DISCRETE SPECTRUM IN GAPS OF THE CONTINUOUS SPECTRUM FOR FINITE DIFFERENCE OPERATORS ON THE LATTICE AND THE GRAPHENE OPERATOR

by

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ABSTRACT

SIYU GAO. Discrete spectrum in gaps of the continuous spectrum for finite difference operators on the lattice and the graphene operator. (Under the direction of DR. OLEG SAFRONOV)

The primary object of study of this dissertation is the distribution of discrete eigenvalues of operators. Specifically, we investigate the number and location of discrete eigenvalues within the gaps of the continuous spectrum of three representative operators, two of which are discrete and one is continuous. The discrete operators are both defined on the lattice $\mathbb{Z}^d$ and have a potential V. However, one has a positive coupling constant and the other does not. For the first discrete operator, we compute the asymptotics of the number of eigenvalues passing through a fixed point inside a spectral gap when the coupling constant increases from 0 to infinity. For the second discrete operator, we estimate the total number of discrete eigenvalues. This estimate is a generalization of the celebrated CLR inequality in dimension $d \geq 3$. The continuous operator is the bilayer graphene operator acting on the Sobolev space $H^2(\mathbb{R}^2, \mathbb{C}^2)$. This operator is used to describe the two-layer graphene structure. The potential V in this model is multiplied by a coupling constant α. We prove that the eigenvalue asymptotics is contingent upon the integrability of the function V. In particular, if V is not integrable, then the asymptotics is determined by the rate of decay of V at infinity. If V is integrable, then it is determined by the integral of V.

DEDICATION

This dissertation is dedicated to everyone who has helped me throughout this stimulating yet rewarding journey.

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LIST OF ABBREVIATIONS

CLR inequality An acronym for Cwikel-Lieb-Rozenblum inequality.

CHAPTER 1: BACKGROUND

In mathematics, operator is a very common concept. It is defined as a function or mapping that transforms elements of one space (domain) to elements of another space (range). Oftentimes, these two spaces are identical. Many operators have eigenvalues. However, the spectrum of an operator does not have to consist of eigenvalues alone.

The spectrum of an operator T on a Hilbert space refers to the set consisting of all $\lambda \in \mathbb{C}$ for which the operator $T - \lambda I$ is not invertible. Here, I denotes the identity operator. Note that the spectrum does not only include all eigenvalues of the operator, but also contains values of λ at which the operator fails to be surjective or its range is not a closed set, if any. The spectrum of any operator is a closed set. The complement of the spectrum is open and is called the resolvent set. The spectrum of an operator provides objective interpretations of the operator's properties, particularly invertibility and eigen-structure.

Some operators have purely continuous spectrum, like the whole real line $\mathbb{R}$, while the other have purely discrete spectrum, like $\{1, 2, 3, ...\}$. Also, there are operators that have complicated spectra which might be challenging to describe. It turns out that the spectra of a huge number of operators extensively studied in physics consist of two parts - one continuous and one discrete. This is the case that we will be dealing with in the upcoming chapters. Sometimes the continuous spectrum of an operator is the complement of an open and bounded interval of the real line $\mathbb{R}$, and that interval is typically referred to as the spectral gap. Obviously, all the discrete eigenvalues will be located inside that gap.

The study of discrete eigenvalues inside spectrum gaps has a long history and is yet a popular topic. M. Birman [3] was a pioneer in that field. In particular, he developed the method of quadratic forms and applied it to the case when the spectral gap coincides with the negative half-line $(-\infty, 0)$ and the potential V is non-regular. For regular potentials $V \in L^{d/2}(\mathbb{R}^d)$ with $d \geq 3$, the number of negative eigenvalues can be estimated as follows:

$$N \leq C_d \int_{\mathbb{R}^d} |V|^{d/2} \, dx, \tag{1.1}$$

where the constant C_d depends on the dimension d. The bound (1.1) is the well-celebrated Cwikel-Lieb-Rozenblum (CLR) inequality whose proof equips us with some groundbreaking ideas regarding the computation of the number of discrete eigenvalues inside the spectral gap. Years later, Alama, Deift and Hempel [1] made some more generalized efforts in the study of eigenvalues of a perturbed periodic operator. They explored the motion of eigenvalues of the operator $H + \lambda W$ inside a bounded gap and drew valuable conclusions. Besides the above-mentioned results, Molchanov and Vainberg [13] did some joint work and identified an upper and a lower bound for the number of negative eigenvalues of one- or two-dimensional Schrödinger operators.

The spectrum of Schrödinger operators remains a central topic in mathematical physics, with applications from solid-state physics to quantum mechanics. A particularly active area of research concerns the behavior of eigenvalues when perturbed by a potential, especially in the large coupling constant limit. This limit is physically significant as it describes scenarios where the potential energy strongly dominates the kinetic energy. Early foundational work in spectral theory for Schrödinger operators on Euclidean space $\mathbb{R}^d$ was established by figures such as M. Birman, E. Lieb, G. Rozenblum, B. Simon, and M. Solomyak, who investigated the asymptotic behavior of the number of bound states (negative eigenvalues) for various classes of potentials. These studies often employed semi-classical methods and established Weyl-type asymptotic formulas, relating the number of eigenvalues to the phase space volume occupied by the region where the potential is negative. The focus on discrete Schrödinger operators, acting on lattices such as $\mathbb{Z}^d$, introduces unique challenges and phenomena compared to their continuous counterparts. Discrete models are essential for understanding phenomena in photonic crystals and other periodic systems, exhibiting features like spectral gaps and spectral bands. The behavior of eigenvalues within spectral gaps under perturbations has been a subject of considerable interest. Alama, Deift, and Hempel, in a series of works, investigated the existence and behavior of eigenvalues entering spectral gaps of a continuous Schrödinger operator when a large coupling constant is introduced. M.Birman and his collaborators extended these ideas to the discrete spectrum of other operators.

The results presented in this dissertation contribute to this body of work by focusing on the asymptotic behavior of the number of eigenvalues of a periodic discrete Schrödinger operator $\mathbb{Z}^d$, perturbed by a decaying negative potential in the large coupling constant limit.

Our primary objective in this dissertation is to provide accurate estimates for the number of discrete eigenvalues located inside the spectral gap of a given operator. We employ a combination of analytical approaches to make our results precise, reliable, and convincing.

CHAPTER 2: EIGENVALUES OF THE DISCRETE SCHRÖDINGER OPERATOR IN THE LARGE COUPLING CONSTANT LIMIT

2.1 Introduction and main results

In this chapter, we study the operator $A-\alpha V$, where A and V are described below. The operator $A = -\Delta + f$ is defined on $\ell^2(\mathbb{Z}^d)$ by

$$[Au]_n = -\sum_{|m-n|=1} u_m + 2du_n + f(n)u_n,$$

where $f : \mathbb{Z}^d \to \mathbb{R}$ is a bounded function on the lattice $\mathbb{Z}^d$. Let $V : \mathbb{Z}^d \to [0,\infty)$ be a real potential having the property

$$V(n) \sim \Psi(\theta)|n|^{-p}, \qquad \text{as } |n| \to \infty, \tag{2.1}$$

where Ψ is a continuous function on the unit sphere $\{x \in \mathbb{R}^d : \ |x| = 1\}$ and $\theta = n/|n|$. Then the operator of multiplication by the function V is compact.

Define the operator $A(\alpha)$ to be

$$A(\alpha) = A - \alpha V, \qquad \alpha > 0.$$

Assume that the spectrum of A has a bounded gap $\Lambda = (\lambda_-, \lambda_+)$, that is, $\Lambda \cap \sigma(A) = \emptyset$ and $\lambda_\pm \in \sigma(A)$. By Weyl's theorem, the spectrum of operator $A(\alpha)$ is discrete in Λ, so it consists of isolated eigenvalues of finite multiplicity. These eigenvalues move from the right to the left with the growth of α and depend on α continuously. For a point $\lambda \in \Lambda$, we define $N(\lambda, \alpha)$ to be the number of eigenvalues $N(\lambda, \alpha)$ of $A(t)$ that pass through λ as t increases from 0 to α.

Also, for any self-adjoint operator $T = T^*$, we set $N(T, \lambda) = \mathrm{rank}(E_T(-\infty, \lambda))$, where E_T is the spectral measure of T.

In order to state our main theorem, we need to introduce the so-called density of states for the unperturbed operator A. For that purpose, for any domain $\Omega \subset \mathbb{R}^d$, we consider the operator A_Ω defined on $\ell^2(\Omega \cap \mathbb{Z}^d)$ by

$$\big[A_\Omega u\big]_n = - \sum_{|m-n|=1,\, m\in\Omega\cap\mathbb{Z}^d} u_m + 2du_n + f(n)u_n, \qquad \forall n \in \Omega \cap \mathbb{Z}^d.$$

Denote by $N(A_\Omega, \lambda)$ the number of eigenvalues of operator A_Ω that are strictly smaller than $\lambda \in \mathbb{R}$. For $\beta > 0$, we define $\beta\Omega$ to be the domain obtained from Ω by scaling:

$$\beta\Omega = \{x \in \mathbb{R}^d : \; x = \beta y \quad \text{for some} \;\; y \in \Omega\}.$$

Then we consider the number $N(A_{\beta\Omega}, \lambda)$ and study its behavior for large values of β.

Condition. Assume that the following limit exists for each $\lambda \in \mathbb{R}$,

$$\rho(\lambda) = \lim_{\beta\to\infty} \frac{N(A_{\beta\Omega}, \lambda)}{\beta^d \mathrm{vol}\Omega}.$$

This limit is called the density of states.

It is known that, ρ is an increasing function of λ having the property

$$\rho(\lambda) = \begin{cases} 0, & \text{for} \quad \lambda < -\|f\|_\infty; \\ 1, & \text{for} \quad \lambda > 4d + \|f\|_\infty. \end{cases}$$

Our main result establishes the asymptotics of $N(\lambda, \alpha)$ as $\alpha \to \infty$.

Theorem 2.1. *Let the potential V obey the condition* (2.1)*. Assume that $\lambda \in \Lambda$. Let $N(\lambda, \alpha)$ be the number of eigenvalues of $A(t)$ passing through λ as t increases from* 0 *to* α. *Then*

$$N(\lambda,\alpha) \sim \alpha^{d/p} \int_{\mathbb{R}^d} \Big(\rho(\lambda + \Psi(\theta)|x|^{-p}) - \rho(\lambda)\Big)\, dx, \qquad as \quad \alpha \to \infty. \tag{2.2}$$

Operators with periodic potentials f play a very important role in the so-called one-electron approximation model used in the quantum theory of crystals. According to this theory, the "allowed" energies of an electron moving in a crystal lie in $\sigma(A)$, the spectrum of A, consisting of bands. For a typical insulator, there is at least one gap in $\sigma(A)$. The Al_2O_3-crystal is one of examples of periodic media with this property. The theory also says that a photon whose energy is smaller than the length of the gap can not be absorbed by the crystal, because all states in the first band of the spectrum are already filled (see also [6]). That is the reason why Corundum (i.e. Al_2O_3) is colorless. However, replacing some of the Al^{3+}- ions by either Cr^{3+} or Ti^{3+}-ions, one obtains Ruby or Sapphire which absorb green and yellow colors. That is the reason why Ruby or Sapphire look red and blue respectively. Due to the fact that the crystal is not pure, there are additional isolated energy levels in spectral gaps of the pure crystal. Since the distances from these levels to the edges of the gap are smaller than the length of the gap, it is easier for the light to be absorbed by a crystal with impurities. In our model, the function V plays the role of the impurity potential.

While such problems on the lattice are considered for the first time, similar problems involving continuous operators on $\mathbb{R}^d$ have been studied before. For instance the main result of [9] says that, if V decays sufficiently fast at infinity, then the number $N(\lambda,\alpha)$ of eigenvalues of the continuous Schrodinger operator $H(t) = -\Delta + f - tV$ passing through a regular point $\lambda \notin \sigma(H)$ as t increases from 0 to α satisfies

$$N(\lambda,\alpha) \sim (2\pi)^{-d} \omega_d \alpha^{d/2} \int_{\mathbb{R}^d} V^{d/2} dx, \qquad \text{as} \quad \alpha \to \infty. \tag{2.3}$$

Here ω_d is the volume of the unit ball in $\mathbb{R}^d$. An interesting short proof of (2.3) was given by M. Birman in [2]. The author showed in [2] that the asymptotics of $N(\lambda, \alpha)$ does not depend on the point λ and the potential f. Consequently, one can take $\lambda = -1$ and set $f = 0$. After that, one can use previously known results.

The articles [6] and [11] are probably the two earliest publications in which the authors discussed the eigenvalues of $H(\alpha)$ in a bounded spectral gap of H. Perturbations in [6] were not necessarily sign-definite and the question was whether the quantity $N(\lambda, \alpha)$ is positive for some $\alpha > 0$. Similar questions were studied in [8]. While the papers that we mentioned answered some important questions, they did not contain asymptotic formulas for the quantity $N(\lambda, \alpha)$, which appeared later in [1] and [9]. Among the other results of [1], this work of S. Alama, P. Deift, and R. Hempel contains an asymptotic formula for $N(\lambda, \alpha)$ in the case $V(x) \sim -c|x|^{-p}$ as $|x| \to \infty$, with $p, c > 0$. In the corresponding statement of [1], the potential $V \leq 0$ is nonpositive, and the eigenvalues of $H(\alpha)$ move from the left to the right. The methods that are close to the ones of the paper [1] were used by R. Hempel in [9] and [10]. In particular, he proved (2.3) for λ that belongs to a finite gap (see [9]).

An interesting phenomenon remotely related to our study was discovered by the authors of [7] (by F. Gesztesy et al.). It turns out that, if $V \geq 0$ is compactly supported and $d = 1$, then the eigenvalues of $H(\alpha)$ move slowly near some very specific points of the gap and then move faster once they pass these points. This type of behavior of eigenvalues is called the "cascading" in [7].

2.2 Birman-Schwinger Principle

Here we describe the way to reduce the study of eigenvalues of operator $A(\alpha)$ in Λ to the study of the spectrum of the compact operator

$$X(\lambda) = \sqrt{V}(A - \lambda I)^{-1}\sqrt{V}. \tag{2.4}$$

Namely, for a self-adjoint operator $T = T^*$, we define $n_+(s, T)$ to be the number of eigenvalues of T that are greater than $s > 0$.

Lemma 2.1. *Assume that $\lambda \in \Lambda$. Let $X(\lambda)$ be defined in (2.4) and $N(\lambda, \alpha)$ be the number of eigenvalues of $A(t)$ passing through λ as t increases from 0 to α. Then*

$$N(\lambda, \alpha) = n_+(s, X(\lambda)), \qquad s = \alpha^{-1}. \tag{2.5}$$

The proof of the lemma can be found in [2]. This lemma is called the Birman-Schwinger Principle.

