child of the vertex labeled $\pi(m)$.

Conversely, the permutation associated to a Foata–Strehl tree is obtained by reading off the labels on the vertices using the *inorder traversal*. This recursively defined process calls for reading the labels in the left subtree of the root, then the label of the root, and then the labels in the right subtree of the root. For example the Foata–Strehl tree shown on the left hand side of Figure 3 represents the permutation $(2, 1, 4, 5, 6, 8, 7, 9, 3, 10, 11)$.

The *Foata–Strehl action* is most easily defined on the Foata–Strehl trees representing the permutations: the operation ϕ_x exchanges the left and right subtrees of the vertex labeled x. For example

$$\phi_1\phi_5(2, 1, 4, 5, 6, 8, 7, 9, 3, 10, 11) = \phi_1(2, 1, 4, 6, 8, 7, 9, 5, 3, 10, 11) = (4, 6, 8, 7, 9, 5, 3, 10, 11, 1, 2),$$

where the associated trees are displayed in Figure 3. Clearly ϕ_i and ϕ_j commute. Hence the Foata–Strehl action is a $\mathbb{Z}_2^{n-1}$-action on the set of all permutations of the set $[n]$. The numbers of the orbits are the tangent and secant numbers, and *André permutations* (of various kinds) may be used as a set of distinct orbit representatives.

Motivated by Brändén's definition [4], we define the *restricted Foata–Strehl action* as follows: we set $\psi_x = \phi_x$ if the vertex labeled x has exactly one child and we set ψ_x to be the identity operation otherwise. The permutations represented by right-adjusted Foata–Strehl trees are a natural choice of distinct orbit representatives for this action. Brändén's *modified Foata–Strehl action* is obtained from this restricted action by replacing each label i with $n+1-i$, thus turning the Foata–Strehl trees into *decreasing trees*: the label on each child is less than the label on its parent. Brändén's paper avoids introducing tree representations. It instead operates with descents and ascents.

Definition 2.9. *Let $\pi = \pi(1)\cdots\pi(n)$ be a permutation of $[n]$. The index $i < n$ is a* descent*, respectively an* ascent*, if $\pi(i) > \pi(i+1)$, respectively $\pi(i) < \pi(i+1)$ holds. The index $2 \leq i \leq n-1$ is a* peak*, a* valley*, a* double descent*, or a* double ascent*, respectively, if $\pi(i-1) < \pi(i) > \pi(i+1)$, $\pi(i-1) > \pi(i) < \pi(i+1)$, $\pi(i-1) > \pi(i) > \pi(i+1)$, or $\pi(i-1) < \pi(i) < \pi(i+1)$ hold, respectively.*

The equivalence between the above construction and Brändén's is straightforward in light of the following facts; see [34, Section 1.5].

Lemma 2.10. *Consider a permutation $\pi = \pi(1)\cdots\pi(n)$. The index $2 \leq i \leq n-1$ is a peak, a valley, a double descent or a double ascent respectively, if the vertex labeled $\pi(i)$ in its Foata–Strehl tree has no children, two children, only a left child, only a right child, respectively.*

Lemma 2.10 implies the following characterization of the orbit representatives of the restricted Foata–Strehl action.

Corollary 2.11. *The Foata–Strehl tree of a permutation π is right-adjusted if and only if the permutation π has no double descents and no final descent.*

3. The toric g-polynomial in terms of the γ-vector

Recall an n-dimensional polytope P is *simple* if every vertex is incident to n facets (maximal faces). Let $f_i = f_i(P)$ be the number of i-dimensional faces of the polytope P for $0 \leq i \leq n$. The associated h-polynomial is given by

$$\sum_{i=0}^{n} h_i \cdot x^i = \sum_{i=0}^{n} f_i \cdot (x-1)^i. \tag{3.1}$$

The Dehn-Sommerville relations state that this polynomial is palindromic, that is, $h_i = h_{n-i}$ for $0 \leq i \leq n$. The *γ-vector* $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor})$ of an n-dimensional simple polytope was introduced by Gal [14, Definition 2.1.4] and is given by

$$\sum_{i=0}^{n} h_i \cdot x^i = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_j \cdot x^j \cdot (1+x)^{n-2j}. \tag{3.2}$$

The fact that the γ-vector is well-defined follows from the Dehn-Sommerville relations.

Consider the coefficient of x^i in equation (3.2). For $0 \leq i \leq n/2$, we have

$$h_i = \sum_{j=0}^{i} \binom{n-2j}{i-j} \cdot \gamma_j.$$

Taking the backward difference, we obtain

$$\begin{aligned} h_i - h_{i-1} &= \gamma_i + \sum_{j=0}^{i-1} \left(\binom{n-2j}{i-j} - \binom{n-2j}{i-j-1} \right) \cdot \gamma_j \\ &= \sum_{j=0}^{i} C(n-2j, i-j) \cdot \gamma_j, \end{aligned} \tag{3.3}$$

for $1 \leq i \leq n/2$, where we used the entries of the Catalan triangle (2.2). Introducing $C(n, 0, x) = 1$ and

$$C(n,i,x) = \sum_{k=1}^{i} \frac{n+1-2i}{k} \cdot \binom{n-i}{k-1} \cdot \binom{i-1}{k-1} \cdot x^k \tag{3.4}$$

for $1 \leq i \leq n/2$, we may write the toric g-polynomial of an n-dimensional simple polytope P in terms of its h-vector [17, Corollary 6.9] as follows:

Lemma 3.1 (Hetyei)**.** *The toric g-polynomial of an n-dimensional simple polytope P is given by*

$$g(P,x) = h_0 + \sum_{i=1}^{\lfloor n/2 \rfloor} (h_i - h_{i-1}) \cdot C(n,i,x). \tag{3.5}$$

Combining Lemma 3.1 with equation (3.3) (and taking into account $\gamma_0 = h_0$), we obtain

$$\begin{aligned} g(P,x) &= \gamma_0 + \sum_{i=1}^{\lfloor n/2 \rfloor} \sum_{j=0}^{i} C(n-2j, i-j) \cdot C(n,i,x) \cdot \gamma_j \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{i} C(n-2j, i-j) \cdot C(n,i,x) \cdot \gamma_j \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_j \cdot \sum_{i=j}^{\lfloor n/2 \rfloor} C(n-2j, i-j) \cdot C(n,i,x). \end{aligned} \tag{3.6}$$

Krattenthaler's remark [17, Remark 6.12] may be rephrased as follows.

Proposition 3.2 (Krattenthaler)**.** *For $1 \leq i \leq n/2$, the coefficient of x^k in the polynomial $C(n,i,x)$ is the number of Dyck paths from $(0,0)$ to $(n, n-2i)$ having k peaks.*

Proof. Reflecting the Dyck path over the line $y = x$ and then rotating it clockwise by $\pi/4$ radians in [17, Remark 6.12], Krattenthaler's remark can be restated as the sought after coefficient is the number of all Dyck paths from $(0,0)$ to $(n+1, n+1-2i)$ containing k valleys such that there are no additional down steps after the last valley. Such a lattice path necessarily has also k peaks, and the last step must be an up step. Removing the final up step yields a Dyck path from $(0,0)$ to $(n, n-2i)$ having k peaks (and k or $k-1$ valleys). The inverse of this operation is adding a final up step to Dyck paths from $(0,0)$ to $(n, n-2i)$ having k peaks. In the resulting lattice path there are k valleys, and there are no peaks after the last valley. □

Lemma 3.3. *The coefficient of x^k in $\sum_{i=j}^{\lfloor n/2 \rfloor} C(n-2j, i-j) \cdot C(n, i, x)$ is the number of Dyck paths of semilength $n-j$ with k peaks whose first coordinate is at most $n-1$.*

Proof. Let $(0,0) = (x_0, y_0), (x_1, y_1), \ldots, (x_{2n-2j}, y_{2n-2j}) = (2n-2j, 0)$ be any Dyck path of semilength $n-j$. Let $i = (n-y_n)/2$. Note that we must be able to return to level 0 from level y_n in $n-2j$ steps. Hence $0 \leq y_n \leq n-2j$ and the parity of y_n is the same as the parity of $n-2j$. In turn, this is the same as the parity of n. We obtain that i is an integer satisfying $j \leq i \leq n/2$ and that $y_n = n-2i$. After fixing i, the number of possible lattice paths $(0,0) = (x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n) = (n, n-2i)$ from $(0,0)$ to $(n, 2n-i)$ with k peaks is the coefficient of x^k in $C(n, i, x)$. Furthermore, there are $C(n-2j, i-j)$ ways to select the lattice path $(n, n-2i) = (x_n, y_n), (x_{n+1}, y_{n+1}), \ldots, (x_{2n-2j}, y_{2n-2j}) = (2n-2j, 0)$. □

Define the *toric g-contribution polynomial $g_{n,j}(x)$* as follows:

$$g_{n,j}(x) = \sum_{k=0}^{\min(\lfloor n/2 \rfloor, n-j)} C_{n-k-j} \cdot \binom{n-k}{k} \cdot (x-1)^k, \tag{3.7}$$

for $0 \leq j \leq n$. We remark that the polynomial $g_{n,0}(x)$ is the toric g-polynomial of the n-dimensional cube [16]. The next result shows that for $0 \leq j \leq n/2$ the polynomial $g_{n,j}(x)$ is the contribution of the γ-vector entry γ_j to the toric g-polynomial of an n-dimensional simple polytope.

Theorem 3.4. *The toric g-polynomial of an n-dimensional simple polytope P is given by*

$$g(P, x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_j \cdot g_{n,j}(x), \tag{3.8}$$

where $(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor})$ is the γ-vector of the simple polytope P.

Proof. Replacing x by $x+1$ in equations (3.6) and (3.7) and fixing j, we need to show the following equivalent identity:

$$\sum_{i=j}^{\lfloor n/2 \rfloor} C(n-2j, i-j) \cdot C(n, i, x+1) = \sum_{k=0}^{\lfloor n/2 \rfloor} C_{n-j-k} \cdot \binom{n-k}{k} \cdot x^k.$$

By Lemma 3.3 the contribution of γ_j to $g(P, x)$ is the total weight of Dyck paths of semilength $n-j$ where each peak whose first coordinate is in the interval $[1, n-1]$ contributes a factor of x to the weight of the lattice path. When we substitute $x+1$ into x, this corresponds to marking some of these peaks and we count only the marked peaks in the exponent of x.

Subject to this rephrasing, it suffices to show the following statement: The number of Dyck paths from $(0,0)$ to $(2n-2j, 0)$ having k marked peaks with first coordinate at most $n-1$ is $C_{n-k-j} \cdot \binom{n-k}{k}$. Let $w = w_1 w_2 \cdots w_{2n-2j}$ be the associated Dyck word.

Hence we are counting Dyck words of semilength $n-j$ with some marked factors UD among the first n letters. Let us remove these factors. If we remove k such marked factors from a Dyck word of semilength $n-j$, we obtain a Dyck word $v = v_1 v_2 \cdots v_{2n-2j-2k}$ of semilength $n-j-k$. Note that the marked peaks belong to the interval $[1, n-1]$ if and only if all marked factors occurred before the letter v_{n-2k+1}. There are C_{n-j-k} ways to select the Dyck word v. To reconstruct w we must insert k factors before v_{n-2k+1}. In other words, we must line up the letters $v_1, v_2, \ldots, v_{n-2k}$ (in this relative order) and k factors UD. This may be performed in $\binom{n-2k+k}{k} = \binom{n-k}{k}$ ways. □

4. Compatible Dyck words

In this section we give several interpretations of the coefficients of the toric g-contribution polynomial $g_{n,j}(x)$. The two first interpretations will be used in Sections 6 and 7.

Recall that a set of integers is *sparse* if it does not contain a pair of consecutive integers. Furthermore, a *labeled Dyck word* w of semilength n is a Dyck word where the letters U, respectively the letters D, are indexed from left to right by 1 through n. For instance, the Dyck word $U_1U_2D_1U_3D_2D_3U_4D_4$ is labeled. We use the convention that a letter has no index if the index of a letter is irrelevant at the moment.

Definition 4.1. *Let A and B be two sparse subsets of the set $[n-1]$ and let w be a labeled Dyck word of semilength n. We say that the word w is (A,B)-compatible if*

(a) *for each $a \in A$ the word $U_aU_{a+1}D$ is a factor in w, and*
(b) *for each $b \in B$ the word UD_bD_{b+1} is a factor in w.*

Let $\mathcal{D}_n$ denote the set of all Dyck words of semilength n. Furthermore, let $\mathcal{D}_{n,A,B}$ be the set of all Dyck words of semilength n that are (A,B)-compatible. Observe that if a Dyck word is both (A_1,B_1)-compatible and (A_2,B_2)-compatible then it is also $(A_1 \cup A_2, B_1 \cup B_2)$-compatible.

Theorem 4.2. *The number of Dyck words of semilength n that are (A,B)-compatible is given by the Catalan number $C_{n-|A|-|B|}$.*

Proof. We present an explicit bijection $\phi_{n,A,B} : \mathcal{D}_{n,A,B} \longrightarrow \mathcal{D}_{n-|A|-|B|}$. Let w be a labeled Dyck word in the set $\mathcal{D}_{n,A,B}$. First replace each occurrence of factors of the form $U_aU_{a+1}D_bD_{b+1}$ where $a \in A$ and $b \in B$ by the empty word 1. Note that this is the only way the two factors of the form $U_aU_{a+1}D$ and UD_bD_{b+1} can overlap. Second, replace each remaining factor of the form $U_aU_{a+1}D$ where $a \in A$ by U. Finally, replace each remaining factor of the form UD_bD_{b+1} where $b \in B$ by D. The resulting word is a Dyck word of semilength $n-|A|-|B|$.

The idea for establishing the inverse map of $\phi_{n,A,B}$ is to consider the left-most replacement. Let a, respectively b, be the minimal elements of the two sparse sets A and B. If the left-most replacement was $U_aU_{a+1}D \longmapsto U$ then the letter U_a will precede the letter D_{b-1} in the relabeled word. Similarly, if the left-most replacement was $UD_bD_{b+1} \longmapsto D$ then D_b will precede U_{a-1}. Finally, if the replacement was $U_a U_{a+1} D_b D_{b+1} \longmapsto 1$ then the two letters U_{a-1} and D_{b-1} precede the two letters U_a and D_b in the image of w. These are three distinct cases.

By induction on $|A| + |B|$ we construct the inverse map $\psi_{n,A,B} : \mathcal{D}_{n-|A|-|B|} \longrightarrow \mathcal{D}_{n,A,B}$. When $|A| + |B| = 0$, the two sets $\mathcal{D}_{n-|A|-|B|}$ and $\mathcal{D}_{n,A,B}$ are equal and there is nothing to prove. Let w be a labeled Dyck word in $\mathcal{D}_{n-|A|-|B|}$. Note in what follows the Dyck word $\psi_{n,A,B}(w)$ is constructed from the word w by reading w left to right.

Next we study the case when the set A is empty and B is nonempty. Let b be the minimal element of the set B. Replace the occurrence of D_b in w with $UD_b D_{b+1}$. After relabeling the indices we obtain a Dyck word v in $\mathcal{D}_{n-|B|+1}$ Now set $\psi_{n,\emptyset,B}(w) = \psi_{n,\emptyset,B-\{b\}}(v)$. Observe that by induction this word is $(\emptyset, B - \{b\})$-compatible. Furthermore, by the construction it is also is $(\emptyset, \{b\})$-compatible. In conclusion, it is $(\emptyset, B)$-compatible, completing this case.

The case when B is empty and A is non-empty is similar.

Now consider the case when A and B are both nonempty. Let $a = \min(A)$ and $b = \min(B)$. Consider the relative order of the four letters U_{a-1}, U_a, D_{b-1} and D_b in the labeled word w. If $a = 1$ or $b = 1$ view the two non-existent letters U_0 and D_0 as standing in front of the word w. There are six possible arrangements that we arrange in three cases, one of which has four subcases:

– The two letters U_{a-1} and D_{b-1} appear in w before the two letters U_a and D_b, reading left to right. That is, we have one of the following four subcases:

$$U_{a-1} \cdots D_{b-1} \cdots U_a \cdots D_b,$$
$$U_{a-1} \cdots D_{b-1} \cdots D_b \cdots U_a,$$
$$D_{b-1} \cdots U_{a-1} \cdots U_a \cdots D_b,$$
$$D_{b-1} \cdots U_{a-1} \cdots D_b \cdots U_a.$$

Furthermore, note that the second and third letters in each of these subcases have to be adjacent, since no U can be between U_{a-1} and U_a, and similarly, no letter D between D_{b-1} and D_b. Hence we insert the factor $U_a U_{a+1} D_b D_{b+1}$ between the second and third letter and relabel to obtain the Dyck word v. Now define the map ψ by $\psi_{n,A,B}(w) = \psi_{n,A-\{a\},B-\{b\}}(v)$.

– The two letters U_{a-1} and U_a appear in w before the two letters D_{b-1} and D_b appear.

$$U_{a-1} \cdots U_a \cdots D_{b-1} \cdots D_b.$$

Replace U_a with $U_a U_{a+1} D$ and relabel to obtain the Dyck word v. Now the map ψ is given by $\psi_{n,A,B}(w) = \psi_{n,A-\{a\},B}(v)$.

– The last case is that the letters D_{b-1} and D_b appear to the left of the letters U_{a-1} and U_a, that is, we have

$$D_{b-1} \cdots D_b \cdots U_{a-1} \cdots U_a.$$

Similar to the previous case, replace D_b with $UD_b D_{b+1}$ and relabel to obtain the Dyck word v and set $\psi_{n,A,B}(w) = \psi_{n,A,B-\{b\}}(v)$.

In each of these three cases observe that the calculation of $\psi_{n,A',B'}(v)$ will not change the initial segment of the word v. Hence since v is $(\{a\},\{b\})$-compatible, $(\{a\},\emptyset)$-compatible, and $(\emptyset,\{b\})$-compatible in each respective case, we obtain that the image $\psi_{n,A',B'}(v)$ is also so. Thus the Dyck word $\psi_{n,A',B'}(v)$ is (A,B)-compatible. By this construction we have that $\psi_{n,A,B}$ is the inverse of $\phi_{n,A,B}$ and the result follows. □

By using the same proof ideas we also obtain the corresponding result for balanced words.

Proposition 4.3. *The number of balanced words of semilength n that are (A,B)-compatible is given by the central binomial coefficient $\binom{2\cdot(n-|A|-|B|)}{n-|A|-|B|}$.*

Using Krattenthaler's bijection between 123-avoiding permutations on the set $[n]$ and Dyck paths of semilength n presented in Subsection 2.2, Theorem 4.2 has a remarkable consequence for the ascent statistics of 123-avoiding permutations. A 123-avoiding permutation π has a sparse ascent set, and the same holds for the inverse permutation π^{-1}, which is also 123-avoiding.

Corollary 4.4. *Let A and B be sparse subsets of $[n-1]$. Then the number of 123-avoiding permutations π in $\mathfrak{S}_n$ such that $B \subseteq \mathrm{Asc}(\pi)$ and $A \subseteq \mathrm{Asc}(\pi^{-1})$ is given by the Catalan number $C_{n-|A|-|B|}$.*

For a 123-avoiding permutation π, it is easy to verify that i is an ascent of π if and only if $\pi(i)$ is a left-to-right minimum and $\pi(i+1)$ is not. Equivalently, in the corresponding Dyck path, the peak UD labeled $\pi(i)$ must be part of a factor UDD. Similarly, $j = \pi(i)$ is an ascent of the inverse π^{-1} if and only if $j+1$ precedes j in the word $\pi(1)\cdots\pi(n)$. This happens if and only $j = \pi(i)$ is a left-to-right maximum and $j+1$ is not. Equivalently, in the corresponding Dyck path, the peak UD labeled $\pi(i)$ must be part of a factor UUD.

Now we obtain two more interpretations of the coefficients of the toric g-contribution polynomial $g_{n,j}(x)$.

Proposition 4.5. *Let B be a sparse subset of the set $[n-1]$. The coefficient of x^k in the toric g-contribution polynomial $g_{n,|B|}(x)$ is given by:*

(a) *the number of all $(\emptyset,B)$-compatible Dyck words of semilength n that contain exactly k copies of the word UUD as a factor.*
(b) *the number of 123-avoiding permutations $\pi \in \mathfrak{S}_n$ such that $B \subseteq \mathrm{Asc}(\pi)$ and the inverse permutation π^{-1} has exactly k ascents.*

Proof. Mimicking the proof of Theorem 3.4, let us compute $g_{n,j}(x+1)$. It suffices to show that the coefficient of x^k in $g_{n,j}(x+1)$ is the number of Dyck words v where we have marked k of the factors UUD. Label the Dyck word v. Then these marked factors correspond to a sparse set A of size k in the set $[n-1]$. The number of sparse sets of cardinality k is given by the binomial coefficient $\binom{n-k}{k}$. Hence the sought after number of $(\emptyset,B)$-compatible Dyck words is $\binom{n-k}{k}\cdot C_{n-k-j}$, which is the coefficient of x^k in $g_{n,j}(x+1)$. The second interpretation follows from Krattenthaler's bijection between Dyck paths and 123-avoiding permutations. □

By setting the set B to be empty in Proposition 4.5 and switching to the inverse permutation, we obtain the following corollary.