2.3 Splitting principle

In this section, we justify the so-called splitting principle described below. Let $0 < \varepsilon_1 < \varepsilon_2 < \infty$. We decompose $\mathbb{R}^d$ into three regions:

$$\begin{aligned}
\Omega_1(\alpha) &= \{x \in \mathbb{R}^d : \ |x| < \varepsilon_1 \alpha^{1/p}\}, \\
\Omega_2(\alpha) &= \{x \in \mathbb{R}^d : \varepsilon_1 \alpha^{1/p} \le |x| \le \varepsilon_2 \alpha^{1/p}\}, \\
\Omega_3(\alpha) &= \{x \in \mathbb{R}^d : \ |x| > \varepsilon_2 \alpha^{1/p}\}.
\end{aligned}$$

We choose ε_2 so that $\varepsilon_2 > (\|\Psi\|_\infty / |\lambda_+ - \lambda| + \|\Psi\|_\infty / |\lambda_- - \lambda|)^{1/p}$. For each region $\Omega_k(\alpha)$, we consider the operator $A_{\Omega_k(\alpha)}$, where $k = 1, 2, 3$. For $\lambda \in \mathbb{R}$, we define

$$N_k(\lambda, \alpha) = \text{rank } \mathrm{E}_{A_{\Omega_k}(\alpha) - \alpha V}(-\infty, \lambda) - \text{rank } \mathrm{E}_{A_{\Omega_k}(\alpha)}(-\infty, \lambda), \qquad k = 1, 2. \tag{2.6}$$

Proposition 2.1. *Let $\varepsilon_2 > (\|\Psi\|_\infty / |\lambda_+ - \lambda| + \|\Psi\|_\infty / |\lambda_- - \lambda|)^{1/p}$ and $N_k(\lambda, \alpha)$ be defined above for $k = 1, 2$. Then*

$$N(\lambda, \alpha) = N_1(\lambda, \alpha) + N_2(\lambda, \alpha) + O(\alpha^{(d-1)/p}), \qquad \text{as} \quad \alpha \to \infty.$$

Proof. First, observe that since $\mathbb{Z}^d$ is the disjoint union of the sets $\Omega_k(\alpha) \cap \mathbb{Z}^d, k =$

1, 2, 3, we have

$$l^2(\mathbb{Z}^d) = l^2(\Omega_1(\alpha) \cap \mathbb{Z}^d) \oplus l^2(\Omega_2(\alpha) \cap \mathbb{Z}^d) \oplus l^2(\Omega_3(\alpha) \cap \mathbb{Z}^d).$$

This decomposition allows one to consider the operator

$$B_0 = A_{\Omega_1(\alpha)} \oplus A_{\Omega_2(\alpha)} \oplus A_{\Omega_3(\alpha)}.$$

This operator might have eigenvalues inside (λ_-, λ_+). However, the operator

$$B = B_0 - (\lambda_+ - \lambda_-) E_{B_0}(\lambda_-, \lambda_+)$$

does not have this property: the spectrum of B in (λ_-, λ_+) is empty. We set now $B(\alpha) = B - \alpha V$, for $\alpha > 0$. Observe that $A - B_0$ is an operator of finite rank, and the rank of this operator does not exceed the double number of the "links" that intersect one of the spheres $\{x \in \mathbb{R}^d : |x| = \varepsilon_k \alpha^{1/p}\}$, $k = 1, 2$. By a "link", we mean any interval connecting two points $n, m \in \mathbb{Z}^d$ such that $|n - m| = 1$. The number of these links is a quantity of order $O(\alpha^{(d-1)/p})$ as $\alpha \to \infty$. Denote $r(\alpha) = \text{rank}(B_0 - A)$. Then $r(\alpha) = O(\alpha^{(d-1)/p})$ as $\alpha \to \infty$.

According to the perturbation theory,

$$\text{rank}(E_{B_0}(\lambda_-, \lambda_+)) \leq r(\alpha).$$

Therefore, $\text{rank}(B - B_0) \leq r(\alpha)$. Thus, $\text{rank}(B - A) \leq \text{rank}(B - B_0) + \text{rank}(B_0 - A) \leq 2r(\alpha)$.

Define $\tilde{X}(\lambda) = \sqrt{V}(B - \lambda I)^{-1}\sqrt{V}$ for $\lambda \in \Lambda$. Then

$$X(\lambda) - \tilde{X}(\lambda) = \sqrt{V}(A - \lambda I)^{-1}(B - A)(B - \lambda I)^{-1}\sqrt{V},$$

which implies that

$$\operatorname{rank}(X(\lambda) - \tilde{X}(\lambda)) \leq \operatorname{rank}(B - A) \leq 2r(\alpha).$$

Consequently, for any $\lambda \in \Lambda$,

$$|n_+(\alpha^{-1}, X(\lambda)) - n_+(\alpha^{-1}, \tilde{X}(\lambda))| \leq 2r(\alpha). \tag{2.7}$$

The operator $\tilde{X}(\lambda)$ is the Birman-Schwinger operator with A replaced by B. Therefore, according to the Birman-Schwinger principle, $n_+(\alpha^{-1}, \tilde{X}(\lambda))$ is the number of eigenvalues of $B - tV$ that pass through λ as t increases from 0 to α. Note now that the operator B can be decomposed to the orthogonal sum $B = B_1 \oplus B_2 \oplus B_3$, where

$$B_k = A_{\Omega_k(\alpha)} - (\lambda_+ - \lambda_-)E_{A_{\Omega_k(\alpha)}}(\Lambda), \qquad k = 1, 2, 3.$$

Hence,

$$(B - \lambda I)^{-1} = (B_1 - \lambda I)^{-1} \oplus (B_2 - \lambda I)^{-1} \oplus (B_3 - \lambda I)^{-1},$$

which leads to the decomposition of the Birman-Schwinger operator

$$\tilde{X}(\lambda) = \tilde{X}_1(\lambda) \oplus \tilde{X}_2(\lambda) \oplus \tilde{X}_3(\lambda), \qquad \text{where } \tilde{X}_k(\lambda) = \sqrt{V}(B_k - \lambda I)^{-1}\sqrt{V}.$$

So we finally conclude that

$$n_+(\alpha^{-1}, \tilde{X}(\lambda)) = n_+(\alpha^{-1}, \tilde{X}_1(\lambda)) + n_+(\alpha^{-1}, \tilde{X}_2(\lambda)) + n_+(\alpha^{-1}, \tilde{X}_3(\lambda)).$$

It remains to note that

$$n_+(\alpha^{-1}, \tilde{X}_1(\lambda)) = N_1(\lambda, \alpha), \qquad n_+(\alpha^{-1}, \tilde{X}_2(\lambda)) = N_2(\lambda, \alpha)$$

and show that

$$n_+(\alpha^{-1}, \tilde{X}_3(\lambda)) = 0. \tag{2.8}$$

Put differently,

$$n_+(\alpha^{-1}, \tilde{X}(\lambda)) = N_1(\lambda, \alpha) + N_2(\lambda, \alpha). \tag{2.9}$$

Combining (2.5), (2.7) and (2.9), we get

$$N(\lambda, \alpha) - (N_1(\lambda, \alpha) + N_2(\lambda, \alpha)) \leq r(\alpha).$$

Let us prove (2.8). First we choose $\delta > 0$ so that

$$\varepsilon_2 > \left(\|\Psi\|_\infty \left(\frac{1}{|\lambda - \lambda_+|} + \frac{1}{|\lambda - \lambda_-|} \right) (1 + \delta) \right)^{1/p}. \tag{2.10}$$

After that, we choose $0 > 0$ so large that

$$|V(n)| \leq \frac{\|\Psi\|_\infty (1 + \delta)}{|n|^p}, \qquad \forall n \in \Omega_3(\alpha), \qquad \alpha > \alpha_0. \tag{2.11}$$

Then it follows from (2.11) that

$$\begin{aligned} \|\tilde{X}_3(\lambda)\| &\leq \sup_{n \in \Omega_3(\alpha)} |V(n)| \cdot \|(B_3 - \lambda I)^{-1}\| \\ &\leq \frac{\|\Psi\|_\infty (1 + \delta)}{\varepsilon_2^p \alpha} \left(\frac{1}{|\lambda - \lambda_+|} + \frac{1}{|\lambda - \lambda_-|} \right). \end{aligned} \tag{2.12}$$

Thus, by (2.10) and (2.12), $\|\tilde{X}_3(\alpha)\| < \alpha^{-1}$, which implies (2.8). $\square$

We see from this proposition that to obtain the asymptotic formula for $N(\lambda, \alpha)$, it is enough to get asymptotic formulas for $N_1(\lambda, \alpha)$ and $N_2(\lambda, \alpha)$. Later we will prove that

$$\limsup_{\alpha\to\infty} \alpha^{-d/p} N_1(\lambda, \alpha) \le 2\omega_d \varepsilon_1^d, \tag{2.13}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. We will also show that

$$\lim_{\alpha\to\infty} \alpha^{-d/p} N_2(\lambda, \alpha) = \int_{\varepsilon_1<|x|<\varepsilon_2} \big(\rho(\lambda + \Psi(\theta)|x|^{-p}) - \rho(\lambda)\big)\, dx, \tag{2.14}$$

for $\varepsilon_2 > (\|\Psi\|_\infty/|\lambda_+ - \lambda| + \|\Psi\|_\infty/|\lambda_- - \lambda|)^{1/p}$.

2.4 Estimate of $N_1(\lambda, \alpha)$

In this section, we estimate the function $N_1(\lambda, \alpha)$. The dimension of the space $l^2(\Omega_1(\alpha) \cap \mathbb{Z}^d)$ coincides with the number of the points in $\Omega_1(\alpha) \cap \mathbb{Z}^d$. On the other hand, the number of these points cannot be larger than the volume of the ball $\{x \in \mathbb{R}^d : |x| < \varepsilon_1 \alpha^{1/p}\}$. Thus, $\dim\big(l^2(\Omega_1(\alpha) \cap \mathbb{Z}^d)\big) \le \operatorname{vol}\{x \in \mathbb{R}^d : |x| < \varepsilon_1\alpha^{1/p} + \sqrt{d}\} = \omega_d(\varepsilon_1\alpha^{1/p} + \sqrt{d})^d$, where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Since $N_1(\lambda, \alpha) \le 2\dim\big(l^2(\Omega_1(\alpha)) \cap \mathbb{Z}^d\big)$, we obtain (2.13).

2.5 Asymptotics of $N_2(\lambda, \alpha)$

Let $\tilde{\Omega} = \{x \in \mathbb{R}^3 : \varepsilon_1 < |x| < \varepsilon_2\}$. In order to obtain an asymptotic formula for $N_2(\lambda, \alpha) = N(A_{\Omega_2(\alpha)} - \alpha V, \lambda) - N(A_{\Omega_2(\alpha)}, \lambda)$, we divide $\tilde{\Omega}$ into a finite number of disjoint sets $\{Q_i\}$ given by $Q_i = (\delta\mathbb{Q}_i) \cap \tilde{\Omega}$, where $\mathbb{Q}_i = [0, 1)^d + i, i \in \mathbb{Z}^d$ and $\delta > 0$. Some of Q_i's are cubes and some of them are not. Observe that since $\Omega_2(\alpha) \cap \mathbb{Z}^d$ is the disjoint union of sets $\alpha^{1/p}Q_i \cap \mathbb{Z}^d$, we have

$$l^2(\Omega_2(\alpha) \cap \mathbb{Z}^d) = \oplus\text{-}\sum_i l^2(\alpha^{1/p}Q_i \cap \mathbb{Z}^d).$$

This orthogonal decomposition allows one to consider the operator

$$D(\alpha) = \oplus\text{-}\sum_i (A_{\alpha^{1/p}Q_i} - \alpha V).$$

Then $A_{\Omega_2(\alpha)} - D(\alpha)$ is an operator of finite rank, and the rank of this operator does not exceed the double number of the "links" that intersect the boundaries of the sets $\alpha^{1/p}Q_i$. The number of these links is a quantity of order $O(\alpha^{(d-1)/p})$ as $\alpha \to \infty$. Therefore,

$$|N_2(\lambda, \alpha) - N(D(\alpha), \lambda)| \leq \text{rank}(A_{\Omega_2(\alpha)} - D(\alpha)) = O(\alpha^{(d-1)/p}), \qquad \text{as} \quad \alpha \to \infty.$$

Since $N(D(\alpha), \lambda) = \sum_i N(A_{\alpha^{1/p}Q_i} - \alpha V, \lambda)$, we obtain

$$N(A_{\Omega_2(\alpha)} - \alpha V, \lambda) = \sum_i N(A_{\alpha^{1/p}Q_i} - \alpha V, \lambda) + O(\alpha^{(d-1)/p}), \qquad \text{as} \quad \alpha \to \infty.$$

Suppose first that the equality $V(x) = \Psi(\theta)/|x|^p$ holds (not only asymptotically, but) exactly for $|x| > 1$. For each Q_i, define $x_i^{\max}$ and $x_i^{\min}$ to be the points for which

$$\max_{x \in Q_i} V(x) = V(x_i^{\max}), \qquad \min_{x \in Q_i} V(x) = V(x_i^{\min}).$$

Then for any $n \in \alpha^{1/p}Q_i$, we have

$$V(x_i^{\min}) \leq \alpha V(n) \leq V(x_i^{\max}), \tag{2.15}$$

where both the lower and upper bounds of $\alpha V(n)$ are independent of α.

By the Minimax principle,

$$N(A - \alpha V, \lambda) = \max_F \dim(F),$$

where the maximum is taken over all subspaces $F \subseteq l^2$ on which $\langle (A - \alpha V)u, u\rangle \leq \lambda \|u\|^2$ for all $u \in F$. This implies that if V_1 and V_2 are two potentials defined on $\mathbb{Z}^d$ such that $V_1 \leq V_2$, then $N(A - \alpha V_1, \lambda) \leq N(A - \alpha V_2, \lambda)$. Similarly, we have $N(A_{\alpha^{1/p}Q_i} - \alpha V_1, \lambda) \leq N(A_{\alpha^{1/p}Q_i} - \alpha V_2, \lambda)$ if $V_1 \leq V_2$.