Corollary 4.6. *The kth entry in the toric g-vector of the n-dimensional cube is the number of* 123-*avoiding permutations in the symmetric group $\mathfrak{S}_n$ with exactly k ascents.*

Proposition 4.5 maybe also restated in terms of noncrossing partition statistics, namely in terms of counting nonsingleton blocks and filler points. Recall a partition of $[n]$ is *noncrossing* if for every four elements $1 \leq a < b < c < d \leq n$ if a and c occur in the same block and b and d occur in the same block, then a, b, c and d all lie within the same block. The notion of filler points was introduced by Denise and Simion [8, Definition 2.5].

Definition 4.7. *Let π be a noncrossing partition of the set $[n]$. An element $2 \leq i \leq n$ is a* filler *if one of the following conditions holds:*

(1) i is the largest element of its block and $i-1$ belongs to the same block as i.
(2) i forms a singleton block and $i-1$ is not the largest element of its block.

The set of all filler points of a noncrossing partition must be sparse: if i is a filler then $i-1$ is not a filler point. Proposition 4.5 may be restated as follows.

Proposition 4.8. *Let J be a sparse subset of the interval $[2, n]$. Then the coefficient of x^k in $g_{n,|J|}(x)$ is the number of noncrossing partitions of $[n]$ containing exactly k nonsingleton blocks whose set of filler points contains J.*

Proof. We use a variant of a well-known bijection between noncrossing partitions of the set $[n]$ and Dyck paths of semilength n, presented for example in [31, Fig. 2 (f)]. Given a Dyck path of semilength n from $(0,0)$ to $(2n,0)$ using U steps $(1,1)$ and D steps $(1,-1)$, reading from right to left let us label the down steps from 1 to n. Treating the down steps as right parentheses and the up steps as left parentheses, we may pair each up step with a down step, and transfer the label on the down step to the matching up step. The labels on the longest runs of contiguous up steps then form the blocks of a noncrossing partition. Under this bijection, nonsingleton blocks correspond to contiguous runs of U steps ending with a factor UUD. Conversely, each factor UUD marks the end of a run of U steps corresponding to a nonsingleton block. We are left to show that filler points correspond exactly to the factors UDD under this correspondence, and the statement will then follow from Proposition 4.5.

Consider first a factor UDD. Since we labeled the D steps in the right to left order, adding the labeling yields a factor UD_iD_{i-1}. A Dyck word cannot begin with a factor UDD, so there is at least one letter X immediately preceding our factor UD_iD_{i-1}. If $X = U$ holds, that is, we have a factor UUD_iD_{i-1}, then the letters U in this factor are matched to the letters D_i and D_{i-1}: both i and $i-1$ belong to the same block and i is the maximum element of this block, hence i is a filler of type (i). If $X = D$ holds, that is, we have a factor DUD_iD_{i-1} then the U in this factor is matched to D_i and $\{i\}$ is a singleton block. The letter D_{i-1} is matched to an earlier U. This earlier U cannot be immediately followed by a D, hence $i-1$ is not the largest element of its block. Therefore i is a filler of type (ii). The converse is straightforward and left to the reader. □

Substituting $j = 0$ into Proposition 4.8 yields a new proof of [16, Lemma 6.4].

Corollary 4.9 (Hetyei)**.** *The kth entry in the toric g-vector of the n-dimensional cube is the number of noncrossing partitions on the set $[n]$ with exactly k nonsingleton blocks.*

It is worth noting that reading Dyck words backwards turns UUD factors into UDD factors and vice versa. Substituting $j = 0$ into the reverse variant of Proposition 4.8 yields the following result of Denise and Simion [8], first pointed out in [16, Lemma 6.7].

Corollary 4.10 (Denise–Simion)**.** *The kth entry in the toric g-vector of the n-dimensional cube is the number of noncrossing partitions on the set $[n]$ with exactly k fillers.*

It has been pointed out in [16] that the work of Denise and Simion [8, Remark 3.4] implicitly contains the description of a simplicial complex whose face numbers are the coefficients of the powers of x in $g_{n,0}(x)$. The first such construction was given by Billera, Chan and Liu [6]. Here we add a third possibility. It is a direct consequence of Corollary 4.9.

Corollary 4.11. *The kth entry in the toric g-vector of the n-dimensional cube is the number of $(k-1)$-dimensional faces in the following simplicial complex:*

1. *The vertices are the subsets of $[n]$ containing at least two elements.*
2. *A collection $\{S_1, S_2, \ldots, S_k\}$ is a face if and only if there is a noncrossing partition on the set $[n]$ whose nonsingleton blocks are exactly the sets $S_1, S_2, \ldots, S_k$.*

5. Generating functions of the toric g-contributions

In this section we compute the ordinary generating function of the toric g-contribution polynomials $g_{n,j}(x)$ defined in equation (3.7). Note that (3.7) is a valid definition for all $0 \leq j \leq n$ even if we use only the polynomials $g_{n,j}(x)$ satisfying $0 \leq j \leq n/2$ in our toric g-polynomial formulas. We extend the definition of $g_{n,j}(x)$ to all $j \geq 0$ by setting $g_{n,j}(x) = 0$ whenever $j > n$. Keeping this extension in mind, we obtain the following recurrence.

Lemma 5.1. *The toric g-contribution polynomials $g_{n,j}(x)$ satisfy the recurrence*

$$g_{n,j}(x) = g_{n-1,j-1}(x) + (x-1) \cdot g_{n-2,j-1}(x) \tag{5.1}$$

for $n \geq 2$ and $j \geq 1$.

Proof. We prove the equivalent expression

$$g_{n,j}(x+1) = g_{n-1,j-1}(x+1) + x \cdot g_{n-2,j-1}(x+1) \tag{5.2}$$

for $n \geq 2$ and $j \geq 1$. The coefficient of x^k is $C_{n-j-k}\binom{n-k}{k}$ on the left-hand side of (5.2) and $C_{n-j-k}\binom{n-1-k}{k} + C_{n-j-k}\binom{n-1-k}{k-1}$ on the right-hand side. The statement is a direct consequence of the Pascal recursion $\binom{n-k}{k} = \binom{n-1-k}{k} + \binom{n-1-k}{k-1}$. □

We introduce the generating function $G_j(x,t)$ of the toric g-contribution polynomials $g_{n,j}(x)$.

$$G_j(x,t) = \sum_{n \geq 2j} g_{n,j}(x) \cdot t^n. \tag{5.3}$$

Lemma 5.1 implies

$$G_j(x,t) = (t + (x-1) \cdot t^2) \cdot G_{j-1}(x,t) = t \cdot (1 - t + xt) \cdot G_{j-1}(x,t)$$

and hence

$$G_j(x,t) = t^j \cdot (1 - t + xt)^j \cdot G_0(x,t).$$

The table of the coefficients of $G_0(x,t)$ is sequence A091156 in [24]. It is stated in [24] that the generating function $G_0(x,t)$ is the solution of the quadratic equation

$$G_0(x,t) = t \cdot (1 - t + xt) \cdot G_0(x,t)^2 + 1. \tag{5.4}$$

This statement is equivalent to the following recurrence.

Proposition 5.2. *The toric g-contribution polynomials $g_{n,0}(x)$ satisfy*

$$g_{n,0}(x) = (x-1) \cdot \sum_{m=2}^{n-1} g_{m-2,0}(x) \cdot g_{n-m,0}(x) + \sum_{m=1}^{n-1} g_{m-1,0}(x) \cdot g_{n-m,0}(x) \quad \textit{for } n \geq 1.$$

Proof. Recall that the coefficient of x^k in $g_{n,0}(x)$ is the number of Dyck words of semilength n containing exactly k factors UUD. Let m be the least positive integer such that the associated Dyck path touches the horizontal axis at $(2m, 0)$. In this case the Dyck word factors as $UuDv$ where u is a Dyck word of semilength $m-1$ and v is a Dyck word of semilength $n-m$. The words u and v may be chosen independently. As v ranges over all possible choices, we obtain a factor of $g_{n-m,0}(x)$.

Case 1: The word u begins with a factor UD, hence we have the word $UUDu'Dv$. In this case the first factor UUD contributes a factor of x, and u' is a Dyck word of semilength $m-2$, which may be chosen independently. The contribution of all Dyck words belonging to this case is $x \cdot g_{m-2,0}(x) \cdot g_{n-m,0}(x)$ (where $m \geq 2$ must hold).

Case 2: The word u does not begin with a factor UD. If we choose the word u in all possible ways, we obtain a contribution of $g_{m-1,0}(x) \cdot g_{n-m,0}(x)$. From this we must subtract the contribution of the words u beginning with a factor UD. Based on the previous case, $g_{m-2,0}(x) \cdot g_{n-m,0}(x)$ must be subtracted (when $m \geq 2$ holds). □

The solution of equation (5.4) is

$$G_0(x,t) = \frac{1 - \sqrt{4(1-x)t^2 - 4t + 1}}{2t(1 - t + xt)}. \tag{5.5}$$

This expression, combined with (5.3) yields

$$G_1(x,t) = \frac{1 - \sqrt{4(1-x)t^2 - 4t + 1}}{2}. \tag{5.6}$$

The table of the coefficients of $G_1(x,t)$ is sequence A091894 in [24]. Among other interpretations listed there, the entry tabulates the number of Dyck words of semilength n containing k factors DDU. It should be noted that the generating function given there (using our variable assignment) is

$$\frac{1-\sqrt{4(1-x)t^2-4t+1}}{2xt},$$

which is $G_1(x,t)/(xt)$ in our notation.

Another way to compute the toric g-contribution polynomials $g_{n,j}(x)$ is by introducing the *peak polynomials.*

Definition 5.3. *Given a pair (n,m) of nonnegative integers satisfying $0 \le m \le 2n$, the* peak polynomial *$p_{n,m}(x)$ is the total weight of all Dyck words of semilength n, where the weight of each Dyck word is x to the power of the number of UD factors occurring in the length m prefix of the word.*

Lemma 3.3 and Theorem 3.4 imply

$$g_{n,j}(x) = p_{n-j,n}(x). \tag{5.7}$$

Let $N_k(x)$ denote the Narayana polynomial

$$N_k(x) = \frac{1}{k}\cdot\sum_{j=1}^{k}\binom{k}{j}\cdot\binom{k}{j-1}\cdot x^j,$$

for $k > 0$ and set $N_0(x) = x$.

Proposition 5.4. *The peak polynomial $p_{n,m}(x)$ satisfies the recurrence*

$$p_{n,m}(x) = \sum_{k=0}^{\lfloor (m-2)/2\rfloor} N_k(x)\cdot p_{n-k-1,m-2k-2}(x) + \sum_{k=\lceil (m-1)/2\rceil}^{n-1} p_{k,m-1}(x)\cdot C_{n-k-1}$$

for $1 \le m \le 2n$, and from the initial conditions $p_{n,0}(x) = C_n$ for $n \ge 0$.

Proof. The stated initial conditions are direct: we do not count peaks when $m = 0$, and the number of all Dyck words of semilength n is the Catalan number C_n. From now on we may assume that $m \ge 1$ holds and we consider a Dyck word w of semilength $n \ge m/2$. Let $k+1 > 0$ be the semilength of the shortest nonempty prefix u in which the number of letters U equals the number of letters D. The word u is then of the form $Uu'D$ where u' is a Dyck word of semilength k. Furthermore w factors as $w = uv$ where v is also a Dyck word. We distinguish two cases.

Case 1: The inequality $2k+1 \le m-1$ holds. Equivalently, we have $k \le \lfloor (m-2)/2\rfloor$. Note that this case does not occur when $m = 1$. In this case all peaks belonging to the word u' contribute a factor of x. As u' varies over all Dyck words of semilength k, their total contribution is the Narayana polynomial $N_k(x)$. Note that the observation also holds when $k = 0$ and $u = UD$ hold, because we fixed the value of $N_0(x)$ to be x. The second factor v is an independently selected Dyck word of semilength $n-k-1$ where we count the peaks among the first $m-2k-2$ letters. They contribute a factor of $p_{n-k-1,m-2k-2}(x)$.

Case 2: The inequality $2k+1 \geq m$, that is, $k \geq \lceil (m-1)/2 \rceil$ holds. In this case all peaks contributing a factor of x belong to the factor u'. As u' varies over all Dyck words of semilength k, their total contribution is $p_{k,m-1}(x)$. The second factor v is an independently selected Dyck word of semilength $n-k-1$ whose peaks are not counted. They contribute a factor of C_{n-k-1}. □

The generating function $N(x,t) = \sum_{n \geq 0} N_n(x) t^n$ may be obtained from the well-known generating function formula of the Narayana polynomials [31, Eq. (9)] by adding $x-1$ to its constant term. We have

$$N(x,t) = \frac{1+t-xt-\sqrt{1-2(1+x)t+(1-x)^2t^2}}{2t} + x - 1$$
$$= \frac{1+xt-t-\sqrt{1-2(1+x)t+(1-x)^2t^2}}{2t}. \tag{5.8}$$

Substituting $x=1$ into $N_n(x)$ yields the Catalan number C_n and hence $N(1,t) = C(t)$.

We introduce the generating function $P(x,y,z)$ for the peak polynomials $p_{n,m}(x)$

$$P(x,y,z) = \sum_{m \geq 0} \sum_{n \geq \lceil m/2 \rceil} p_{n,m}(x) \cdot y^n z^m.$$

Proposition 5.4 implies the following identity.

Theorem 5.5. *The generating function $P(x,y,z)$ is given by*

$$P(x,y,z) = \frac{C(y)}{1 - yz^2 \cdot N(x,yz^2) - yz \cdot C(y)}.$$

Proof. Let us assume that $m \geq 1$ and $n \geq \lceil m/2 \rceil$ hold. By replacing k with $n-1-k$ in the second sum of the recurrence stated in Proposition 5.4, we have the following equivalent recurrence:

$$p_{n,m}(x) = \sum_{k=0}^{\lfloor (m-2)/2 \rfloor} N_k(x) \cdot p_{n-k-1,m-2k-2}(x) + \sum_{k=0}^{n-1-\lceil (m-1)/2 \rceil} p_{n-k-1,m-1}(x) \cdot C_k.$$

Multiplying both sides with $y^n z^m$, we obtain

$$y^n z^m \cdot p_{n,m}(x) = \sum_{k=0}^{\lfloor (m-2)/2 \rfloor} N_k(x) \cdot (yz^2)^{k+1} \cdot y^{n-k-1} z^{m-2k-2} \cdot p_{n-k-1,m-2k-2}(x)$$
$$+ \sum_{k=0}^{n-1-\lceil (m-1)/2 \rceil} C_k \cdot y^k \cdot yz \cdot y^{n-k-1} z^{m-1} \cdot p_{n-k-1,m-1}(x).$$

Summing over all pairs (n,m) satisfying $m \geq 1$ and $n \geq \lceil m/2 \rceil$ we obtain all terms of $P(x,y,z)$ except for those with $m=0$. The sum of these terms is $\sum_{n \geq 0} C_n y^n = C(y)$. Hence we obtain

$$P(x,y,z) = C(y) + \left(yz^2 \cdot N(x,yz^2) + yz \cdot C(y)\right) \cdot P(x,y,z).$$

Solving this equation for $P(x,y,z)$ yields the expression stated in the theorem. □

As a consequence of Theorem 5.5 and equation (3.7) we may write $G_j(x,t)$ as

$$G_j(x,t) = \sum_{n\geq 0} p_{n-j,n}(x)\cdot t^n = \sum_{n\geq 0}[y^{n-j}z^n]P(x,y,z)\cdot t^n = \sum_{n\geq 0}[y^n z^n]y^j\cdot P(x,y,z)\cdot t^n.$$

In particular, substituting $j=0$ yields $G_0(x,t) = \sum_{n\geq 0}[y^n z^n]P(x,y,z)\cdot t^n$. Here we use the notation $[\cdot]$ to denote extracting the coefficient from the expression that follows.

6. The toric g-polynomial of the associahedron

We now turn our attention to the associahedron. The n-dimensional associahedron is a well-studied simple polytope where the number of vertices is given by the Catalan number C_{n+1}. The polytope first discovered by Tamari and rediscovered by Stasheff. There are many ways to realize this polytope. For a survey we refer to the article by Ceballos, Santos and Ziegler [7].

Postnikov, Reiner and Williams computed the γ-vector of the n-dimensional associahedron [25, Proposition 11.14].

Proposition 6.1 (Postnikov–Reiner–Williams)**.** *The jth entry of the γ-vector of the n-dimensional associahedron given by*

$$\gamma_{n,j} = C_j\cdot\binom{n}{2j}.$$

Hence the generating function of the gamma polynomial of the n-dimensional associahedron is

$$\begin{aligned}\sum_{n\geq 0}\gamma_n(x)\cdot t^n &= \sum_{n\geq 0}\sum_{j=0}^{\lfloor n/2\rfloor} C_j\cdot\binom{n}{2j}\cdot x^j\cdot t^n = \sum_{j\geq 0} C_j\cdot x^j\cdot\sum_{n\geq 2j}\binom{n}{2j}\cdot t^n \\ &= \sum_{j\geq 0} C_j\cdot\frac{x^j t^{2j}}{(1-t)^{2j+1}} = \frac{1}{(1-t)}\cdot C\left(\frac{xt^2}{(1-t)^2}\right),\end{aligned}$$

where $C(u)$ is the generating function of the Catalan numbers. Using expression (2.1), the above equation simplifies to

$$\sum_{n\geq 0}\gamma_n(x)\cdot t^n = \frac{1-t-\sqrt{(1-t)^2-4xt^2}}{2xt^2}.$$

Lemma 6.2. *The toric g-polynomial of the associahedron is given by*

$$g_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\cdot(x-1)^k\cdot\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}\cdot C_j\cdot C_{n-k-j}.$$

n \ j	0	1	2	3	4
1	1				
2	1	2			
3	1	10			
4	1	37	10		
5	1	126	105		
6	1	422	714	70	
7	1	1422	4032	1176	
8	1	4853	20628	11928	588

TABLE 1. The toric g-vector of the associahedron up to dimension 8.

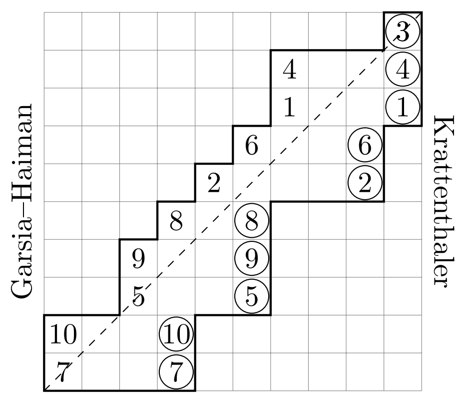

FIGURE 4. The pair of Dyck paths representing the parking function $(7, 5, 10, 7, 3, 6, 1, 4, 3, 1)$.

Proof. We have

$$\begin{aligned} g_n(x) &= \sum_{j=0}^{\lfloor n/2 \rfloor} C_j \cdot \binom{n}{2j} \cdot g_{n,j}(x) \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} C_j \cdot \binom{n}{2j} \cdot \sum_{k=0}^{\lfloor n/2 \rfloor} C_{n-k-j} \cdot \binom{n-k}{k} \cdot (x-1)^k \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \cdot (x-1)^k \cdot \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \cdot C_j \cdot C_{n-k-j}. \end{aligned}$$

□

The toric g-polynomial of the associahedron has a combinatorial interpretation in terms of the ascent statistics of 123-avoiding parking functions. As reported in [1], the number of all 123-avoiding parking functions was first computed by Qiu and Remmel [27]. We refine Qiu's formulas [26] to the following result.

Theorem 6.3. *The coefficient of x^k in the toric g-polynomial of the n-dimensional associahedron is the number of* 123*-avoiding parking functions on the set* $[n]$ *having exactly k ascents.*

Proof. The key tools are the Garsia–Haiman and the Krattenthaler bijections presented in Section 2. Observe that we have combined the two pictures in Figures 1 and 2 into Figure 4. We begin by characterizing 123-avoiding parking functions in terms of their Garsia–Haiman representation.

Given a parking function $f : [n] \longrightarrow [n]$, the Garsia–Haiman bijection yields the pair (π, v). It is almost an immediate consequence of the construction and the definitions that the function f is 123-avoiding if and only if the permutation π avoids the pattern 123: there is no triple $1 \leq i_1 < i_2 < i_3 \leq n$ satisfying $\pi(i_1) < \pi(i_2) < \pi(i_3)$. Indeed, the existence of such a triple (i_1, i_2, i_3) is equivalent to stating that the horizontal coordinates of the labels $\pi(i_1)$, $\pi(i_2)$ and $\pi(i_3)$ are weakly increasing which is the same as having $f(\pi(i_1)) \leq f(\pi(i_2)) \leq f(\pi(i_3))$. Before moving on to the discussion of the other Dyck path in Figure 4, let us note that the permutation π being 123-avoiding precludes the presence of the factor UUU in the Dyck word v.