It follows from (2.15) that

$$N(A_{\alpha^{1/p}Q_i} - V(x_i^{\min}), \lambda) \leq N(A_{\alpha^{1/p}Q_i} - \alpha V, \lambda) \leq N(A_{\alpha^{1/p}Q_i} - V(x_i^{\max}), \lambda).$$

Observe that

$$\begin{aligned} N(A_{\alpha^{1/p}Q_i} - V(x_i^{\max}), \lambda) &= N(A_{\alpha^{1/p}Q_i}, \lambda + V(x_i^{\max})) \qquad \text{and} \\ N(A_{\alpha^{1/p}Q_i} - V(x_i^{\min}), \lambda) &= N(A_{\alpha^{1/p}Q_i}, \lambda + V(x_i^{\min})), \end{aligned}$$

thus, we have, as $\alpha \to \infty$,

$$\begin{aligned} N(A_{\alpha^{1/p}Q_i}, \lambda + V(x_i^{\max})) &\sim \mathrm{vol}(\alpha^{1/p}Q_i)\rho(\lambda + V(x_i^{\max})) \\ &= \alpha^{d/p}\mathrm{vol}(Q_i)\rho(\lambda + V(x_i^{\max})), \\ N(A_{\alpha^{1/p}Q_i}, \lambda + V(x_i^{\min})) &\sim \mathrm{vol}(\alpha^{1/p}Q_i)\rho(\lambda + V(x_i^{\min})) \\ &= \alpha^{d/p}\mathrm{vol}(Q_i)\rho(\lambda + V(x_i^{\min})). \end{aligned}$$

Consequently, the upper and lower bounds of

$$\lim_{\alpha \to \infty} \frac{N(A_{\Omega_2(\alpha)} - \alpha V, \lambda)}{\alpha^{d/p}}$$

are the Riemann sums

$$\sum_{i=1}^{k} \rho(\lambda + V(x_i^{\max}))\mathrm{vol}(Q_i) \qquad \text{and}$$
$$\sum_{i=1}^{k} \rho(\lambda + V(x_i^{\min}))\mathrm{vol}(Q_i),$$

respectively. Therefore, when $\mathrm{vol}(Q_i) \to 0$ for each i,

$$\lim_{\alpha\to\infty} \frac{N(A_{\Omega_2(\alpha)} - \alpha V, \lambda)}{\alpha^{d/p}} = \int_{\varepsilon_1<|x|<\varepsilon_2} \rho(\lambda + \Psi(\theta)|x|^{-p})dx. \tag{2.16}$$

On the other hand, according to the definition of the density of states,

$$\lim_{\alpha\to\infty} \frac{N(A_{\Omega_2(\alpha)}, \lambda)}{\alpha^{d/p}} = \rho(\lambda)\mathrm{vol}(\tilde{\Omega}) = \int_{\varepsilon_1<|x|<\varepsilon_2} \rho(\lambda)dx. \tag{2.17}$$

Subtracting (2.17) from (2.16), we obtain (2.14).

2.6 End of the proof of Theorem 2.1

It follows from (2.13) and (2.14) that

$$\begin{aligned} \limsup_{\alpha\to\infty} \frac{N(\lambda,\alpha)}{\alpha^{d/p}} &= \limsup_{\alpha\to\infty} \frac{N_1(\lambda,\alpha) + N_2(\lambda,\alpha)}{\alpha^{d/p}} \\ &\leq \omega_d \varepsilon_1^d + \int_{\varepsilon_1<|x|<\varepsilon_2} \big(\rho(\lambda + \Psi(\theta)|x|^{-p}) - \rho(\lambda)\big)\, dx. \end{aligned}$$

Taking the limits as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to \infty$, we obtain

$$\limsup_{\alpha\to\infty} \frac{N(\lambda,\alpha)}{\alpha^{d/p}} \leq \int_{\mathbb{R}^d} \big(\rho(\lambda + \Psi(\theta)|x|^{-p}) - \rho(\lambda)\big)\, dx. \tag{2.18}$$

On the other hand, it also follows from (2.13) and (2.14) that

$$\begin{aligned}\liminf_{\alpha\to\infty}\frac{N(\lambda,\alpha)}{\alpha^{d/p}} &= \liminf_{\alpha\to\infty}\frac{N_1(\lambda,\alpha)+N_2(\lambda,\alpha)}{\alpha^{d/p}}\\ &\geq \int_{\varepsilon_1<|x|<\varepsilon_2}\big(\rho(\lambda+\Psi(\theta)|x|^{-p})-\rho(\lambda)\big)\,dx.\end{aligned}$$

Taking the limits as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to \infty$, we obtain

$$\liminf_{\alpha\to\infty}\frac{N(\lambda,\alpha)}{\alpha^{d/p}} \geq \int_{\mathbb{R}^d}\big(\rho(\lambda+\Psi(\theta)|x|^{-p})-\rho(\lambda)\big)\,dx. \tag{2.19}$$

Combining (2.18) and (2.19), we get (2.2). ■

CHAPTER 3: DISCRETE SPECTRUM OF THE DISCRETE SCHRÖDINGER OPERATOR ON THE LATTICE

3.1 Introduction and main results

In this chapter, we use different notations. Namely, let $H = -\Delta - V$ be a perturbed discrete Schrödinger operator defined on $\mathbb{Z}^d$ given by

$$[Hu]_n = -\sum_{|m-n|=1} u(m) + 2du(n) - V(n)u_n, \tag{3.1}$$

with $d \geq 3$, $V(n) \geq 0$ for all $n \in \mathbb{Z}^d$ and $V \in \ell^{d/2}(\mathbb{Z}^d)$. Then the negative spectrum of the operator H is discrete. We define N to be the number of negative eigenvalues of H. Our goal is to prove the following theorem.

Theorem 3.1. *Let N be the number of discrete negative eigenvalues of operator* (3.1)*. Then there exists a constant C_d depending only on d such that*

$$N \leq C_d \sum_{n \in \mathbb{Z}^d} V(n)^{d/2}. \tag{3.2}$$

It turns out that we can extend this theorem to the case of more general operators, namely, let $F : \ell^2(\mathbb{Z}^d) \to L^2([0,1)^d)$ be the Fourier transform operator defined by

$$\widehat{u}(\xi) = Fu(\xi) = \sum_{n \in \mathbb{Z}^d} u(n) e^{2\pi i n \xi}, \quad \xi \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d, \tag{3.3}$$

where $n \cdot \xi = \sum_{j=1}^d n_j \xi_j$. Then $\|\widehat{u}\| = \|u\|$ for each $u \in \ell^2(\mathbb{Z}^d)$.

For a non-negative function $a(\xi)$ on $[0,1)^d$, define the operator $H : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$

by

$$H = F^*[a(\xi)]F - V,$$

where $[a(\xi)]$ denotes the operator of multiplication by the function $a(\xi)$. We are interested in the number of eigenvalues of the operator H, denoted also by N. The assumptions regarding the potential V will depend on the choice of $a(\xi)$ obeying another condition:

$$\frac{1}{\sqrt{a(\xi)}} \in L^{p,\infty}([0,1)^d), \qquad p > 2, \tag{3.4}$$

where $L^{p,\infty}$ is the class of functions $u : [0,1)^d \to \mathbb{R}$ with

$$\sup_{t>0} \left(t^p \cdot \text{meas} \left\{ x \in [0,1)^d : |u(x)| > t \right\} \right) < \infty.$$

We denote the associated (quasi-)norm by $\|u\|_{p,\infty}$ or $\|u\|_{L^{p,\infty}}$.

Theorem 3.2. *Suppose that $a \geq 0$ satisfies* (3.4)*. Assume that $V \in \ell^{p/2}(\mathbb{Z}^d)$, $V \geq 0$. Then there is a constant $C_{d,p}$ depending only on d and p such that*

$$N \leq C_{d,p} \|a^{-1/2}\|^p_{L^{p,\infty}} \sum_{n \in \mathbb{Z}^d} V(n)^{p/2}.$$

The continuous Schrödinger operator with the same expression as (3.1) defined on $\mathbb{R}^d$ has been studied extensively for decades. One of the most remarkable discoveries concerning the number N of negative eigenvalues of the continuous operator is the celebrated Cwikel-Lieb-Rozenblum (CLR) inequality

$$N \leq C_d \int_{\mathbb{R}^d} V^{d/2}(x)\, dx, \quad d \geq 3, \tag{3.5}$$

where C_d is a constant that depends solely on d. The full proof of (3.5) can be found in [5]. Lieb [12] and Rozenblum [14] proved this result for the continuous Schrödinger operator.

However, the discrete operator (3.1) has not been analyzed in depth so far. In this paper, we primarily aim at demonstrating a discrete version of (3.5), as detailed in Theorem 3.2.

3.2 Preliminaries

We say that a compact operator T belongs to the Schatten class $\mathfrak{S}_p$ with $p > 0$ if

$$\sum_{k=1}^{\infty} s_k^p(T) < \infty, \tag{3.6}$$

where $\{s_k(T)\}_{k=1}^{\infty}$ is the set of all singular values of T arranged in a non-increasing order (counting multiplicities), and the Schatten norm corresponding to (3.6) is denoted by $\|T\|_p$.

For a compact and self-adjoint operator $T = T^*$ on a Hilbert space $\mathfrak{H}$, define

$$n_+(s, T) = \#\{k : \lambda_k(T) > s\}, \quad s > 0,$$

where $\lambda_k(T)$ are eigenvalues of T. Similarly, for a compact operator T, we set

$$n(s, T) = \#\{k : s_k(T) > s\}, \quad s > 0,$$

Consider the operator X defined by

$$X = WF^* \left[\frac{1}{a(\xi)}\right] FW, \quad \text{where } W = \sqrt{V}.$$

Proposition 3.1. *Under the assumptions of Theorem 3.2,*

$$N = n_+(1, X).$$

This statement is known as the Birman-Schwinger principle. See [2] for the proof.

Corollary 3.1. *Under the assumptions of Theorem 3.2,*

$$N = n(1, Y),$$

where $Y = a^{-1/2} F W$.

3.3 Proof of Theorem 3.2

Consider two functions $u : [0,1)^d \to \mathbb{C}, g : \mathbb{Z}^d \to \mathbb{C}$, and assume that $u \in L^{p,\infty}([0,1)^d), g \in \ell^p(\mathbb{Z}^d)$. We study a modulated Fourier operator $B_{u,g} : \ell^2 \to L^2([0,1)^d)$ given by

$$[B_{u,g} h](\xi) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i n \xi} u(\xi) g(n) h(n). \tag{3.7}$$

Observe that Y in Corollary 3.1 equals $B_{u,g}$ with $u = a^{-1/2}$ and $g = W$. Thus, Theorem 3.2 is a consequence of the following result.

Theorem 3.3. *Suppose that* $u \in L^{p,\infty}([0,1)^d)$ *and* $g \in \ell^p$. *Then*

$$\sup_{k>0} \left(k^{1/p} \left(\frac{1}{k} \sum_{m=1}^{k} s_m^2(B_{u,g}) \right)^{1/2} \right) \leq C_p \|u\|_{p,\infty} \cdot \|g\|_{\ell^p}, \tag{3.8}$$

for some constant C_p *that depends purely on* p.

Proof. Our proof will follow the pattern of the proof of the main theorem of [5].

Observe that

$$B_{u,g} = uFg = (u)|a|F|g|(g).$$

Hence,

$$B_{u,g} = T_1 B_{|u|,|g|} T_2,$$

where T_1 and T_2 are both unitary operators. Therefore, without loss of generality, assume that $u, g \geq 0$. Additionally,

$$\frac{B_{u,g}}{\|u\|_{p,\infty}\|g\|_{\ell^p}} = \frac{u}{\|u\|_{p,\infty}} F \frac{g}{\|g\|_{\ell^p}}. \tag{3.9}$$

To simplify the computations, suppose that $\|u\|_{p,\infty} = \|g\|_{\ell^p} = 1$ in (3.9).

We define the sets

$$G_k = \{n \in \mathbb{Z}^d : t2^{k-1} < g(n) \leq t2^k\} \quad \text{and} \quad U_k = \{\xi \in [0,1)^d : 2^{k-1} < u(\xi) \leq 2^k\}.$$

The characteristic functions of the sets G_k and U_k will be denoted by χ_{G_k} and χ_{U_k}, respectively. We can write $B_{u,g}$ as $B_{u,g} = B_t + C_t$, where B_t is the operator with kernel

$$b(\xi, n) = e^{2\pi i \xi n} u(\xi) g(n) \sum_{k+m\geq 1} \chi_{U_k}(\xi)\chi_{G_m}(n),$$

and C_t is the operator with kernel

$$c(\xi, n) = e^{2\pi i \xi n} u(\xi) g(n) \sum_{k+m\leq 0} \chi_{U_k}(\xi)\chi_{G_m}(n).$$

For the operator B_t, we have

$$\|B_t\|_2^2 = \sum_{k=1}^{\infty} |s_k(B_t)|^2 = \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d} |b(\xi, n)|^2 \, d\xi.$$

We are interested in an estimate of the Hilbert-Schmidt norm of B_t. First, the definition of B_t implies

$$|b(\xi,n)| \leq u(\xi)g(n)\chi_{\{(\xi,n):u(\xi)g(n)>t2^{-1}\}}.$$

Additionally, for every $\alpha > 0$, we introduce set $E_\alpha = \{(\xi,n) : u(\xi)g(n) > \alpha\}$, whose product measure is calculated as

$$\sum_{n\in\mathbb{Z}^d} \text{meas}\{\xi : (\xi,n) \in E_\alpha\} = \sum_{n\in\mathbb{Z}^d} \text{meas}\{\xi : u(\xi)g(n) > \alpha\}$$
$$= \sum_{n\in\mathbb{Z}^d} \text{meas}\{\xi : u(\xi) > \alpha/g(n)\} \leq \alpha^{-p} \sum_{n\in\mathbb{Z}^d} |g(n)|^p = \alpha^{-p}\|g\|_{\ell^p} = \alpha^{-p},$$

based on the prescribed assumption that $\|g\|_{\ell^p} = 1$. It is also evident that

$$\{(\xi,n) : |b(\xi,n)| > \alpha\} \subseteq \begin{cases} E_{t2^{-1}}, & \text{if } \alpha < t2^{-1}; \\ E_\alpha, & \text{if } \alpha \geq t2^{-1}. \end{cases} \tag{3.10}$$

Now let ν be a measure on a set Ω, and let $f \in L^p(\Omega)$.

$$\mu(\alpha) = \nu\{x \in \Omega : |f(x)| > \alpha\},$$

with $\alpha \geq 0$. Then

$$\int_{\mathbb{R}} |f|^p\,dx = p\int_0^\infty \mu(t)t^{p-1}\,dt.$$

In particular, if $\Omega = \mathbb{R}^{2d}$ and ν is the product of the Lebesgue measure on $\mathbb{R}^d$ and the discrete measure with atoms of mass 1 at the points of $\mathbb{Z}^d$. Then

$$\sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d} |b(\xi,n)|^2\,d\xi = 2\int_0^\infty \mu(\alpha)\alpha\,d\alpha,$$

where

$$\mu(\alpha) = \nu\{(\xi, n) \in \mathbb{R}^2 : |b(\xi, n)| > \alpha\}.$$

Also note that (3.10) yields

$$\mu(\alpha) \leq \begin{cases} \nu(E_{t2^{-1}}) = t^p 2^{-p}, & \text{if } \alpha < t2^{-1}; \\ \nu(E_\alpha) = \alpha^p, & \text{if } \alpha \geq t2^{-1}, \end{cases}$$

which results in

$$\sum_{\mathbb{Z}^d} \int_{\mathbb{R}^d} |b(\xi, n)|^2 \, d\xi = 2 \left(\int_0^{t2^{-1}} \mu(\alpha)\alpha \, d\alpha + \int_{t2^{-1}}^{\infty} \alpha^{-p+1} \, d\alpha \right).$$