Since the permutation π is 123-avoiding, we apply the Krattenthaler bijection to obtain the Dyck path below $y = x$ encoded by the Dyck word w. Hence we have the pair of Dyck paths (v, w).

Not any UUU-avoiding Dyck path above the line $y = x$ can be paired with any Dyck path below it due to the following requirement: whenever $f(\pi(i)) = f(\pi(i+1))$ holds, we must have $\pi(i) < \pi(i+1)$. This follows because in the Garsia–Haiman encoding of a parking function the labels with the same horizontal coordinate must increase in the vertical order. After observing that $\pi(i)$ labels the ith letter U in the Dyck word v, we obtain the following *compatibility criterion*: if U_i and U_{i+1} are consecutive letters of the Dyck word v then D_i and D_{i+1} must be consecutive letters in the Dyck word w immediately preceded by a letter U. In other words, if $U_i U_{i+1}$ is a factor in the Dyck word v, then $U D_i D_{i+1}$ must be a factor in the Dyck word w. Indeed, the index i is an ascent of a 123-avoiding permutation π if and only if $\pi(i)$ is a left-to-right minimum of π and $\pi(i+1)$ is not. Conversely, if the compatibility criterion is satisfied, a permutation π encoded by the Dyck word w provides a valid labeling for the compatible Dyck word v, yielding the Garsia–Haiman encoding of a 123-avoiding parking function.

Fix j as the number of factors $U_i U_{i+1}$ in the Dyck word v. By Lemma 2.1 part (a) we know that the number of such Dyck words is $C_j \cdot \binom{n}{2j}$. This is also the entry γ_j in the γ-vector of the associahedron. To complete the proof using Proposition 4.5 we only need to make the following observation: i is an ascent of the parking function f if and only if it is an ascent of the *inverse* of the permutation π. Indeed $f(i) \leq f(i+1)$ is equivalent to stating that i precedes $i+1$ in the permutation π which is equivalent to $\pi^{-1}(i) < \pi^{-1}(i+1)$. In terms of the Dyck word w, the label i must precede the label $i+1$ on a D step: this is only possible if i is a left-to-right minimum in π and the left-to-right minimum immediately preceding i is not $i+1$. Equivalently, the UD factor whose D step is labeled i is immediately preceded by a letter U. The converse is also true, hence the number of ascents of f is the same as the number of factors UUD in the Dyck path encoding π.

The statement now follows from Proposition 4.5. □

$n \backslash j$	0	1	2	3	4
1	1				
2	1	3			
3	1	16			
4	1	65	20		
5	1	246	225		
6	1	917	1659	175	
7	1	3424	10192	3136	
8	1	12861	56664	34104	1764

TABLE 2. The toric g-vector of the cyclohedron up to dimension 8.

7. The toric g-polynomial of the cyclohedron

The n-dimensional cyclohedron is a type B version of the associahedron. The cyclohedron was constructed as combinatorial object by Bott and Taubes [3]. Polytopal realizations were given by Markl [22], Simion [32] and the authors [10]. Postnikov, Reiner and Williams computed the γ-vector of the n-dimensional cyclohedron [25, Proposition 11.15].

In this section we give a combinatorial interpretation of the toric g-vector of the cyclohedron in terms of 123-avoiding *functions*. The computation has a similar favor for the cyclohedron to that of the associahedron.

Proposition 7.1 (Postnikov–Reiner–Williams). *The jth entry of the γ-vector of the n-dimensional cyclohedron given by*

$$\gamma_{n,j} = \binom{2j}{j} \cdot \binom{n}{2j}.$$

Lemma 7.2. *The generating function for the γ-polynomials of the cyclohedra is given by*

$$\sum_{n \geq 0} \gamma_n(x) \cdot t^n = \frac{1}{\sqrt{(1-t)^2 - 4xt^2}}.$$

Proof. The computation is as follows:

$$\begin{aligned}\sum_{n \geq 0} \gamma_n(x) \cdot t^n &= \sum_{n \geq 0} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{2j}{j} \cdot \binom{n}{2j} \cdot x^j \cdot t^n = \sum_{j \geq 0} \binom{2j}{j} \cdot x^j \cdot \sum_{n \geq 2j} \binom{n}{2j} \cdot t^n \\ &= \sum_{j \geq 0} \binom{2j}{j} \cdot \frac{x^j t^{2j}}{(1-t)^{2j+1}} = \frac{1}{(1-t) \cdot \sqrt{1 - 4\frac{xt^2}{(1-t)^2}}} \\ &= \frac{1}{\sqrt{(1-t)^2 - 4xt^2}}.\end{aligned}$$

□

We have the following combinatorial interpretation for the toric g-vector of the cyclohedron.

Theorem 7.3. *The coefficient of x^k in the toric g-polynomial of the cyclohedron is the number of 123-avoiding functions $f : [n] \longrightarrow [n]$ having exactly k ascents.*

Proof. The proof idea is analogous to the argument for Theorem 6.3. The only difference is that the Garsia–Haiman lattice path does not need to be a Dyck path. Instead it is a lattice path described in Lemma 2.1 part (b). This lemma states that the number of such paths is $\binom{n}{j} \cdot \binom{2j}{j}$, which is the γ-vector of the cyclohedron. □

8. Parking trees

In this section we introduce *parking trees* and characterize that represent 123-avoiding parking functions. The idea of a parking tree is suggested in an exercise due to Sagan [28, Ch 1. Exercise (32)(c)]. A related approach to the same idea can be found in [9, Section 8] and in [18].

Definition 8.1. *A* parking tree *is a bilabeled rooted plane tree on $n+1$ vertices whose vertices, respectively edges, are bijectively labeled with the elements of the set $[n+1]$, respectively $[n]$, subject to the following conditions:*

(1) The labeling on the vertices is increasing*: the label of each vertex is less than the label of any of its children.*
(2) If i and j satisfying $i < j$ are labels of edges connecting the same parent vertex to different children then the edge labeled i lies to the left *of the edge labeled j.*

Using Exercise 1.63 in [34] we note that there are $n!^2$ parking trees on $n+1$ nodes.

Each parking tree may be used to encode a parking function using the fact that every edge in a rooted plane tree connects a parent vertex to its child.

Definition 8.2. *Given a parking tree on $n+1$ vertices, we associate a parking function $f : [n] \longrightarrow [n]$ to it as follows. For each $i \in [n]$ we set $f(i)$ to be the label of the parent vertex incident to the edge labeled i.*

Definition 8.3. *We define the* depth-first search*, respectively* breadth-first search labeling*, of the vertices of a rooted plane tree as follows.*

(1) The root of the tree is labeled 1.
(2) Assume we have assigned the labels 1 through j.
  - *For depth-first search, let v be the vertex with the largest label that has at least one unlabeled child.*
  - *For breadth-first search, let v be the vertex with the smallest label that has at least one unlabeled child.*

  Label the leftmost unlabeled child of v by $j+1$.

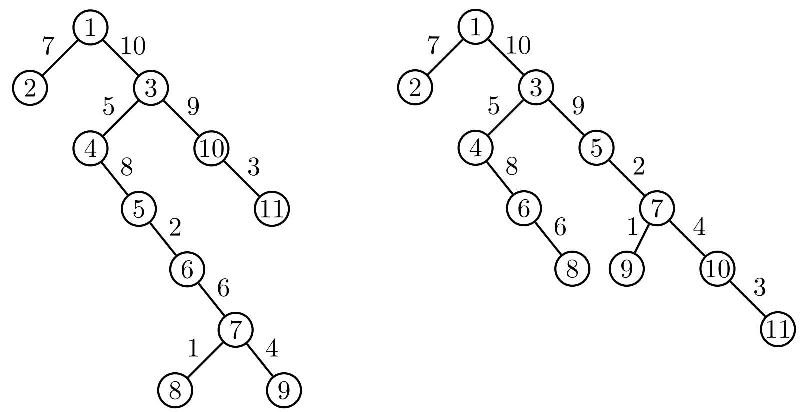

FIGURE 5. The depth-first search and breadth-first search parking trees of the same parking function $(7, 5, 10, 7, 3, 6, 1, 4, 3, 1)$.

A pair of parking trees encoding the same parking function is shown in Figure 5. The tree on the left has the depth-first search labeling of the nodes, whereas the tree on the right has the breadth-first search labeling.

Remark 8.4. Textbooks usually define the depth-first search and breadth-search process for a connected graph in general, and use it to build a spanning tree rooted at the starting point of the process. In such a description of the breadth-first search process, the notion of the *level* is important: the starting vertex of the process has level 0, its neighbors have level 1, and so on, all vertices that are added as neighbors of a vertex of level i have level $i + 1$. We will not use this notion of level, but it is worth noting that the breadth-first labeling process we defined in Definition 8.3 labels the vertices of a plane tree level by level, where the level of a vertex is simply its distance from the root. Finally, it should be emphasized that our processes number the children of each vertex in the left-to-right order and the resulting labeled tree can be drawn in exactly one way in the plane such that this condition is satisfied. This aspect of our process is not part of the general definition of a depth-first search or a breadth-first search process as the graph considered may not even be planar.

The trees shown in Figure 5 are parking trees of the parking function whose Garsia–Haiman representation is shown in Figure 2. If we read off the labels on the parking tree in the breadth-first search order, we obtain the same permutation $\pi = (7, 10, 5, 9, 8, 2, 6, 1, 4, 3)$ that is represented by the second Dyck path in Figure 2.

Definition 8.5. *Consider a parking tree on $n+1$ vertices. We define its* edge labeling permutation π *as the list of labels on the edges in the following order:*

(1) If $i < j$ holds then the edges connecting the vertex labeled i to its children precede the labels of the edges connecting the vertex labeled j to its children.
(2) The labels of the edges connecting the same vertex to its children are listed in left-to-right (that is, increasing) order.

It is direct from the definitions that the edge labeling permutation of a parking tree is identical to the permutation associated to the encoded parking function. We only need to show that Definition 8.2 always defines a parking function.

This fact is part of the main result in this section:

Theorem 8.6. *Each parking tree on $n+1$ vertices encodes a parking function on $[n]$. Conversely, every parking function may be represented by a parking tree in at least one way. Furthermore, the correspondence between parking functions of $[n]$ and those parking trees on $n+1$ vertices that are labeled in the depth first (breadth-first) search order, is a bijection.*

Proof. By definition, each parking tree on $n+1$ vertices encodes a function $f : [n] \longrightarrow [n]$. (Note that $n+1$ is not in the range of f: it labels a vertex with no children due to the fact that the labeling of the vertices is increasing.) For each $i \leq n$, the set $f^{-1}([i])$ is the set of edge labels whose parent vertex is labeled with an element of $[i]$. The corresponding edges form the edge set of a tree rooted at the vertex labeled 1: this is a consequence of the fact that the labeling of the vertices is increasing. Not all vertices of this tree are labeled with elements of the set $[i]$ because the vertex labeled $i+1$ is the child vertex of an edge that belongs to this tree. Hence the tree has at least $i+1$ vertices and at least i edges. Therefore $|f^{-1}([i])| \geq i$ holds for each $i \leq n$.

To prove the converse it suffices to show that there is a unique parking tree whose vertices are labeled in the depth-first (breadth-first) search order that encodes a given parking function. We prove the statement for the depth-first search labeling. The breadth-first search variant is completely analogous and left to the reader. Consider a parking function $f : [n] \longrightarrow [n]$ corresponding to the pair (π, v) via the Garsia–Haiman bijection, described in Subsection 2.3. Here the associated permutation π is the ordered list of the elements of $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(n)$, listed in this order such that the elements belonging to the same fiber are listed in increasing order, and v is the Dyck word $v = U^{q_1}DU^{q_2}D \cdots U^{q_n}D$ in which $q_i = |f^{-1}(i)|$ for $i = 1, 2, \ldots, n$. A parking tree encodes this parking function if and only if for each $i \in [n]$ the vertex labeled i satisfies the following two conditions:

(1) the number of children of the vertex is $q_i = |f^{-1}(i)|$;
(2) the edge labeling permutation π of the parking tree is the same as the permutation π associated to f by the Garsia–Haiman bijection.

Clearly, if a rooted plane tree whose vertices are labeled in an increasing order satisfies Condition (1) then there is only one way to label the edges in such a way that the labeling satisfies Condition (2). Keeping in mind that our balanced words $U^{q_1}DU^{q_2}D \cdots U^{q_n}D$ of length $2n$ correspond bijectively to Łukasziewicz words $f_{q_1}f_{q_2} \cdots f_{q_n}f_0$ of length $n+1$, our statement is a consequence of the well-known result stating that for each Łukasziewicz word $f_{q_1}f_{q_2} \cdots f_{q_n}f_0$ of length $n+1$ there is a unique rooted plane tree whose vertex labeled i in the depth-first search order has q_i children for $i = 1, 2, \ldots, n$. A proof of this statement may be found [11, Section I.5.3]. The analogous statement for the breadth-first search order may be shown in a similar fashion, see Proposition A.1 in the Appendix. □

Next we characterize the parking trees of 123-avoiding parking functions.

Proposition 8.7. *A parking tree on $n+1$ vertices represents a 123-avoiding parking function if and only if the following criteria are satisfied:*

(1) Each vertex has at most two children.
(2) The edge labeling permutation π is 123-avoiding.

The proof is a direct consequence of the definitions.

Definition 8.8. *A* plane 0-1-2 tree *is a rooted plane tree such that each vertex has at most two children. A vertex of a rooted plane tree is a* fork *if it has more than one child.*

Observe that a plane 0-1-2 tree has one more leaf than forks. As noted in Lemma 2.1 part (b) and in the proof of Theorem 6.3, the entry $\gamma_j = C_j \cdot \binom{n}{2j}$ in the γ-vector of the associahedron is the number of Motzkin paths of length n with j up steps. Next we observe that this number is also the number of 0-1-2 trees with j forks, by specializing the operation associating a Łukasziewicz word to each parking function.

Definition 8.9. *Let T be a plane 0-1-2 tree with $n+1$ vertices labeled in some increasing order. We define the* associated Motzkin word w_T *as the word $x_1 x_2 \cdots x_n$ in which*

$$x_i = \begin{cases} U & \text{if the vertex labeled } i \text{ is a fork,} \\ D & \text{if the vertex labeled } i \text{ is a leaf,} \\ H & \text{otherwise.} \end{cases}$$

Proposition 8.10. *Let T be a plane 0-1-2 tree having $n+1$ vertices labeled in an increasing order. Then the associated Motzkin word w_T encodes a Motzkin path of length n where each letter U represents an up step, each letter D represents a down step and each letter H represents a horizontal step.*

Proof. Let us label the edges of T with the elements of $[n]$ bijectively in such a way that the labels on edges incident to the same parent vertex increase left to right. This transforms T into a parking tree encoding a parking function $f : [n] \longrightarrow [n]$ such that for each $i \in [n]$ the number $q_i = |f^{-1}(i)|$ is the number of children of the vertex labeled i. Hence the associated Łukasziewicz word $f_{q_1} f_{q_2} \cdots f_{q_n} f_0$ satisfies $f_{q_i} \in \{0, 1, 2\}$ for each $i \in [n]$. After deleting the last letter f_0, replacing each f_2 with U, each f_1 with H and each f_0 with D, we obtain the associated Motzkin word that encodes a Motzkin path of length n. □

In light of Theorem 6.3, Proposition 4.5 may be rephrased as follows.

Definition 8.11. *Let B be a sparse subset of $[n-1]$ and let T be a plane 0-1-2 tree on $n+1$ vertices, labeled in an increasing order. Let us turn T into a parking tree by using the identity permutation of $[n]$ as the edge labeling permutation π. We say that the vertex-labeled tree T has* sibling type B *if the numbers b and $b+1$ label edges sharing the same parent vertex if and only if $b \in B$ holds.*

Corollary 8.12. *Let B be a sparse subset of $[n-1]$ and let T be a plane 0-1-2 tree on $n+1$ vertices, labeled in an increasing order. The coefficient of x^k in $g_{n,|B|}(x)$ is the number of parking trees obtained by adding an edge labeling to T in such a way that the encoded parking function is 123-avoiding and has exactly k ascents.*

Corollary 8.12 motivates the question of determining whether two vertex-labeled plane 0-1-2 trees have the same sibling type.

Proposition 8.13. *Let T and T' be two plane 0-1-2 trees on $n+1$ vertices labeled in an increasing order. These vertex-labeled trees then have the same number of vertices having the same sibling type if and only if, after deleting all letters D from their associated Motzkin words w_T and $w_{T'}$, the two words become identical.*

Indeed, as seen in the proof of Proposition 8.10, the Motzkin word w_T carries the same information as the associated Łukasziewicz word $f_{q_1} f_{q_2} \cdots f_{q_n} f_0$ in which $q_i = 0$ corresponds to the vertex labeled i having no children. As we list the edges whose parent vertex is labeled 1, 2, and so on, and label them consecutively with the elements of $[n]$ in increasing order, no edge is labeled when we read a letter D in the Motzkin word that corresponds to an f_0 in the Łukasziewicz word.

9. The toric g-polynomial of the permutahedron

The n-dimensional *permutahedron* is the convex hull of the $(n+1)!$ vectors $(\pi_1, \pi_2, \ldots, \pi_{n+1})$ in $\mathbb{R}^{n+1}$ where $(\pi_1, \pi_2, \ldots, \pi_{n+1})$ ranges over all permutations in the symmetric group $\mathfrak{S}_{n+1}$. Note that this polytope lies in the hyperplane $x_1 + x_2 + \cdots + x_{n+1} = \binom{n+2}{2}$. An alternative definition is that the permutahedron is the Minkowski sum of the line segments $[e_i, e_j]$ for $1 \leq i < j \leq n$.

It is well-known that the h-polynomials of the permutahedra are the Eulerian polynomials [25, Eq. (4)]. For the n-dimensional permutahedron we have

$$\sum_{i=0}^{n} h_i \cdot x^i = \sum_{\pi \in \mathfrak{S}_{n+1}} x^{\mathrm{des}(\pi)}.$$

As pointed out in [25], the γ-vector of the permutahedron was first described in terms of permutation statistics by Shapiro, Getu and Woan [29]. In [25, Theorem 11.1] Postnikov, Reiner and Williams rephrased [29, Proposition 4] as follows.

Theorem 9.1 (Shapiro–Getu–Woan)**.** *The γ-polynomial of the n-dimensional permutahedron is given by*

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_{n,j} \cdot x^j = \sum_{\pi \in \widehat{\mathfrak{S}}_{n+1}} x^{\mathrm{des}(\pi)},$$

where $\widehat{\mathfrak{S}}_{n+1}$ denotes the set of permutations of the set $[n+1]$ not containing a final descent or a double descent.

The technique of slides used to prove their formulas was generalized to chordal nestohedra in [25]; see Section 10. As pointed out in [4], the $\mathbb{Z}_2^{n-1}$ action introduced in [29] is the same as Brändén's modified Foata–Strehl group action discussed in Subsection 2.4. Recall that this action *reverses* the labeling on the associated Foata–Strehl trees, turning them into decreasing trees. On the other hand, the actions of *hops* in [25] define precisely the restricted Foata–Strehl action. As noted in Subsection 2.4, we may choose the orbit representatives to be the right-adjusted Foata–Strehl trees. By Corollary 2.11 these are exactly the Foata–Strehl trees representing permutations having no double descents and no final descent. Using Definition 2.7 we may identify these with the labeled plane increasing 0-1-2 trees. This is how they are called in [5]. As stated in [5], $\gamma_{n,j}$ is the number of labeled plane increasing 0-1-2 trees on $n+1$ vertices having j forks.

The γ-vector entries of the permutahedron are given as sequence A101280 in OEIS [24]. Introducing $\gamma_{n,j}$ for the jth entry of the γ-vector of the n-dimensional permutahedron, these entries satisfy the initial condition $\gamma_{1,0} = 1$ and the following recurrence:

$$\gamma_{n,j} = (j+1) \cdot \gamma_{n-1,j} + (2n+2-4j) \cdot \gamma_{n-1,j-1}, \tag{9.1}$$

for $j \leq n/2$.

A generating function formula for the γ-vector may be found using the calculation in [12, Section 7]. Note that the numbers $D_{n,k}$, whose generating function Foata and Han computed, count those orbits of the Foata–Strehl group action where each tree has k nonroot leaves. The modified Foata–Strehl group action has smaller orbits. To obtain the γ-vector entries we need to multiply each $D_{n,k}$ by the appropriate power of 2. Subject to these modifications one may derive the following result; see sequence A101280 in OEIS [24].

Proposition 9.2 (Peter Bala)**.** *Introducing* $r(x) = \sqrt{1-4x}$ *and* $w(x) = \frac{1-r(x)}{1+r(x)}$*, we have*

$$\sum_{n \geq 0} \gamma_n(x) \cdot \frac{t^{n+1}}{(n+1)!} = \frac{1}{2x} \cdot \left(r(x) \cdot \frac{1 + w(x) \cdot e^{t \cdot r(x)}}{1 - w(x) \cdot e^{t \cdot r(x)}} - 1 \right).$$

Keeping in mind Corollary 8.12 we can express the toric g-polynomial of the permutahedron in terms of an ascent statistics on parking trees as follows.