It can be shown that

$$\|B_t\|_2^2 \leq (t2^{-1})^p (t2^{-1})^2 + \frac{2}{p-2}(t2^{-1})^{2-p} = t^{2-p} 2^{p-2} \left(1 + \frac{2}{p-2}\right).$$

Suppose that $T : \ell^2(\mathbb{Z}^d) \to L^2([0,1)^d)$ is a bounded linear operator. If for any two functions $\phi \in \ell^2(\mathbb{Z}^d)$ and $\psi \in L^2([0,1)^d)$, we have the inequality

$$|\langle T\phi, \psi \rangle| \leq C\|\phi\| \cdot \|\psi\|,$$

for some constant $C \in \mathbb{R}$, then $\|T\| \leq C$. We are going to use this statement to estimate the norm of C_t. For every $k \in \mathbb{N}$, we define ϕ_k and ψ_k as

$$\phi_k = 2^{-k} g\phi\chi_{G_k} \quad \text{and} \quad \psi_k = 2^{-k} u\psi\chi_{U_k},$$

respectively. We have

$$\begin{aligned}
\langle C_t\phi, \psi\rangle &= \sum_n \int_{\mathbb{R}^d} c(\xi, n)\cdot \phi(n)\psi(\xi)\, d\xi \\
&= \sum_n \int_{\mathbb{R}^d} e^{2\pi i \xi_n} \sum_{k+m\le 0} g(n)\phi(n)\chi_{G_k} u(\xi)\psi(\xi)\chi_{u_k}\, d\xi \\
&= \sum_n \int_{\mathbb{R}^d} e^{2\pi i \xi_n} \sum_{k+m\le 0} 2^{m+k}\phi_m(n)\psi_k(\xi)\, d\xi. \qquad (3.11)
\end{aligned}$$

This leads to the inequality

$$\begin{aligned}
|\langle C_t\phi, \psi\rangle| &= \left|\int_{\mathbb{R}^d} \sum_{m+k\le 0} 2^{m+k} \sum_n e^{2\pi i \xi n}\phi_m(n)\psi_k(\xi)\, d\xi\right| \\
&= \left|\int_{\mathbb{R}^d} \sum_{m+k\le 0} 2^{m+k}\widehat{\phi}_m(\xi)\psi_k(\xi)\, d\xi\right| \le \sum_{m+k\le 0}\left|\int_{\mathbb{R}^d} 2^{m+k}\widehat{\phi}_m(\xi)\psi_k(\xi)\, d\xi\right| \\
&\le \sum_{m+k\le 0} 2^{m+k}\|\widehat{\phi}_m(\xi)\|_{L^2}\|\widehat{\psi}_k(\xi)\|_{L^2} = \sum_{m+k\le 0} 2^{m+k}\|\phi_m\|_{\ell^2}\|\psi_k\|_{L^2}. \qquad (3.12)
\end{aligned}$$

Then we perform a substitution by introducing $j = m + k$, which transforms (3.12) to

$$\begin{aligned}
&\sum_{m+k\le 0} 2^{m+k}\|\phi_m\|_{\ell^2}\cdot\|\psi_k\|_{L^2} = \sum_{j=-\infty}^{0}\sum_{k=-\infty}^{\infty} 2^j\|\phi_{j-k}\|_{\ell^2}\cdot\|\psi_k\|_{L^2} \\
&= \sum_{j=-\infty}^{0} 2^j \sum_{k=-\infty}^{\infty}\|\phi_{j-k}\|_{\ell^2}\cdot\|\psi_k\|_{L^2} \le \sum_{j=-\infty}^{0} 2^j\cdot t\|\phi\|_{\ell^2}\cdot\|\psi\|_{L^2} \\
&= 2t\|\phi\|_{\ell^2}\cdot\|\psi\|_{L^2}, \qquad (3.13)
\end{aligned}$$

due to

$$\|\phi_m\|_{\ell^2}^2 = \sum_{n\in\mathbb{Z}^d}|\phi_m(n)|^2 \le \sum_{n\in\mathbb{Z}^d} t^2\chi_{G_m}|\phi(n)|^2 = t^2\|\phi\|_{\ell^2}^2\chi_{G_m},$$

and a similar inequality for $\|\psi_k\|_{L^2}$. In summary, we obtain that

$$\|B_t\|_2^2 \leq t^{2-p}2^{p-2}\left(1+\frac{2}{p-2}\right) \qquad \text{and} \qquad \|C_t\| \leq 2t. \tag{3.14}$$

It could be rewritten in terms of singular values of the operators as follows

$$\sum_{k=1}^{\infty}|s_k(B_t)|^2 \leq t^{2-p}2^{p-2}\left(1+\frac{2}{p-2}\right) \qquad \text{and} \qquad s_1(C_t) \leq 2t. \tag{3.15}$$

We intend to apply the basic inequality for singular values

$$s_{m+k-1}(B_{u,g}) \leq s_m(B_t) + s_k(C_t),$$

which holds for all $m, k \in \mathbb{N}$. Letting $k = 1$, we get

$$s_m(B_{u,g}) \leq s_m(B_t) + s_1(C_t). \tag{3.16}$$

As a straightforward consequence of (3.15) and (3.16), we compute that

$$\begin{aligned} \left(\sum_{m=1}^{k} s_m^2(B_{u,g})\right)^{1/2} &\leq \left(\sum_{m=1}^{k}(s_m(B_t)+s_1(C_t))^2\right)^{1/2} \\ &\leq \left(\sum_{m=1}^{k} s_m^2(B_t)\right)^{1/2} + \sqrt{k}s_1(C_t) \leq \|B_t\|_2 + \sqrt{k}\|C_t\| \\ &= \left(t^{2-p}2^{p-2}\left(1+\frac{2}{p-2}\right)\right)^{1/2} + 2\sqrt{k}t, \end{aligned} \tag{3.17}$$

where the resulting expression can be considered as a function of t. We denote the right-hand side of (3.17) by

$$f(t) = t^{1-p/2}2^{2/p-1}\left(1+\frac{2}{p-2}\right)^{1/2} + 2\sqrt{k}t.$$

Then its derivative function equals

$$f'(t) = \left(1 - \frac{p}{2}\right) C_1 t^{-p/2} + 2\sqrt{k}, \tag{3.18}$$

where $C_1 = 2^{2/p-1}(1 + \frac{2}{p-2})^{1/2}$ are constants and $p > 2$. Substituting into (3.18) and setting $f'(t) = 0$, we conclude that f has a unique global minimum that appears at

$$t_{\min} = \left(\frac{4}{(\frac{p}{2} - 1)^2 C_1^2}\right) k^{-1/p}. \tag{3.19}$$

In view of (3.19), we conclude that there is a constant C_2 such that

$$\left(\sum_{m=1}^{k} s_m^2(B_{u,g})\right)^{1/2} \leq C_2 k^{-\frac{1}{p}+\frac{1}{2}},$$

which, after rearrangements, becomes

$$\left(\frac{1}{k}\sum_{m=1}^{k} s_m^2(B_{u,g})\right)^{1/2} \leq C_2 k^{-1/p} \|u\|_{L^{p,\infty}} \|g\|_{\ell^p}. \tag{3.20}$$

The inequality (3.20) is equivalent to (3.8). □

We now prove Theorem 3.2. For that purpose, we set $k = N = n_+(1, B_{u,g})$ in (3.20) and get

$$\frac{1}{N}\sum_{m=1}^{N} 1 = 1 \leq \left(\frac{1}{N}\sum_{m=1}^{N} s_m^2(B_{u.g})\right)^{1/2} \leq C_2 N^{-1/p} \|u\|_{L^{p,\infty}} \cdot \|g\|_{\ell^p}, \tag{3.21}$$

which is equivalent to

$$N \leq (C_2 \|u\|_{L^{p,\infty}} \|g\|_{\ell^p})^p = C_2^p \|u\|_{L^{p,\infty}}^p \|g\|_{\ell^p}^p. \tag{3.22}$$

It remains to apply Corollary 3.1 with $u = a^{-1/2}$ and $g = W$. □

To prove Theorem 3.1, we set

$$u(\xi) = \left(2d - \sum_{i=1}^{d} \cos(2\pi\xi i)\right)^{-1/2} \qquad \text{and} \qquad g(n) = \sqrt{V(n)}.$$

By Theorem 3.2,

$$N \leq C_4^d \|u\|_{L^{d,\infty}}^d \cdot \sum_{n\in\mathbb{Z}^d} |V(n)|^{d/2}.$$

Thus, if we set $C_5 = C_4^d \|u\|_{L^{p,\infty}}^d$, then we will have

$$N \leq C_5 \sum_{n\in\mathbb{Z}^d} V(n)^{d/2}.$$

This result is identical to (3.1) and thereby finishes the proof of Theorem 3.1. □

CHAPTER 4: DISCRETE SPECTRUM OF THE BILAYER GRAPHENE OPERATOR

4.1 Introduction

Chapter 4 is essentially different from the proceeding chapters. Instead of a discrete operator on the lattice $\mathbb{Z}^d$, we study a differential operator on $\mathbb{R}^2$. The spectrum of this operator will have only one bounded spectral gap. More specifically, let the free bilayer graphene operator D_m be defined on the space $L^2(\mathbb{R}^2, \mathbb{C}^2)$ by

$$D_m = \begin{pmatrix} m & \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)^2 \\ \left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)^2 & -m \end{pmatrix}, \quad \text{where } m > 0.$$

While this particular operator was introduced by Ferrulli, Laptev and Safronov in [16], the study of bilayer graphene and its properties, including the development of theoretical models and operators describing its electronic structure, has been ongoing for a longer period of time (since 2004).

The spectrum of the operator D_m coincides with the set $(-\infty, -m] \cup [m, \infty)$. Thus, the interval $(-m, m)$ is a gap in the spectrum. Let $V : \mathbb{R}^2 \to [0, \infty)$ be a non-negative potential on $\mathbb{R}^2$ that decays as $|x| \to \infty$. In this paper, we analyze the discrete spectrum of the perturbed operator

$$D(\alpha) = D_m - \alpha V, \quad \alpha > 0. \tag{4.1}$$

The operator V in (4.1) is understood as the matrix operator

$$V \cdot I = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}. \tag{4.2}$$

The constraints that we impose on the potential V guarantee that the spectrum of $D(\alpha)$ is discrete in $(-m, m)$. That implies that it consists of isolated eigenvalues of finite multiplicity, which move monotonically from the right to the left with the growth of α.

Let us fix a point $\lambda \in (-m, m)$ and introduce $N(\lambda, \alpha)$ as the number of eigenvalues of $D(t)$ passing through λ when t increases from 0 to α. We investigate the asymptotic behavior of $N(\lambda, \alpha)$ for sufficiently large values of α.

We define the cubes $\mathbb{Q}_n = [0, 1)^2 + n$ with $n \in \mathbb{Z}^2$, and set

$$\|V\|^q_{L^q(\mathbb{Q}_n)} = \int_{\mathbb{Q}_n} |V|^q \, dx, \qquad \text{for } q > 0.$$

Our first result deals with the case

$$\sum_{n \in \mathbb{Z}^2} \|V\|_{L^q(\mathbb{Q}_n)} < \infty, \qquad q > 1. \tag{4.3}$$

Theorem 4.1. *Let V satisfy* (4.3). *Then*

$$N(\lambda, \alpha) \sim \frac{\alpha}{4\pi} \int_{\mathbb{R}^2} V \, dx, \quad \textit{as } \alpha \to \infty.$$

Now we consider the case where $V \notin L^1(\mathbb{R}^2)$. Instead, we assume that V is a bounded function obeying the condition

$$V(x) \sim \frac{\Psi(\theta)}{|x|^p}, \quad \text{as } |x| \to \infty, \tag{4.4}$$

where $\theta = x/|x|$ and $0 < p < 2$. The function $\Psi \geq 0$ is assumed to be continuous on the unit circle $\mathbb{S} = \{x \in \mathbb{R}^2 : |x| = 1\}$.

Theorem 4.2. *Let $0 < p < 2$ and let V satisfy* (4.4). *Then*

$$N(\lambda, \alpha) \sim \frac{\alpha^{2/p}}{4\pi} \int_{\mathbb{R}^2} [((\lambda + \Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\, dx, \quad \text{as } \alpha \to \infty.$$

Here, a_+ denotes the positive part of the number a, i.e.,

$$a_+ = \max\{a, 0\}.$$

Let us give a short summary of the past research in the area. Eigenvalues in gaps of the continuous spectrum were studied extensively for Schrödinger operators. For instance, R. Hempel proved in [9] that the number of eigenvalues of the operator $-\Delta + f - \alpha V$ passing through the point λ of a gap in the spectrum of the periodic operator $-\Delta + f$ obeys Weyl's law:

$$N(\lambda, \alpha) \sim (2\pi)^{-d} \omega_d \alpha^{d/2} \int_{\mathbb{R}^d} V^{d/2}\, dx, \qquad \text{as } \alpha \to \infty, \tag{4.5}$$

where ω_d is volume of the unit ball in $\mathbb{R}^d$. Then M. Birman proved (4.5) through a different approach in [2]. Other relevant results could also be found in [1, 10]. Such results could be linked to the study of crystal color. When some ions in a crystal lattice are replaced by impurity ions, the resulting perturbation, described by the potential αV, can create new energy levels. These energy levels are exactly the eigenvalues of $-\Delta + f - \alpha V$. They are responsible for the absorption of the light of specific wavelengths, leading to the observed color of the crystal. Based on these remarks, the problem addressed in the present paper can be associated with the study of the color of graphene.

Several mathematicians have studied eigenvalues in gaps of the spectrum with non-signdefinite perturbations (see [6] and [15]). In this case, the eigenvalues of the operator are not monotone functions of α, and the quantity $N(\lambda, \alpha)$ should be defined as the difference of the number of eigenvalues passing λ in two opposite directions. Using this definition, Safronov [15] generalized the formula (4.5) to the case of non-signdefinite perturbations. The only change is that V has to be replaced by V_+ on the right hand side of (4.5).

The graphene operator D_m was introduced in the paper by Ferrulli, Laptev and Safronov [16]. However, the authors of [16] analyzed complex eigenvalues of the perturbed operator $D(\alpha)$ with a non-self adjoint perturbation V and a fixed value of α. Our results differ from theorems in [16] in a critical way: the perturbations that we consider are self-adjoint and $\alpha \to \infty$.