Theorem 9.3. *The coefficient of* x^k *in the toric* g*-polynomial of the* n*-dimensional permutahedron is the number of parking trees on* $n+1$ *nodes encoding* 123*-avoiding parking function having exactly* k *ascents.*

Proof. As noted above, $\gamma_{n,j}$ is the number of (vertex-)labeled plane increasing 0-1-2 trees T on $n+1$ vertices having j forks [5]. The tree T has j forks if and only if its sibling type is B for some j-element sparse subset of $[n-1]$. By Corollary 8.12, the coefficient of x^k in $g_{n,|B|}(x)$ is the number of parking trees obtained by adding an edge labeling to T in such a way that the encoded parking function is 123-avoiding and has k ascents. The statement is now a direct consequence of Theorem 3.4. □

The toric g-vectors of the permutahedra up to dimension 8 are listed in Table 3.

The proof of Theorem 9.3 may be adapted to show several analogous results, if we focus on the following key ideas: a parking tree consists of an increasing vertex labeling, and a compatible edge labeling, the latter one may be added after fixing the vertex labeling and summing over all possible edge labelings results in a contribution of $g_{n,j}(x)$ by Corollary 8.12. If there is an expression of γ_j in

n \ j	0	1	2	3	4
1	1				
2	1	3			
3	1	20			
4	1	115	40		
5	1	714	735		
6	1	5033	10101	1225	
7	1	40312	131068	45304	
8	1	362871	1723716	1143996	67956

TABLE 3. The toric g-vector of the permutahedron up to dimension 8. See also [33, Page 195].

the γ-vector of the polytope as a number of vertex-labeled increasing plane 0-1-2 trees then there is an analogous result for the toric g-polynomials.

Postnikov, Reiner and Williams showed [25, Section 10.2] that the entry γ_j in the γ-vector of the n-dimensional associahedron is the number of 312-avoiding permutations in $\mathfrak{S}_{n+1}$ having exactly j descents, no double descents and no final descent. They also note that a permutation is 312-avoiding if and only if the vertices of its Foata–Strehl tree are labeled in the depth-first-search order. Hence, adapting the proof of Theorem 9.3 yields the following result:

Theorem 9.4. *The coefficient of x^k in the gamma polynomial of the n-dimensional associahedron is the number of parking trees whose vertices are labeled in the depth-first search order that encode a* 123-*avoiding parking function having exactly k ascents.*

This statement is equivalent to Theorem 6.3 since by Theorem 8.6 every parking function may be uniquely encoded by a parking tree whose vertices are labeled in the depth-first search order.

10. TORIC g-POLYNOMIALS OF CHORDAL NESTOHEDRA

In this section we review some of the terminology and results of Postnikov, Reiner and Williams [25] and outline how their formula for the γ-vectors of a chordal nestohedra can be extended to results on the toric g-polynomials of these simple polytopes. Some cited facts and definitions also appear in earlier papers. We refer the reader to [25] for a more complete bibliography.

Definition 10.1. *Given a finite set S, a* building set *$\mathcal{B}$ on a set S is a collection of nonempty subsets of S such that*

(1) if $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$ then $I \cup J \in \mathcal{B}$,
(2) $\mathcal{B}$ contains the singleton $\{i\}$ for all $i \in S$.

The building set $\mathcal{B}$ on S is connected *if $S \in \mathcal{B}$. The connected components of a building set $\mathcal{B}$ are the maximal subsets under inclusion that are contained in $\mathcal{B}$. For any subset $T \subseteq S$ we define the* T-restricted building set $\mathcal{B}|_T = \{I \in \mathcal{B} : I \subseteq T\}$.

For a graph G on the vertex set S with no loops and no multiple edges, the graphical building set $\mathcal{B}(G)$ is the family of all nonempty sets $I \subseteq S$ where the induced graph $G|_I$ is connected. Given a connected building set $\mathcal{B}$, we have the associated *nestohedron* $P_{\mathcal{B}}$ defined as the Minkowski sum

$$P_{\mathcal{B}} = \sum_{I \in \mathcal{B}} \Delta_I$$

where Δ_I is the convex hull of the basis vectors e_i satisfying $i \in I$. The nestohedron $P_{\mathcal{B}}$ is known to be a simple polytope. A connected building set $\mathcal{B}$ is *chordal* if, for any of the sets $I = \{i_1 < i_2 < \cdots < i_r\}$ in $\mathcal{B}$, all subsets of the form $\{i_s, i_{s+1}, \ldots, i_r\}$ also belong to $\mathcal{B}$. For chordal building sets the h-vector and the γ-vector of the nestohedron $P_{\mathcal{B}}$ may be expressed in terms of descent statistics of *$\mathcal{B}$-permutations*, defined as follows [25, Definition 8.7].

Definition 10.2. *The set of $\mathcal{B}$-permutations of a connected building on the set $[n]$ is the set of all permutations π of $[n]$ such that for any $i \in [n]$, the two elements $\pi(i)$ and $\max(\pi(1), \pi(2), \ldots, \pi(i))$ lie in the same connected component of the restricted building set $B|_{\{\pi(1),\ldots,\pi(i)\}}$. We denote the set of $\mathcal{B}$-permutations by $\mathfrak{S}_n(\mathcal{B})$.*

As stated in [25, Corollary 9.6], the h-polynomial of the chordal nestohedra satisfy

$$h_{\mathcal{B}}(t) = \sum_{w \in \mathfrak{S}_n(\mathcal{B})} t^{\operatorname{des}(w)}. \tag{10.1}$$

Remark 10.3. More generally, Postnikov, Reiner and Williams also have a formula for the h-vector of any nestohedron associated to a connected building set [25, Corollary 8.4]. This formula uses the descent statistics of *$\mathcal{B}$-trees* whose definition we omit here (see Section 8 of [25]). The descent statistics of $\mathcal{B}$-trees and $\mathcal{B}$-permutations coincide for chordal nestohedra [25, Proposition 9.5]. We only wish to highlight that the *$\mathcal{B}$-trees* are unrelated to the trees we will consider in this section.

For our purposes, the most important result of Postnikov, Reiner and Williams is the following identity [25, Theorem 11.6].

Theorem 10.4 (Postnikov–Reiner–Williams)**.** *The γ-polynomial of a chordal nestohedron $P_{\mathcal{B}}$ is given by*

$$\gamma_{\mathcal{B}}(t) = \sum_{w \in \widehat{\mathfrak{S}}_n(\mathcal{B})} t^{\operatorname{des}(w)}, \tag{10.2}$$

where $\widehat{\mathfrak{S}}_n(\mathcal{B})$ is the set of all $\mathcal{B}$-permutations containing no double descents and no final descents.

The proof of this theorem relies on considering a $\mathbb{Z}_2^{n-1}$-action of *$\mathcal{B}$-hops* that is similar to the restricted Foata–Strehl action. In case of the permutahedron, the $\mathcal{B}$-hops coincide with the restricted Foata–Strehl action. For the definition and detailed study of the $\mathcal{B}$-hops, we refer the reader to [25, Section 11]. For us it suffices to observe the following direct consequence of equation (10.2).

Corollary 10.5. *In the γ-vector of a chordal nestohedron $P_{\mathcal{B}}$ the entry γ_j is the number of right-adjusted Foata-Strehl trees representing a $\mathcal{B}$-permutation and having j forks.*

Using Corollary 10.5 we obtain the following generalization of Theorem 9.3.

Theorem 10.6. *Let $\mathcal{B}$ be a connected chordal building set on $[n+1]$. The coefficient of x^k in the toric g-polynomial of the nestohedron $P_{\mathcal{B}}$ is the number of parking trees T satisfying the following two conditions:*

(1) T encodes a 123-avoiding parking function $f : [n] \longrightarrow [n]$ having exactly k ascents.
(2) Removing the labeling on the edges of T results in the right-adjusted Foata–Strehl tree of a $\mathcal{B}$-permutation having no double descents and no final descent.

In the above statement we transform each 0-1-2 tree into a right-adjusted Foata–Strehl tree by declaring the only child of a parent to be a right child.

The proof is a straightforward adaptation of the proof of Theorem 9.3 and therefore omitted.

Example 10.7. The n-dimensional cube is combinatorially equivalent to the *Stanley–Pitman polytope* of the same dimension. As shown in [25, Subsection 10.5], the Stanley–Pitman polytope is a chordal nestohedron $P_{\mathcal{B}}$, where the building set is given by

$$\mathcal{B} = \{\{i\}, [i, n+1] : 1 \leq i \leq n+1\}.$$

The $\mathcal{B}$-permutations are exactly the permutations $\pi \in \mathfrak{S}_{n+1}$ satisfying $\pi(1) < \pi(2) < \cdots < \pi(k) > \pi(k+1) > \cdots > \pi(n+1)$ for some $k \in [n+1]$. The only $\mathcal{B}$-permutation with no double descent and no final descent is the identity permutation. Its Foata-Strehl tree is the right-adjusted path 1–2–$\cdots$–$(n+1)$. Labeling the edges amounts to selecting a permutation τ of $[n]$ where $\tau(i)$ is the label of the edge $(i, i+1)$. The encoded parking function is 123-avoiding if and only if the permutation τ is 123-avoiding. Hence we recover Corollary 4.6. Keeping Remark 10.3 in mind, notice that all Foata–Strehl trees associated to $\mathcal{B}$-permutations are paths, whereas $\mathcal{B}$-trees T_I defined in [25, Subsection 10.5] have definitely more than one path when a proper subset I of $[n]$ is considered.

Example 10.8. The associahedron is discussed in [25, Subsection 10.5], where it is represented as a chordal nestohedron $P_{\mathcal{B}}$ with the building set given by

$$\mathcal{B} = \{[i, j] : 1 \leq i \leq j \leq n+1\}.$$

In this case $\mathcal{B}$-permutations are exactly the 312-avoiding permutations. It is easy to check, and left to the reader, that a permutation is 312-avoiding if and only if the vertices of the corresponding Foata–Strehl tree are labeled in the depth-first search order. In the proof of Theorem 8.6 we established that every parking function has a unique parking tree representation such that the vertices are labeled in the depth-first search order. Parking trees of 123-avoiding parking functions are plane 0–1–2 trees which we may identify with right-adjusted Foata–Strehl trees. Hence we recover Theorem 6.3. Keeping Remark 10.3 in mind, it may be somewhat confusing that the $\mathcal{B}$-trees coincide with the Foata–Strehl trees of $\mathcal{B}$-permutations in this case.