The current paper overcomes the main technical difficulty appearing when one studies operators having only bounded gaps in the spectrum: one cannot use Dirichlet-Neumann bracketing as suggested in the paper by Alama, Deift, and Hempel [1]. While Dirichlet-Neumann bracketing is a powerful technique for analyzing the spectrum of semi-bounded operators, it cannot be applied when dealing with operators that are not semi-bounded.

4.2 Preliminaries

Here, we provide necessary background information that is needed to understand the subsequent sections of the paper.

Let $\mathfrak{H}$ be a separable Hilbert space. The class of compact operators on $\mathfrak{H}$ will be denoted by $\mathfrak{S}_\infty$. For a compact operator $T \in \mathfrak{S}_\infty$, the symbols $s_k(T)$ denote the singular values of T enumerated in the non-increasing order ($k \in \mathbb{N}$) and counted in accordance with their multiplicity. Observe that $s_k^2(T)$ are eigenvalues of the operator T^*T. We set

$$n(s, T) = \#\{k : s_k(T) > s\}, \qquad s > 0.$$

For a self-adjoint compact operator T, we also set

$$n_\pm(s,T) = \#\{k : \pm\lambda_k(T) > s\}, \qquad s > 0,$$

where $\lambda_k(T)$ are eigenvalues of T. It follows that (see [4])

$$n_\pm(s_1+s_2, T_1+T_2) \le n_\pm(s_1,T_1) + n_\pm(s_2,T_2), \qquad s_1, s_2 > 0.$$

A similar inequality holds for the function n. Also,

$$n(s_1 s_2, T_1 T_2) \le n(s_1,T_1) + n(s_2,T_2), \qquad s_1, s_2 > 0.$$

The class of compact operators T whose singular values satisfy

$$\|T\|^p_{\mathfrak{S}_p} := \sum_k s_k^p(T) < \infty, \qquad p > 0,$$

is called the Schatten class $\mathfrak{S}_p$.

Besides the classes $\mathfrak{S}_p$, we will be dealing with the so-called weak Schatten classes Σ_p of compact operators T obeying the condition

$$\|T\|^p_{\Sigma_p} := \sup_{s>0} s^p n(s,T) < \infty.$$

We also introduce the class Σ_p^0 as the collection of compact operators T such that

$$n(s,T) = o(s^{-p}), \quad \text{as } s \to 0.$$

Note that $\mathfrak{S}_p \subset \Sigma_p^0$. The following proposition is very well known (see Theorem 9 of Section 11.6 in [4]).

Proposition 4.1. *Let $T_1 \in \Sigma_p$ and $T_2 \in \Sigma_q$, where $p > 0$ and $q > 0$. Then $T_1 T_2 \in \Sigma_r$,* *where* $1/r = 1/p + 1/q$, *and*

$$\|T_1T_2\|_{\Sigma_r} \le 2^{1/r}\|T_1\|_{\Sigma_p}\|T_2\|_{\Sigma_q}.$$

Proof. Since $s = s^{r/p}s^{r/q}$, we have

$$\begin{aligned} n(s, T_1T_2) &\le n(s^{r/p}, T_1) + n(s^{r/q}, T_2) \\ &\le s^{-r}(\|T_1\|^p_{\Sigma_p} + \|T_2\|^q_{\Sigma_q}). \end{aligned}$$

Therefore,

$$\|T_1T_2\|^r_{\Sigma_r} \le \|T_1\|^p_{\Sigma_p} + \|T_2\|^q_{\Sigma_q}. \tag{4.6}$$

Clearly, the inequality (4.6) implies the estimate

$$\left\| \frac{1}{\|T_1\|_{\Sigma_p}\|T_2\|_{\Sigma_q}} T_1T_2 \right\|_{\Sigma_r} \le 2^{1/r},$$

which can alternatively be written in the form $\|T_1T_2\|_{\Sigma_r} \le 2^{1/r}\|T_1\|_{\Sigma_p}\|T_2\|_{\Sigma_q}$. □

For self-adjoint operators $T = T^* \in \Sigma_p$, we introduce the functionals

$$\Delta^\pm_p(T) := \limsup_{s\to 0} s^p n_\pm(s, T), \qquad \delta^\pm_p(T) := \liminf_{s\to 0} s^p n_\pm(s, T).$$

The following theorem is due to H. Weyl.

Theorem 4.3. *If* T_1, T_2 *are self-adjoint operators,* $T_1, T_2 \in \Sigma_p$, *and* $T_1 - T_2 \in \Sigma^0_p$, *then*

$$\Delta^\pm_p(T_1) = \Delta^\pm_p(T_2), \qquad \delta^\pm_p(T_1) = \delta^\pm_p(T_2).$$

For $\lambda \in (-m, m)$, we define the operator $X(\lambda)$ by

$$X(\lambda) = W(D_m - \lambda I)^{-1} W, \qquad W = \sqrt{V}. \tag{4.7}$$

The next statement is widely known as the Birman-Schwinger principle. It allows one to transform the spectral problem involving the unbounded operator $D(\alpha)$ into a spectral problem for the compact operator $X(\lambda)$.

Proposition 4.2. *Let $\alpha > 0$ and $X(\lambda)$ be defined by (4.7). Then*

$$N(\lambda, \alpha) = n_+(s, X(\lambda)), \qquad \textit{for } s = \alpha^{-1}.$$

For the proof of this statement, see [2].

We will also need the following theorem (see M. Birman [2]) in which $-\Delta$ is the operator on $\mathbb{R}^d$ with an arbitrary $d \in \mathbb{N}$.

Theorem 4.4. *Let W be a real-valued function on $\mathbb{R}^d$. Let $X = W((-\Delta)^l + I)^{-1} W$, $V = W^2$ and $p = d/2l$. Assume that $[V]_p < \infty$, where*

$$[V]_p = \begin{cases} \left(\sum_{n \in \mathbb{Z}^d} \|V\|^p_{L^1(\mathbb{Q}_n)}\right)^{1/p}, & \textit{if } 0 < p < 1; \\ \sum_{n \in \mathbb{Z}^d} (\int_{\mathbb{Q}_n} V^q \, dx)^{1/q}, \quad \textit{for some} \quad q > 1, & \textit{if } p = 1; \\ \|V\|_p, & \textit{if } p > 1, \end{cases}$$

and $\mathbb{Q}_n = [0,1)^d + n$ for $n \in \mathbb{Z}^d$. Then $X \in \Sigma_p$ and $\|X\|_{\Sigma_p} \leq C[V]_p$ for some constant $C > 0$ independent of V.

With $[V]_p$ defined as in Theorem 4.4, we formulate the following corollary.

Corollary 4.1. *Assume that $X = W((-\Delta)^l + I)^{-1} \tilde{W}, V = W^2, \tilde{V} = \tilde{W}^2$ and $p = d/2l$. Then $\|X\|_{\Sigma_p} \leq C_0 [V]_p^{1/2} [\tilde{V}]_p^{1/2}$ for some constant $C_0 > 0$ independent of V and*

$\tilde{V}$.

4.3 Proof of Theorem 4.1

Let F be the Fourier transform on $L^2(\mathbb{R}^2)$ defined by

$$[Fu](\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^d} e^{-i\xi x}u(x)\,dx.$$

Then F diagonalizes the graphene operator in the sense that

$$D_m = \begin{pmatrix} m & (\frac{\partial}{\partial x_1} + \frac{1}{i}\frac{\partial}{\partial x_2})^2 \\ (\frac{\partial}{\partial x_1} - \frac{1}{i}\frac{\partial}{\partial x_2})^2 & -m \end{pmatrix} = F^* \begin{pmatrix} m & (i\xi_1+\xi_2)^2 \\ (i\xi_1-\xi_2)^2 & -m \end{pmatrix} F.$$

For convenience, we denote

$$\hat{D}_m(\xi) = \begin{pmatrix} m & (i\xi_1+\xi_2)^2 \\ (i\xi_1-\xi_2)^2 & -m \end{pmatrix} = \begin{pmatrix} m & -(\xi_1-i\xi_2)^2 \\ -(\xi_1+i\xi_2)^2 & -m \end{pmatrix},$$

and call it the symbol of D_m. Observe that

$$\begin{aligned} (D_m-\lambda I)^{-1} &= F^*[(\hat{D}_m(\xi)-\lambda I)^{-1}]F = F^* \begin{pmatrix} m-\lambda & -(\xi_1-i\xi_2)^2 \\ -(\xi_1+i\xi_2)^2 & -m-\lambda \end{pmatrix}^{-1} F \\ &= F^* \frac{1}{\lambda^2-m^2-|\xi|^4} \begin{pmatrix} -m-\lambda & (\xi_1-i\xi_2)^2 \\ (\xi_1+i\xi_2)^2 & m-\lambda \end{pmatrix} F. \qquad (4.8) \end{aligned}$$

We now show that it is enough to prove Theorem 4.1 for the case where $W \in C_0^\infty(\mathbb{R}^2)$.

Let $W \notin C_0^\infty(\mathbb{R}^2)$ satisfy the conditions of Theorem 4.1. For an arbitrary $\varepsilon > 0$, let W_ε be a smooth compactly supported approximation of W with the property

$$[(W-W_\varepsilon)^2]_1 < \varepsilon.$$

Then apparently, the operator $X_\varepsilon(\lambda) = W_\varepsilon(D_m - \lambda I)^{-1}W_\varepsilon$ approximates $X(\lambda) = W(D_m - \lambda I)^{-1}W$ in the class Σ_1.

In order to illustrate that, we apply Corollary 4.1, which leads us to the following statement.

Proposition 4.3. *Let W and $\tilde{W}$ be two real-valued functions on $\mathbb{R}^2$ such that*

$$[W^2]_1 < \infty \quad \textit{and} \quad [\tilde{W}^2]_1 < \infty,$$

Then

$$\|W(D_m - \lambda I)^{-1}\tilde{W}\|_{\Sigma_1} \le C[W^2]_1^{1/2}[\tilde{W}^2]_1^{1/2},$$

for some constant $C \ge 0$ that is independent of both W and $\tilde{W}$.

Proof. Indeed, because

$$(D_m - \lambda I)^{-1} = (-\Delta + I)^{-1/2}(-\Delta + I)^{1/2}(D_m - \lambda I)^{-1}(-\Delta + I)^{1/2}(-\Delta + I)^{-1/2},$$

it suffices to show that the operator

$$B(\lambda) = (-\Delta + I)^{1/2}(D_m - \lambda I)^{-1}(-\Delta + I)^{1/2}$$

initially defined on $C_0^\infty(\mathbb{R}^2)$ has a bounded extension to all of $L^2(\mathbb{R}^2)$. The boundedness of $B(\lambda)$ is equivalent to the assertion that the matrix-valued function

$$(|\xi|^2 + 1)^{1/2}(\hat{D}_m(\xi) - \lambda I)^{-1}(|\xi|^2 + 1)^{1/2}$$

is bounded. The latter follows straightforwardly from (4.8). □

As a direct consequence of Proposition 4.3, we obtain

$$\begin{aligned}\|X(\lambda)-X_\varepsilon(\lambda)\|_{\Sigma_1} &\le \|(W-W_\varepsilon)(D_m-\lambda I)^{-1}W\|_{\Sigma_1}+\|W_\varepsilon(D_m-\lambda I)^{-1}(W-W_\varepsilon)\|_{\Sigma_1}\\ &\le C([(W-W_\varepsilon)^2]_1^{1/2}[W^2]_1^{1/2}+[W_\varepsilon^2]_1^{1/2}[(W-W_\varepsilon)^2]_1^{1/2})\\ &\le C\sqrt{\varepsilon}([W^2]_1^{1/2}+[W_\varepsilon^2]_1^{1/2}]) =: \delta(\varepsilon),\end{aligned}$$

which tends to 0 when $\varepsilon \to 0$. Moreover, $X(\lambda) = X_\varepsilon(\lambda) + (X(\lambda) - X_\varepsilon(\lambda))$. By Ky Fan inequality, it follows that

$$\begin{aligned}n_+(s, X(\lambda)) &\le n_+((1-\varepsilon_0)s, X_\varepsilon(\lambda)) + n_+(\varepsilon_0 s, X(\lambda)-X_\varepsilon(\lambda))\\ &\le n_+((1-\varepsilon_0)s, X_\varepsilon(\lambda)) + (\varepsilon_0 s)^{-1}\|X(\lambda)-X_\varepsilon(\lambda)\|_{\Sigma_1}\\ &\le n_+((1-\varepsilon_0)s, X_\varepsilon(\lambda)) + (\varepsilon_0 s)^{-1}\delta(\varepsilon), \end{aligned}\tag{4.9}$$

for any $0 < \varepsilon_0 < 1$. Suppose that Theorem 4.1 holds for the potential $V_\varepsilon = W_\varepsilon^2$, that is,

$$\lim_{s\to 0} sn_+(s, X_\varepsilon(\lambda)) = \frac{1}{4\pi}\int_{\mathbb{R}^2} V_\varepsilon\, dx.$$

Then we deduce from (4.9) that

$$\begin{aligned}&\limsup_{s\to 0} sn_+(s, X(\lambda)) \le \limsup_{s\to 0} sn_+((1-\varepsilon_0)s, X_\varepsilon(\lambda)) + \delta(\varepsilon)/\varepsilon_0\\ &= \frac{1}{1-\varepsilon_0}\limsup_{s\to 0} sn_+(s, X_\varepsilon(\lambda)) + \delta(\varepsilon)/\varepsilon_0 = \frac{1}{4\pi(1-\varepsilon_0)}\int_{\mathbb{R}^2} V_\varepsilon\, dx + \delta(\varepsilon)/\varepsilon_0.\end{aligned}$$

By taking the limit as $\varepsilon \to 0$, we infer that

$$\limsup_{s\to 0} sn_+(s, X(\lambda)) \le \frac{1}{4\pi(1-\varepsilon_0)}\int_{\mathbb{R}^2} V\, dx.$$

Due to the fact that $\varepsilon_0 > 0$ is arbitrary, we obtain

$$\limsup_{s\to 0} sn_+(s, X(\lambda)) \le \frac{1}{4\pi}\int_{\mathbb{R}^2} V\,dx.$$

Similarly, we can establish the inequality

$$\liminf_{s\to 0} sn_+(s, X(\lambda)) \ge \frac{1}{4\pi}\int_{\mathbb{R}^2} V\,dx.$$

Thus, Theorem 4.1 holds for potentials $V \notin C_0^\infty$ as long as it holds for all $V = W^2$ with $W \in C_0^\infty$.

Now we assume that $W \in C_0^\infty(\mathbb{R}^2)$, and choose an appropriate function $\zeta \in C^\infty(\mathbb{R}^2)$ with the property

$$\zeta(\xi) = \begin{cases} 0, & \text{if } |\xi| < \varepsilon; \\ 1, & \text{if } |\xi| > 2, \end{cases}$$

where $\varepsilon > 0$ is sufficiently small. Then we define the matrix-valued function

$$a(\xi) = \frac{\zeta(\xi)}{|\xi|^4}\begin{pmatrix} 0 & -(\xi_1 - i\xi_2)^2 \\ -(\xi_1 + i\xi)^2 & 0 \end{pmatrix}.$$

Additionally, we set

$$\tilde{X}(\lambda) = \sqrt{V}F^*[a(\xi)]F\sqrt{V},$$

where $[a(\xi)]$ denotes the operator of multiplication by the function $a(\xi)$, and prove a valuable proposition below.

Proposition 4.4. *Let* $W \in C_0^\infty(\mathbb{R}^2)$*. Suppose that* $\lim_{s\to 0} sn_+(s, \tilde{X}(\lambda))$ *exists. Then*

$$\lim_{s\to 0} sn_+(s, \tilde{X}(\lambda)) = \lim_{s\to 0} sn_+(s, X(\lambda)).$$

Proof. It is adequate to prove that $\tilde{X}(\lambda) - X(\lambda) \in \Sigma_1^0$. Observe that

$$\tilde{X}(\lambda) - X(\lambda) = WF^*[\beta(\xi)]FW,$$

where

$$\begin{aligned} \beta(\xi) &= (\hat{D}_m(\xi) - \lambda I)^{-1} - \frac{\zeta(\xi)}{|\xi|^4} \begin{pmatrix} 0 & -(\xi_1 - i\xi_2)^2 \\ -(\xi_1 + i\xi_2)^2 & 0 \end{pmatrix} \\ &= \frac{1}{m^2 - \lambda^2 + |\xi|^4} \begin{pmatrix} m + \lambda & 0 \\ 0 & \lambda - m \end{pmatrix} \\ &\quad + \left(\frac{1}{m^2 - \lambda^2 + |\xi|^4} - \frac{\zeta(\xi)}{|\xi|^4} \right) \begin{pmatrix} 0 & -(\xi_1 - i\xi_2)^2 \\ -(\xi_1 + i\xi_2)^2 & 0 \end{pmatrix}. \end{aligned} \tag{4.10}$$

The first term on the right-hand side of (4.10) is an integrable function of ξ. And regarding the second term, we have

$$\frac{1}{m^2 - \lambda^2 + |\xi|^4} - \frac{\zeta(\xi)}{|\xi|^4} = \frac{(1 - \zeta(\xi))|\xi|^4 - \zeta(\xi)(m^2 - \lambda^2)}{|\xi|^4(m^2 - \lambda^2 + |\xi|^4)},$$

which is evidently integrable in some neighborhood of the origin by the definition of ζ and equals $O(|\xi|^{-8})$ as $|\xi| \to \infty$. Thus, $\beta \in L^1(\mathbb{R}^2)$. This implies that the operators $WF^*|\beta(\xi)|^{1/2}$ and $|\beta(\xi)|^{1/2}FW$ are both Hilbert-Schmidt operators. Therefore,

$$\tilde{X}(\lambda) - X(\lambda) = WF^*\left[|\beta(\xi)|^{1/2}\mathrm{sign}(\beta(\xi))|\beta(\xi)|^{1/2}\right]FW$$

belongs to $\mathfrak{S}_1 \subset \Sigma_1^0$. □

The following theorem is a consequence of Theorem 2 from [4].

Theorem 4.5. *Let $W \in C_0^\infty(\mathbb{R}^2)$. Then*

$$\lim_{s\to 0}(sn_+(s,\tilde{X}(\lambda))) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n_+(1,G(x,\xi))\,dx\,d\xi, \tag{4.11}$$

where

$$G(x,\xi) = V(x)|\xi|^{-4}\begin{pmatrix} 0 & -(\xi_1 - i\xi_2)^2 \\ -(\xi_1 + i\xi_2)^2 & 0 \end{pmatrix}. \tag{4.12}$$

Hence, for every fixed $\xi \in \mathbb{R}^2$, the trace of operator $G(x,\xi)$ in (4.12) is

$$\operatorname{tr}(G(x,\xi)) = 0,$$

which implies that the sum of the eigenvalues of $G(x,\xi)$ is zero: $\lambda_- + \lambda_+ = 0$. Besides computing the trace, we calculate the determinant

$$\det(G(x,\xi)) = \lambda_+ \cdot \lambda_- = -\frac{V^2(x)}{|\xi|^8}\cdot|\xi|^4 = -V^2(x)|\xi|^{-4}.$$

Thus, $\lambda_+ = V(x)|\xi|^{-2}, \lambda_- = -V(x)|\xi|^{-2}$, which yields that $n_+(1,G(x,\xi)) \le 1$. More precisely,

$$n_+(1,G(x,\xi)) = \begin{cases} 0, & \text{if } V(x)|\xi|^{-2} \le 1; \\ 1, & \text{if } V(x)|\xi|^{-2} > 1. \end{cases}$$

Consequently, $n_+(1,G(x,\xi))$ is the characteristic function χ_Ω of the set

$$\Omega := \{(x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2 : V(x)|\xi|^{-2} > 1\},$$

which implies

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n_+(1, G(x,\xi))\,dx\,d\xi = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\chi_\Omega(x,\xi)\,d\xi\right)dx$$
$$= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\pi V(x)\,dx = \frac{1}{4\pi}\int_{\mathbb{R}^2}V(x)\,dx. \tag{4.13}$$

Combining (4.11) and (4.13) with Propositions 4.2 and 4.4, we establish the asymptotic relation

$$N(\lambda,\alpha)\sim\frac{\alpha}{4\pi}\int_{\mathbb{R}^2}V(x)\,dx, \quad \text{as } \alpha\to\infty,$$

for every V that obeys (4.3). The proof of Theorem 4.1 is complete. ■

4.4 Proof of Theorem 4.2

For $0<\varepsilon_1<\varepsilon_2<\infty$ and $\alpha>0$, we decompose $\mathbb{R}^2$ into the union of the following three subsets:

$$\begin{aligned}
\Omega_1(\alpha) &= \{x\in\mathbb{R}^2:\ |x|<\varepsilon_1\alpha^{1/p}\},\\
\Omega_2(\alpha) &= \{x\in\mathbb{R}^2: \varepsilon_1\alpha^{1/p}\le|x|\le\varepsilon_2\alpha^{1/p}\},\\
\Omega_3(\alpha) &= \{x\in\mathbb{R}^2:\ |x|>\varepsilon_2\alpha^{1/p}\},
\end{aligned}$$

and set χ_1,χ_2 and χ_3 to be the characteristic function of $\Omega_1(\alpha),\Omega_2(\alpha)$ and $\Omega_3(\alpha)$, respectively. Also, we define the three operators $W_i=\chi_i W$ for $i=1,2,3$. Clearly, $W=\sum_{i=1}^3 W_i$, which leads to

$$X(\lambda) = W(D_m-\lambda I)^{-1}W = \sum_{i=1}^{3}\sum_{j=1}^{3}W_i(D_m-\lambda I)^{-1}W_j. \tag{4.14}$$

Only the three operators of form $W_i(D_m-\lambda I)^{-1}W_i$ in (4.14) contribute to the asymptotics of $n(s,X(\lambda))$. In some sense, the sum on the right-hand side of (4.14) can be replaced by the operator

$$X(\lambda)^+ = W_1(D_m - \lambda I)^{-1}W_1 + W_2(D_m - \lambda I)^{-1}W_2 + W_3(D_m - \lambda I)^{-1}W_3.$$

We will show that the contribution of the remaining six operators in (4.14) to the asymptotics of $n(s, X(\lambda))$ is negligible. Namely, for $i \neq j$, we will prove that

$$n(\varepsilon\alpha^{-1}, W_i(D_m - \lambda I)^{-1}W_j) = o(\alpha^{2/p}), \qquad \text{as } \alpha \to \infty, \quad \forall \varepsilon > 0. \tag{4.15}$$

By Ky Fan inequality, (4.15) would indicate that

$$n(\varepsilon\alpha^{-1}, X(\lambda)^-) = o(\alpha^{2/p}), \qquad \text{as } \alpha \to \infty, \tag{4.16}$$

where

$$X(\lambda)^- = \sum_{i \neq j} W_i(D_m - \lambda I)^{-1}W_j. \tag{4.17}$$

Proposition 4.5. *Let* (4.16) *hold for any* $\varepsilon > 0$. *Assume that for every* $\tau > 0$, $n_+(\tau\alpha^{-1}, X(\lambda)^+) \sim \tau^{-2/p}\alpha^{2/p}J(\lambda, m)$ *as* $\alpha \to \infty$, *where*

$$J(\lambda, m) = \frac{1}{4\pi}\int_{\mathbb{R}^2} [((\lambda + \Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\, dx.$$

Then

$$n_+(\alpha^{-1}, X(\lambda)) \sim \alpha^{2/p}J(\lambda, m), \qquad \textit{as } \alpha \to \infty.$$

Proof. Indeed, for any $\varepsilon > 0$,

$$n_+(\alpha^{-1}, X(\lambda)) \le n_+((1-\varepsilon)\alpha^{-1}, X(\lambda)^+) + n(\varepsilon\alpha^{-1}, X(\lambda)^-),$$

and

$$n_+(\alpha^{-1}, X(\lambda)) \geq n_+(((1-\varepsilon)\alpha)^{-1}, X(\lambda)^+) - n(\frac{\varepsilon}{1-\varepsilon}\alpha^{-1}, X(\lambda)^-).$$

Therefore,

$$\begin{aligned}(1-\varepsilon)^{2/p} J(\lambda, m) &\leq \liminf_{\alpha\to\infty} \alpha^{-2/p} n_+(\alpha^{-1}, X(\lambda)) \\ &\leq \limsup_{\alpha\to\infty} \alpha^{-2/p} n_+(\alpha^{-1}, X(\lambda)) \leq \frac{J(\lambda, m)}{(1-\varepsilon)^{2/p}}.\end{aligned}$$

Taking the limit as $\varepsilon \to 0$, we obtain the objective relation. □

Let us first prove (4.16). Evidently, the operator $W_i(D_m - \lambda I)^{-1}W_j$ is self-adjoint if $i = j$ and non-self-adjoint if $i \neq j$. Also, $W_i(D_m - \lambda I)^{-1}W_j$ is the adjoint of $W_j(D_m - \lambda I)^{-1}W_i$. Hence, $X(\lambda)^+$ and $X(\lambda)^-$ are both self-adjoint operators. Since the singular values of the operators $W_i(D_m - \lambda I)^{-1}W_j$ and $W_j(D_m - \lambda I)^{-1}W_i$ are the same, we need to consider only three operators in (4.17).

4.4.1 Operator $W_1(D_m - \lambda I)^{-1}W_2$

We intend to demonstrate that

$$n(\varepsilon\alpha^{-1}, W_1(D_m - \lambda I)^{-1}W_2) = o(\alpha^{2/p}), \qquad \text{as } \alpha \to \infty. \tag{4.18}$$

We fix $\delta \in (0, \infty)$ and select a function $\zeta \in C^\infty(\mathbb{R})$ satisfying the condition

$$\zeta(t) = \begin{cases} 0, & \text{if } t < 0; \\ 1, & \text{if } t > \delta. \end{cases} \tag{4.19}$$

Then we define another function $\theta \in C^\infty(\mathbb{R}^2)$ by setting $\theta(x) = \zeta(|x| - \varepsilon_1 \alpha^{1/p})$. Observe that

$$\begin{aligned} W_1(D_m - \lambda I)^{-1} W_2 &= W_1(D_m - \lambda I)^{-1}\theta W_2 + W_1(D_m - \lambda I)^{-1}(1-\theta)W_2 \\ &= W_1[(D_m - \lambda I)^{-1}, \theta]\, W_2 + W_1(D_m - \lambda I)^{-1}(1-\theta)W_2, \end{aligned} \tag{4.20}$$

where $[A, B] = AB - BA$. Also, it is easy to verify that $[A^{-1}, B] = A^{-1}(BA - AB)A^{-1}$, provided that A is invertible. Therefore,

$$\begin{aligned} [(D_m - \lambda I)^{-1}, \theta] &= (D_m - \lambda I)^{-1}(\theta(D_m - \lambda I) - (D_m - \lambda I)\theta)(D_m - \lambda I)^{-1} \\ &= (D_m - \lambda I)^{-1}(\theta D_m - D_m \theta)(D_m - \lambda I)^{-1}. \end{aligned} \tag{4.21}$$

A brief computation yields that

$$\theta D_m - D_m \theta = \theta D_0 - D_0 \theta = Y_0 + Y_1,$$

where

$$\begin{aligned} Y_0 &= -\begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix}, \quad Y_1 = -2\begin{pmatrix} 0 & \theta_3(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}) \\ \theta_4(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}) & 0 \end{pmatrix}, \\ \theta_1 &= \frac{\partial^2\theta}{\partial x_1^2} - 2i\frac{\partial^2\theta}{\partial x_1 \partial x_2} - \frac{\partial^2\theta}{\partial x_2^2}, \quad \theta_2 = \frac{\partial^2\theta}{\partial x_1^2} + 2i\frac{\partial^2\theta}{\partial x_1 \partial x_2} - \frac{\partial^2\theta}{\partial x_2^2}, \\ \theta_3 &= \frac{\partial\theta}{\partial x_1} - i\frac{\partial\theta}{\partial x_2}, \quad \theta_4 = \frac{\partial\theta}{\partial x_1} + i\frac{\partial\theta}{\partial x_2}. \end{aligned}$$

Multiplying the equality (4.21) by W_1 and W_2, we obtain

$$\begin{aligned} & W_1((D_m - \lambda I)^{-1}\theta - \theta(D_m - \lambda I)^{-1})W_2 \\ = \;& W_1(D_m - \lambda I)^{-1}(Y_0 + Y_1)(D_m - \lambda I)^{-1}W_2 \\ = \;& X_{1,2} + \tilde{X}_{1,2}, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} X_{1,2} &= W_1(D_m - \lambda I)^{-1} Y_0 (D_m - \lambda I)^{-1} W_2, \\ \tilde{X}_{1,2} &= W_1(D_m - \lambda I)^{-1} Y_1 (D_m - \lambda I)^{-1} W_2. \end{aligned}$$

Let us also denote $\hat{X}_{1,2} = W_1(D_m - \lambda I)^{-1}(1-\theta)W_2$. To prove (4.15) for $i = 1, j = 2$, it is sufficient to establish that $n(\varepsilon\alpha^{-1}, X_{1,2}) = o(\alpha^{2/p}), n(\varepsilon\alpha^{-1}, \tilde{X}_{1,2}) = o(\alpha^{2/p})$ and $n(\varepsilon\alpha^{-1}, \hat{X}_{1,2}) = o(\alpha^{2/p})$ as $\alpha \to \infty$.