Example 10.9. The permutahedron is discussed in [25, Subsection 10.1]. Here the building set is

$$\mathcal{B} = 2^{[n+1]} - \{\emptyset\} = \{I \subseteq [n+1] : I \neq \emptyset\}.$$

In this case every permutation is a $\mathcal{B}$-permutation and Theorem 10.6 specializes to Theorem 9.3. Keeping Remark 10.3 in mind, here $\mathcal{B}$-trees are linear orders (labeled paths), whereas all Foata–Strehl trees appear in this example.

Question 10.10. Consider the graph on the vertex set $[n+1]$ where (i, j) is an edge if $\max(i, j) \geq r+1$. The edge set of this graph is $E(K_{n+1}) - E(K_r)$ where the subgraph K_r is on the vertex set $[r]$. The associated building set is chordal and is given by

$$\mathcal{B} = \{\{i\} : i \in [r]\} \cup \{I \subseteq [n+1] : \max(I) \geq r+1\}.$$

The associated nestohedra interpolate between the stellohedron when $r = n$ (see [2, Subsection 3.5.4], [25, Subsection 10.4] and [30, Section 5]) and the permutahedron when $r = 1$. A permutation π is a $\mathcal{B}$-permutation if its longest initial segment of elements less than or equal to r is increasing. By removing this initial segment, we have a bijection between the set of $\mathcal{B}$-permutations and *partial permutations* τ of $[n+1]$ where the "missing elements" of τ are from the set $[r]$. For example, for $n = 8$, $r = 6$ and $\pi = (1, 5, 7, 4, 9, 2, 6, 3, 8)$ we have $\tau = (7, 4, 9, 2, 6, 3, 8)$. Comparing the Foata–Strehl tree of π with the Foata–Strehl tree of τ, the latter one is obtained from the former by removing the right-adjusted path 1–5–7. Are there explicit expressions for the γ- and the toric g-vectors of the associated nestohedron?

11. Concluding remarks

By a result of Billera, Chan and Liu [6] (see also Corollary 4.11) the coefficients in the toric g-polynomial of the cube are face numbers of a simplicial complex. Can this result be generalized to other simple polytopes? As stated in Conjecture 1.1, we suspect that the answer is "yes": perhaps one can combine the proof of Theorem 9.3 and the constructions of Nevo and Petersen [23] who proved that the γ-vectors of several simple polytopes are f-vectors of simplicial complexes.

Besides Conjecture 1.1 that we suspect to hold for structural reasons, strong numerical evidence supports the following conjecture.

Conjecture 11.1. *(a) The toric g-contribution polynomials $g_{n,j}(x)$ for $0 \leq j \leq \lfloor n/2 \rfloor$ are real-rooted.*
(b) The g-polynomials of the n-dimensional cyclohedron and chordal nestohedra are also real-rooted.

Acknowledgments

The second author thanks the Department of Mathematics at the University of Kentucky for research visits. This work was partially supported by grants from the Simons Foundation (#429370 to Richard Ehrenborg, #245153 and #514648 to Gábor Hetyei), and by the NSF grant DMS-2247382 to Margaret Readdy.

Appendix A. Łukasziewicz words

Proposition A.1. *Given a Łukasziewicz word $f_{q_1} f_{q_2} \cdots f_{q_n} f_0$ of length $n+1$, there is a unique rooted plane tree on $n+1$ vertices whose vertices labeled in the breadth-first search order have the property that the number of children of the vertex labeled i is q_i.*

Proof. We show the statement by analyzing the breadth-first search labeling process. For each $0 \le i \le n$ let $m(i)$ be the number of vertices labeled after visiting the first i vertices and labeling their children. At the beginning of the process only the root vertex is labeled 1, but not yet visited, hence we set $m(0) = 1$ for the empty word ε. The quantity $m(i)$ is given by adding the number of children of each visited vertex, that is, we have

$$m(i) = 1 + w(f_{q_1}) + \cdots + w(f_{q_i}) \quad \text{where } w(f_q) = q - 1. \tag{A.1}$$

After visiting the first i vertices, the number of vertices already labeled but not yet visited is the difference $m(i) - i$. The labeling process can continue only if this number is positive at each stage of the process. Furthermore, at the end of the process we must have $m(n) = n+1$, that is, $m(n) - n = 1$. Conversely, if the numbers $m(i)$ satisfy the above mentioned conditions, we can use the breadth-first search labeling process to reconstruct our rooted plane tree in a unique fashion. Let us define the *level* $\ell(f_{q_1} \cdots f_{q_i})$ of the word $f_{q_1} \cdots f_{q_i}$ by setting

$$\ell(f_{q_1} \cdots f_{q_i}) = m(i) - i - 1.$$

By (A.1) and the above reasoning, the word $f_{q_1} \cdots f_{q_n} f_0$ encodes a rooted plane tree if and only if $\ell(f_{q_1} \cdots f_{q_i}) \ge 0$ holds for $i \in [n]$ and we have $\ell(f_{q_1} \cdots f_{q_n} f_0) = 0$. This is precisely the condition requiring that the word is a Łukasziewicz word of length $n+1$. □

References

[1] A. Adeniran and L. Pudwell, Pattern avoidance in parking functions, *Enumer. Comb. Appl.* **3** (2023), Paper No. S2R17, 21 pp.

[2] N. Aisbett, "Gamma-polynomials of flag homology spheres," University of Sydney, 2013.

[3] R. Bott and C. Clifford, On the self-linking of knots, *J. Math. Phys.* **35** (1994), 5247–5287.

[4] P. Brändén, Actions on permutations and unimodality of descent polynomials, *European J. Combin.* **29** (2008), 514–531.

[5] F. Bergeron, P. Flajolet and B. Salvy, Varieties of increasing trees, CAAP '92 (Rennes, 1992), 24–48, Lecture Notes in Comput. Sci. **581**, Springer, Berlin, 1992.

[6] L. J. Billera, C. S. Chan and N. Liu, Flag complexes, labelled rooted trees, and star shellings, Advances in Discrete and Computational Geometry (South Hadley, MA, 1996), 91–102. Contemp. Math. **223** American Mathematical Society, Providence, RI, 1999.

[7] C. Ceballos, F. Santos and G. Ziegler, Many non-equivalent realizations of the associahedron, *Combinatorica* **35** (2015), 513–551.

[8] A. Denise and R. Simion, Two combinatorial statistics on Dyck paths, *Discrete Math.* **137** (1995), 155–176.

[9] R. Ehrenborg and G. Hetyei, On the structure of the d-indivisible noncrossing partition posets, `arXiv:2407.08577 [math.CO]`.

[10] R. Ehrenborg, G. Hetyei and M. Readdy, Simion's type B associahedron is a pulling triangulation of the Legendre polytope, *Discrete Comput. Geom.* **60** (2018), 98–114.

[11] P. Flajolet and R. Sedgewick, "Analytic combinatorics," Cambridge University Press, Cambridge, 2009.

[12] D. Foata and G-N. Han, Arbres minimax et polynômes d'André, *Adv. in Appl. Math.* **27** (2001), 367–389.

[13] D. Foata and V. Strehl, Euler numbers and variations of permutations, in: Colloquio Internazionale sulle Teorie Combinatorie (Roma, 1973), Tomo I, 119–131. Atti dei Convegni Lincei, No. 17, Accad. Naz. Lincei, Rome, 1976.
[14] S. R. Gal, Real root conjecture fails for five- and higher-dimensional spheres, *Discrete Comput. Geom.* **34** (2005), 269–284.
[15] A. M. Garsia and M. Haiman, A remarkable q,t-Catalan sequence and q-Lagrange inversion, *J. Algebraic Combin.* **5** (1996), 191–244.
[16] G. Hetyei, A second look at the toric h-polynomial of a cubical complex., *Ann. Comb.* **16** (2012), 517–541.
[17] G. Hetyei, The short toric polynomial, *Trans. Amer. Math. Soc.* **365** (2013), 1441–1468.
[18] G. Hetyei, Labeling regions in deformations of graphical arrangements, `arXiv:2312.06513 [math.CO], to appear in` *Comb. Theory.*
[19] S. Kitaev, "Patterns in permutations and words," Monogr. Theoret. Comput. Sci. EATCS Ser., Springer, Heidelberg, 2011.
[20] A. G. Konheim and B. Weiss, An occupancy discipline and applications, *SIAM J. Applied Math.* **14** (1966), 1266–1274.
[21] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, Special issue in honor of Dominique Foata's 65th birthday (Philadelphia, PA, 2000), *Adv. in Appl. Math.* **27** (2001), 510–530.
[22] M. Markl, "Simplex, associahedron, and cyclohedron" in Higher homotopy structures in topology and mathematical physics, *Contemp. Math.*, **227** (1999), 235–265.
[23] E. Nevo and T. K. Petersen, On γ-vectors satisfying the Kruskal-Katona inequalities, *Discrete Comput. Geom.* **45** (2011), 503–521.
[24] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.
[25] A. Postnikov, V. Reiner and L. Williams, Faces of generalized permutohedra, *Doc. Math.* **13** (2008), 207–273.
[26] D. Qiu, Patterns in ordered set partitions and parking functions, Permutation Patterns 2016 (slides). Available at `https://www.math.ucsd.edu/~duqiu/files/PP16.pdf`.
[27] D. Qiu and J. Remmel, Patterns in words of ordered set partitions, *J. Comb.* **10** (2019), 433–490.
[28] B. E. Sagan, "Combinatorics: the art of counting," Grad. Stud. Math., **210** American Mathematical Society, Providence, RI, 2020.
[29] L. W. Shapiro, W. J. Woan and S. Getu, Runs, slides and moments, *SIAM J. Algebraic Discrete Methods* **4** (1983), 459–466.
[30] J. Shareshian and M. Wachs, Gamma-positivity of variations of Eulerian polynomials, *J. Comb.* **11** (2020), 1–33.
[31] R. Simion, Noncrossing partitions, Formal Power Series and Algebraic Combinatorics (Vienna, 1997), *Discrete Math.* **217** (2000), 367–409.
[32] R. Simion, A type-B associahedron, *Adv. in Appl. Math.* **30** (2003), 2–25.
[33] R. Stanley, Generalized H-vectors, intersection cohomology of toric varieties, and related results, Commutative algebra and combinatorics (Kyoto, 1985), 187–213. Adv. Stud. Pure Math., 11 North-Holland Publishing Co., Amsterdam, 1987
[34] R. P. Stanley, "Enumerative Combinatorics, Vol. I," Cambridge University Press, Cambridge, 2012.

Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027.
`http://www.math.uky.edu/~jrge/, richard.ehrenborg@uky.edu.`

Department of Mathematics and Statistics, UNC-Charlotte, Charlotte NC 28223-0001.
`http://www.math.charlotte.edu/ghetyei, ghetyei@charlotte.edu.`

Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027.
`http://www.math.uky.edu/~readdy/, margaret.readdy@uky.edu.`