Let us examine the operator $X_{1,2}$. Let χ_δ denote the characteristic function of the layer $\{x \in \mathbb{R}^2 : \varepsilon_1\alpha^{1/p} < |x| < \varepsilon_1\alpha^{1/p} + \delta\}$. Since the supports of the functions θ_1 and θ_2 are both contained in the layer, one may represent operator $X_{1,2}$ as

$$X_{1,2} = W_1(D_m - \lambda I)^{-1} \chi_\delta Y_0 \chi_\delta (D_m - \lambda I)^{-1} W_2.$$

In addition, by the definition of Y_0, we have $\|Y_0\| \leq \|\theta_1\|_\infty + \|\theta_2\|_\infty$. Based on Hölder's inequality for the Σ_p class stated in Proposition 4.1, we infer that

$$\|X_{1,2}\|_{\Sigma_{1/2}} \leq 4(\|\theta_1\|_\infty + \|\theta_2\|_\infty)\|W_1(D_m - \lambda I)^{-1}\chi_\delta\|_{\Sigma_1} \cdot \|\chi_\delta(D_m - \lambda I)^{-1} W_2\|_{\Sigma_1}.$$

Moreover,

$$F(D_m - \lambda I)^{-1} F^* = \frac{1}{m^2 - \lambda^2 + |\xi|^4} \begin{pmatrix} \lambda + m & -(\xi_1 - i\xi_2)^2 \\ -(\xi_1 + i\xi_2)^2 & \lambda - m \end{pmatrix} \tag{4.23}$$

which tends to zero as $O(|\xi|^{-2})$ as $|\xi| \to \infty$. Therefore, there exists some matrix-valued function $b \in L^\infty(\mathbb{R}^2)$ which fulfills the relation

$$F(D_m - \lambda I)^{-1} F^* = \frac{b(\xi)}{1 + |\xi|^2}.$$

Consequently, we obtain

$$\begin{aligned}\|W_1(D_m-\lambda I)^{-1}\chi_\delta\|_{\Sigma_1} &\le \|W_1F^*(|\xi|^2+1)^{-1/2}b(\xi)(|\xi|^2+1)^{-1/2}F\chi_\delta\|_{\Sigma_1}\\ &\le 2\|b\|_\infty\|W_1F^*(|\xi|^2+1)^{-1/2}\|_{\Sigma_2}\|(|\xi|^2+1)^{-1/2}F\chi_\delta\|_{\Sigma_2}. \end{aligned} \tag{4.24}$$

Through Corollary 4.1, we conclude that

$$\begin{aligned}&\|W_1F^*(|\xi|^2+1)^{-1/2}\|_{\Sigma_2}^2 = \|W_1F^*(|\xi|^2+1)^{-1}FW_1\|_{\Sigma_1}\\ &= \|W_1((-\Delta)+I)^{-1}W_1\|_{\Sigma_1} \le C_0[W_1^2]_1. \end{aligned} \tag{4.25}$$

Likewise, we have

$$\|(|\xi|^2+1)^{-1/2}F\chi_\delta\|_{\Sigma_2}^2 \le C_0[\chi_\delta]_1. \tag{4.26}$$

Combining (4.24), (4.25) and (4.26), we come to the following estimate:

$$\begin{aligned}\|W_1(D_m-\lambda I)^{-1}\chi_\delta\|_{\Sigma_1} &\le 2C_0\|b\|_\infty[\chi_\delta]_1^{1/2}\cdot[W_1^2]_1^{1/2}\\ &= C_1\left(\sum_{n\in\mathbb{Z}^d}\|W_1^2\|_{L^q(\mathbb{Q}_n)}\right)^{1/2}[\chi_\delta]_1^{1/2}\\ &\le C_1\left(\sum_{n\in\mathbb{Z}^d,|n|<\varepsilon_1\alpha^{1/p}}\frac{1}{|n|^p+1}\right)^{1/2}(O(\alpha^{1/p}))^{1/2}\\ &\le C_2\left(\int_{|x|<\varepsilon_1\alpha^{1/p}}\frac{dx}{|x|^p+1}\right)^{1/2}O(\alpha^{1/2p})\\ &\asymp C_2\left(\int_0^{2\pi}\int_{r<\varepsilon_1\alpha^{1/p}}r^{1-p}\,dr\,d\theta\right)^{1/2}O(\alpha^{1/2p})\\ &= \left(2\pi C_2\left[\frac{r^{2-p}}{2-p}\right]\Big|_{r=0}^{r=\varepsilon_1\alpha^{1/p}}\right)^{1/2}O(\alpha^{1/2p})\\ &= (C_3\varepsilon_1^{2-p}\alpha^{\frac{2}{p}-1})^{1/2}O(\alpha^{1/2p}) = C_4\varepsilon_1^{1-\frac{p}{2}}O(\alpha^{\frac{3}{2p}-\frac{1}{2}}),\end{aligned}$$

where C_1, C_2, C_3 and C_4 are non-negative constants. In the estimate above, we used the fact that

$$\begin{aligned}[\chi_\delta]_1 &= \sum_{n\in\mathbb{Z}^d} \|\chi_\delta\|_{L^q(\mathbb{Q}_n)} \asymp \sum_{\varepsilon_1\alpha^{1/p}<|n|<\varepsilon_1\alpha^{1/p}+\delta} 1 \\ &= \text{Area}\{x\in\mathbb{R}^2 : \varepsilon_1\alpha^{1/p} < |x| < \varepsilon_1\alpha^{1/p}+\delta\} \\ &= \pi\left[(\varepsilon_1\alpha^{1/p}+\delta)^2 - (\varepsilon_1\alpha^{1/p})^2\right] = \pi\left(2\varepsilon_1\alpha^{1/p}\delta+\delta^2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|X_{1,2}\|_{\Sigma_{1/2}} &\le 4(\|\theta_1\|_\infty + \|\theta_2\|_\infty)\|W_1(D_m-\lambda I)^{-1}\chi_\delta\|_{\Sigma_1}\cdot\|\chi_\delta(D_m-\lambda I)^{-1}W_2\|_{\Sigma_1} \\ &= O(\alpha^{\frac{3}{p}-1}), \qquad \text{as } \alpha\to\infty.\end{aligned}$$

Due to

$$\sqrt{s}n(s, X_{1,2}) \le \|X_{1,2}\|_{\Sigma_{1/2}}^{1/2},$$

we draw the conclusion that

$$n(\varepsilon\alpha^{-1}, X_{1,2}) \le O(\alpha^{\frac{3}{2p}}) = o(\alpha^{2/p}),$$

as $\alpha\to\infty$. The study of operator $X_{1,2}$ is complete.

Let us deal with the operator $\tilde{X}_{1,2}$. Note that the order of the differential operator Y_1 is 1 and that of the Laplace operator Δ is 2, which implies that the operator $Y_1(-\Delta+I)^{-1/2}$ is bounded. Indeed, through direct computation, we derive the following inequality:

$$\begin{aligned}\|Y_1 u\|^2 &= 4\int_{\mathbb{R}^2}\left|\theta_3\left(\frac{\partial u_2}{\partial x_1} - i\frac{\partial u_2}{\partial x_2}\right)\right|^2 dx + 4\int_{\mathbb{R}^2}\left|\theta_4\left(\frac{\partial u_1}{\partial x_1} + i\frac{\partial u_1}{\partial x_2}\right)\right|^2 dx \\ &\le 8\left(\|\theta_3\|_\infty^2 \int_{\mathbb{R}^2} |\nabla u_2|^2\, dx + \|\theta_4\|_\infty^2 \int_{\mathbb{R}^2} |\nabla u_1|^2\, dx\right). \end{aligned} \tag{4.27}$$

On the other hand, we have

$$\|(-\Delta + I)^{1/2} u\|^2 = \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2)\, dx, \tag{4.28}$$

where

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)^T.$$

Incorporating (4.27) and (4.28), we obtain

$$\begin{aligned}\|Y_1 u\|^2 &\le 8(\|\theta_3\|_\infty^2 + \|\theta_4\|_\infty^2)\cdot\left(\int_{\mathbb{R}^2}\left(|\nabla u_1|^2 + |\nabla u_2|^2\right)\, dx\right) \\ &\le C_\delta\left(\int_{\mathbb{R}^2}\left(|\nabla u_1|^2 + |u_1|^2 + |\nabla u_2|^2 + |u_2|^2\right)\, dx\right) \\ &= C_\delta \|(-\Delta + I)^{1/2} u\|_{L^2}^2, \qquad \forall u \in L^2, \end{aligned} \tag{4.29}$$

where $C_\delta = 8(\|\theta_3\|_\infty^2 + \|\theta_4\|_\infty^2)$ is a constant depending only on δ. Making the substitution $u = (-\Delta + I)^{-1/2} v$ in (4.29), we find that

$$\|Y_1(-\Delta + I)^{-1/2} v\| \le C_\delta^{1/2}\|v\|, \qquad \forall v \in L^2,$$

which means that $Y_1(-\Delta + I)^{-1/2}$ is a bounded operator.

One can also see that the operator $\tilde{X}_{1,2}$ may be written as

$$\tilde{X}_{1,2} = W_1(D_m - \lambda I)^{-1}\chi_\delta[Y_1(-\Delta + I)^{-1/2}](-\Delta + I)^{1/2}(D_m - \lambda I)^{-1} W_2.$$

It follows from Hölder's inequality and Proposition 4.3 that

$$\begin{aligned}\|\tilde{X}_{1,2}\|_{\Sigma_{2/3}} &\leq C\|W_1(D_m-\lambda I)^{-1}\chi_\delta\|_{\Sigma_1}\cdot\|(-\Delta+I)^{1/2}(D_m-\lambda I)^{-1}W_2\|_{\Sigma_2}\\ &\leq CC_0^{1/2}[W_1^2]_1^{1/2}[\chi_\delta]_1^{1/2}\|(-\Delta+I)^{1/2}(D_m-\lambda I)^{-1}W_2\|_{\Sigma_2}\\ &= O(\alpha^{\frac{3}{2p}-\frac{1}{2}})\|(-\Delta+I)^{1/2}(D_m-\lambda I)^{-1}W_2\|_{\Sigma_2},\end{aligned}$$

for some constant $C \geq 0$. To approximate the last factor on the right-hand side, we use the representation

$$\begin{aligned}&(-\Delta+I)^{1/2}(D_m-\lambda I)^{-1}W_2\\ =\ &F^*\frac{(|\xi|^2+1)^{1/2}}{m^2-\lambda^2+|\xi|^4}\begin{pmatrix}\lambda+m & -(\xi_1-i\xi_2)^2\\ -(\xi_1+i\xi_2)^2 & \lambda-m\end{pmatrix}FW_2\\ =\ &F^*\frac{1}{(|\xi|^2+1)^{1/2}}\tilde{b}(\xi)FW_2,\end{aligned}$$

for some matrix-valued $\tilde{b} \in L^\infty(\mathbb{R}^2)$. Hence,

$$\begin{aligned}&\|(-\Delta+I)^{1/2}(D_m-\lambda I)^{-1}W_2\|_{\Sigma_2} = \|F^*\frac{1}{(|\xi|^2+1)^{1/2}}\tilde{b}(\xi)FW_2\|_{\Sigma_2}\\ &\leq \|\tilde{b}\|_\infty\cdot\|(|\xi|^2+1)^{-1/2}FW_2\|_{\Sigma_2} = \|\tilde{b}\|_\infty\cdot\|W_2F^*\frac{1}{\sqrt{|\xi|^2+1}}\frac{1}{\sqrt{|\xi|^2+1}}FW_2\|_{\Sigma_1}^{1/2}\\ &= \|\tilde{b}\|_\infty\|W_2(-\Delta+I)^{-1}W_2\|_{\Sigma_1}^{1/2} \leq C_5[W_2^2]_1^{1/2} \quad \text{(by Corollary 4.1)}\\ &= C_5\sum_{n\in\mathbb{Z}^d}\|W_2^2\|_{L^q(\mathbb{Q}_n)}^{1/2} = O(\alpha^{\frac{1}{p}-\frac{1}{2}}), \quad \text{as } \alpha\to\infty,\end{aligned}$$

where C_5 is a non-negative constant. Therefore, we get

$$\|\tilde{X}_{1,2}\|_{\Sigma_{2/3}} = O(\alpha^{\frac{5}{2p}-1}), \quad \text{as } \alpha\to\infty,$$

which leads to the inequality

$$(\varepsilon\alpha^{-1})^{2/3} n(\varepsilon\alpha^{-1}, \tilde{X}_{1,2}) \leq \|\tilde{X}_{1,2}\|^{2/3}_{\Sigma_{2/3}} = O(\alpha^{\frac{5}{3p}-\frac{2}{3}}).$$

As a result, we establish that $n(\varepsilon\alpha^{-1}, \tilde{X}_{1,2}) = O(\alpha^{\frac{5}{3p}}) = o(\alpha^{2/p})$. The analysis of operator $\tilde{X}_{1,2}$ is accomplished.

We still have to demonstrate that $n(\varepsilon\alpha^{-1}, \hat{X}_{1,2}) = o(\alpha^{2/p})$ as $\alpha \to \infty$. Note that

$$\hat{X}_{1,2} = W_1(D_m - \lambda I)^{-1}(1-\theta)W_2 = W_1 F^* \frac{1}{(\xi^2+1)^{1/2}} b(\xi) \frac{1}{(\xi^2+1)^{1/2}} F(1-\theta)W_2.$$

As a consequence,

$$\begin{aligned} \|\hat{X}_{1,2}\|_{\Sigma_1} &\leq 2\|b\|_\infty \|W_1 F^* \frac{1}{(\xi^2+1)^{1/2}}\|_{\Sigma_2} \cdot \|\frac{1}{(\xi^2+1)^{1/2}} F(1-\theta)W_2\|_{\Sigma_2} \\ &\leq C_0 \|b\|_\infty [W_1^2]_1^{1/2} [(1-\theta)^2 W_2^2]_1^{1/2}. \end{aligned}$$

Recall that we have already illustrated the relation

$$[W_1^2]_1 = O(\alpha^{\frac{2}{p}-1}), \quad \text{as } \alpha \to \infty.$$

Moreover,

$$\begin{aligned} [(1-\theta)^2 W_2^2]_1 &= \sum_{n\in\mathbb{Z}^2} \|(1-\theta)^2 W_2^2\|_{L^q(\mathbb{Q}_n)} \asymp C \int_{\varepsilon_1\alpha^{1/p} < r < \varepsilon_1\alpha^{1/p}+\delta} \frac{r\,dr}{r^p} \\ &= C \left[\frac{r^{2-p}}{2-p}\right] \Big|_{r=\varepsilon_1\alpha^{1/p}}^{r=\varepsilon_1\alpha^{1/p}+\delta} = C_1((\varepsilon_1\alpha^{1/p}+\delta)^{2-p} - (\varepsilon_1\alpha^{1/p})^{2-p}) \\ &= C_1(\varepsilon_1\alpha^{1/p})^{2-p}((1+\frac{\delta}{\varepsilon_1\alpha^{1/p}})^{2/p} - 1) \sim C_1(\varepsilon_1\alpha^{1/p})^{2-p}\left(\frac{2\delta}{p\varepsilon_1\alpha^{1/p}}\right) \\ &= O(\alpha^{\frac{1}{p}-1}). \end{aligned}$$

Thus,

$$\|\hat{X}_{1,2}\|_{\Sigma_1} = O(\alpha^{\frac{3}{2p}-1}),$$

which implies

$$n(\varepsilon\alpha^{-1}, \hat{X}_{1,2}) = O(\alpha^{\frac{3}{2p}}) = o(\alpha^{\frac{2}{p}}), \quad \text{as } \alpha \to \infty.$$

4.4.2 Operator $W_2(D_m - \lambda I)^{-1}W_3$

We employ the same approach as in the preceding subsection to examine the second non-self-adjoint operator in (4.14), specifically, $W_2(D_m - \lambda I)^{-1}W_3$.

Let function $\zeta \in C^\infty(\mathbb{R})$ be defined as in (4.19) and function $\eta : \mathbb{R}^2 \to \mathbb{R}$ be given by $\eta(x) = \zeta(|x| - \varepsilon_2\alpha^{1/p})$. There is no doubt that (4.20), (4.21) and (4.22) would all remain applicable if we replace θ, W_1 and W_2 by η, W_2 and W_3, respectively. In particular,

$$\begin{aligned} W_2(D_m - \lambda I)^{-1}W_3 &= W_2(D_m - \lambda I)^{-1}\eta W_3 + W_2(D_m - \lambda I)^{-1}(1-\eta)W_3 \\ &= W_2[(D_m - \lambda I)^{-1}, \eta]\, W_3 + W_2(D_m - \lambda I)^{-1}(1-\eta)W_3, \quad (4.30) \end{aligned}$$

and the first term on the right-hand side can be written as

$$\begin{aligned} W_2[(D_m - \lambda I)^{-1}, \eta]\, W_3 &= W_2(D_m - \lambda I)^{-1}(\tilde{Y}_0 + \tilde{Y}_1)(D_m - \lambda I)^{-1}W_3 \\ &= X_{2,3} + \tilde{X}_{2,3}, \quad (4.31) \end{aligned}$$

where

$$\begin{aligned}
X_{2,3} &= W_2(D_m-\lambda I)^{-1}\tilde{Y}_0(D_m-\lambda I)^{-1}W_3,\\
\tilde{X}_{2,3} &= W_2(D_m-\lambda I)^{-1}\tilde{Y}_1(D_m-\lambda I)^{-1}W_3,\\
\tilde{Y}_0 &= -\begin{pmatrix} 0 & \eta_1 \\ \eta_2 & 0 \end{pmatrix}, \tilde{Y}_1 = -2\begin{pmatrix} 0 & \eta_3(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}) \\ \eta_4(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}) & 0 \end{pmatrix},
\end{aligned}$$

with η_1, η_2, η_3 and η_4 defined in the same way as $\theta_1, \theta_2, \theta_3$ and θ_4, respectively, except that, each time, θ is replaced by η. We argue in the same way as before to show that $n(s, X_{2,3}) = o(\alpha^{2/p})$ and $n(s, \tilde{X}_{2,3}) = o(\alpha^{2/p})$.

Let us first deal with the operator $X_{2,3}$. Denote by ψ_δ the characteristic function of set $\{x \in \mathbb{R}^2 : \varepsilon_2\alpha^{1/p} < |x| < \varepsilon_2\alpha^{1/p} + \delta\}$. Then

$$X_{2,3} = W_2(D_m-\lambda I)^{-1}\psi_\delta\tilde{Y}_0\psi_\delta(D_m-\lambda I)^{-1}W_3.$$

Referring to the bound $\|\tilde{Y}_0\| \le \|\eta_1\|_\infty + \|\eta_2\|_\infty$ and using Hölder's inequality in the Σ_p-class, we obtain

$$\|X_{2,3}\|_{\Sigma_1} \le (\|\eta_1\|_\infty + \|\eta_2\|_\infty)\|W_2(D_m-\lambda I)^{-1}\psi_\delta\|_{\Sigma_1} \cdot \|\psi_\delta(D_m-\lambda I)^{-1}W_3\|. \quad (4.32)$$

If $b \in L^\infty(\mathbb{R}^2)$ is the function specified in the preceding subsection, then

$$\begin{aligned}
\|W_2(D_m-\lambda I)^{-1}\psi_\delta\|_{\Sigma_1} &\le \|W_2F^*(|\xi|^2+1)^{-1/2}b(\xi)(|\xi|^2+1)^{-1/2}F\psi_\delta\|_{\Sigma_1}\\
&\le 2\|b\|_\infty\|W_2F^*(|\xi|^2+1)^{-\frac{1}{2}}\|_{\Sigma_2}\|(|\xi|^2+1)^{-\frac{1}{2}}F\psi_\delta\|_{\Sigma_2}. \quad (4.33)
\end{aligned}$$

Furthermore, (4.25) and (4.26) hold when W_1 and χ_δ are substituted by W_2 and ψ_δ, respectively. Combining them yields the following estimate:

$$\|W_2(D_m-\lambda I)^{-1}\psi_\delta\|_{\Sigma_1} \;\; \le \;\; C_8\|b\|_\infty[\psi_\delta]_1^{1/2} \cdot [W_2^2]_1^{1/2}.$$

On the other hand,

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d}\|W_2^2\|_{L^q(\mathbb{Q}_n)} &\leq C_9 \sum_{n\in\mathbb{Z}^d,\varepsilon_1\alpha^{1/p}<|n|<\varepsilon_2\alpha^{1/p}} \frac{1}{|n|^p+1}\\ &\leq C_{10}\int_{\varepsilon_1\alpha^{1/p}<|x|<\varepsilon_2\alpha^{1/p}} \frac{dx}{|x|^p+1} \asymp C_{10}\int_0^{2\pi}\int_{\varepsilon_1\alpha^{1/p}}^{\varepsilon_2\alpha^{1/p}} r^{1-p}\,dr\,d\theta\\ &= 2\pi C_{10}\left[\frac{r^{2-p}}{2-p}\right]\Big|_{r=\varepsilon_1\alpha^{1/p}}^{r=\varepsilon_2\alpha^{1/p}} = C_{11}(\varepsilon_2^{2-p}-\varepsilon_1^{2-p})\alpha^{\frac{2}{p}-1},\end{aligned}$$

where C_9, C_{10} and C_{11} are all non-negative constants. Also, we have

$$\begin{aligned}[\psi_\delta]_1 &= \sum_{n\in\mathbb{Z}^d}\|\psi_\delta\|_{L^q(\mathbb{Q}_n)} \leq \sum_{\varepsilon_2\alpha^{1/p}<|n|<\varepsilon_2\alpha^{1/p}+\delta} 1\\ &= \text{Area}\{x\in\mathbb{R}^2 : \varepsilon_2\alpha^{1/p}<|x|<\varepsilon_2\alpha^{1/p}+\delta\} = \pi[(\varepsilon_2\alpha^{1/p}+\delta)^2-(\varepsilon_2\alpha^{1/p})^2]\\ &= \pi(2\varepsilon_2\alpha^{1/p}\delta+\delta^2).\end{aligned}$$

As a consequence,

$$\|W_2(D_m-\lambda I)^{-1}\psi_\delta\|_{\Sigma_1} \leq O(\alpha^{\frac{1}{p}-\frac{1}{2}})O(\alpha^{\frac{1}{2p}}) = O(\alpha^{\frac{3}{2p}-\frac{1}{2}}), \qquad \text{as } \alpha\to\infty.$$

Therefore, since according to (4.32), there is a constant $C_{12}\geq 0$ for which

$$\|X_{2,3}\|_{\Sigma_1} \leq C_{12}\|W_2(D_m-\lambda I)^{-1}\psi_\delta\|_{\Sigma_1}\cdot\|\psi_\delta(D_m-\lambda I)^{-1}W_3\|,$$

it remains to estimate the last factor on the right hand side. Recall that for every $\lambda\in(-m,m)$,

$$\|(D_m-\lambda I)^{-1}\| \leq \frac{1}{\text{dist}(\lambda,\sigma(D_m))},$$

where $\sigma(D_m)$ denotes the spectrum of the operator D_m. Also, there is a constant C

that depends only on λ and satisfies that $W_3(x) \leq C|x|^{-p/2}$, which implies that

$$\|\psi_\delta(D_m - \lambda I)^{-1}W_3\| \leq \frac{C}{\mathrm{dist}(\lambda, \sigma(D_m))}\varepsilon_2^{-p/2}\alpha^{-1/2}.$$

As a consequence, we obtain the following formula:

$$\varepsilon\alpha^{-1}n(\varepsilon\alpha^{-1}, X_{2,3}) \leq \|X_{2,3}\|_{\Sigma_1} = O(\alpha^{\frac{3}{2p}-1}),$$

which leads to

$$n(\varepsilon\alpha^{-1}, X_{2,3}) = O(\alpha^{\frac{3}{2p}}) = o(\alpha^{2/p}), \quad \text{as } \alpha \to \infty.$$

That is exactly the desired relation.

We now study operator $\tilde{X}_{2,3}$. Observe that an alternative expression for $\tilde{X}_{2,3}$ is

$$\tilde{X}_{2,3} = W_2(D_m - \lambda I)^{-1}\psi_\delta[\tilde{Y}_1(-\Delta + I)^{-1/2}](-\Delta + I)^{1/2}(D_m - \lambda I)^{-1}W_3.$$

Using boundedness of the operator $\tilde{Y}_1(-\Delta+I)^{-1/2}$, Hölder's inequality and Corollary 4.1, we estimate the Σ_1-norm of $\tilde{X}_{2,3}$ as follows:

$$\begin{aligned}
\|\tilde{X}_{2,3}\|_{\Sigma_1} &\leq C_{13}\|W_2(D_m - \lambda I)^{-1}\psi_\delta\|_{\Sigma_1} \cdot \|(-\Delta + I)^{1/2}(D_m - \lambda I)^{-1}W_3\| \\
&\leq C_{13}C_0[W_2^2]_1^{1/2}[\psi_\delta]_1^{1/2}\|(-\Delta + I)^{1/2}(D_m - \lambda I)^{-1}W_3\| \text{(by Proposition 4.3)} \\
&= O(\alpha^{\frac{3}{2p}-\frac{1}{2}})\|(-\Delta + I)^{1/2}(D_m - \lambda I)^{-1}W_3\|,
\end{aligned}$$

with some constant $C_{13} \geq 0$. Also, note that

$$\begin{aligned} & (-\Delta + I)^{1/2}(D_m - \lambda I)^{-1} W_3 \\ = \; & F^* \frac{(|\xi|^2+1)^{1/2}}{m^2 - \lambda^2 + |\xi|^4} \begin{pmatrix} \lambda + m & -(\xi_1 - i\xi_2)^2 \\ -(\xi_1 + i\xi_2)^2 & \lambda - m \end{pmatrix} F W_3 \\ = \; & F^* \frac{1}{(|\xi|^2+1)^{1/2}} \tilde{b}(\xi) F W_3, \end{aligned}$$

where $\tilde{b} \in L^\infty(\mathbb{R}^2)$. By the fact that there is some constant $C \geq 0$ such that $|W_3(x)|$ is bounded from above by $C|x|^{-p/2}$, we get

$$\|F^* \frac{1}{(|\xi|^2+1)^{1/2}} \tilde{b}(\xi) F W_3\| \leq \|\tilde{b}\|_\infty \cdot \|W_3\|_\infty \leq C_{14} \|\tilde{b}\|_\infty {\varepsilon_2}^{-p/2} \alpha^{-1/2}.$$

Therefore,

$$\|\tilde{X}_{2,3}\|_{\Sigma_1} \leq O(\alpha^{\frac{3}{2p}-1}), \quad \text{as } \alpha \to \infty,$$

which gives

$$\varepsilon \alpha^{-1} n(\varepsilon \alpha^{-1}, \tilde{X}_{2,3}) \leq \|\tilde{X}_{2,3}\|_{\Sigma_1} = O(\alpha^{\frac{3}{2p}-1}).$$

This is enough to conclude that $n(s, \tilde{X}_{2,3}) = O(\alpha^{3/2p}) = o(\alpha^{2/p})$. By this we complete the work with the operator $\tilde{X}_{2,3}$.

However, according to (4.30), we still have to show that $n(\varepsilon\alpha^{-1}, \hat{X}_{2,3}) = o(\alpha^{2/p})$, as $\alpha \to \infty$, for the operator

$$\hat{X}_{2,3} = W_2 (D_m - \lambda I)^{-1} (1 - \eta) W_3.$$

For this purpose, we observe that

$$\begin{aligned}\hat{X}_{2,3} &= W_2(D_m-\lambda I)^{-1}(1-\eta)W_3\\ &= W_2F^*\frac{1}{(\xi^2+1)^{1/2}}b(\xi)\frac{1}{(\xi^2+1)^{1/2}}F(1-\eta)W_3.\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}\|\hat{X}_{2,3}\|_{\Sigma_1} &\le 2\|b\|_\infty\|W_2F^*\frac{1}{(\xi^2+1)^{1/2}}\|_{\Sigma_2}\cdot\|\frac{1}{(\xi^2+1)^{1/2}}F(1-\eta)W_3\|_{\Sigma_2}\\ &\le 2C_0\|b\|_\infty[W_2^2]_1^{1/2}[(1-\eta)^2W_3^2]_1^{1/2}.\end{aligned}$$

Now we recall that

$$[W_2^2]_1 = O(\alpha^{\frac{2}{p}-1}), \quad \text{as } \alpha\to\infty.$$

Furthermore,

$$\begin{aligned}&[(1-\eta)^2W_3^2]_1 = \sum_{n\in\mathbb{Z}^d}\|(1-\eta)^2W_3^2\|_{L^q(\mathbb{Q}_n)} \asymp C\int_{\varepsilon_2\alpha^{1/p}<r<\varepsilon_2\alpha^{1/p}+\delta}\frac{r\,dr}{r^p}\\ &= C\left[\frac{r^{2-p}}{2-p}\right]\Big|_{r=\varepsilon_2\alpha^{1/p}}^{r=\varepsilon_2\alpha^{1/p}+\delta} = C_1((\varepsilon_2\alpha^{1/p}+\delta)^{2-p}-(\varepsilon_2\alpha^{1/p})^{2-p}) = O(\alpha^{\frac{1}{p}-1}).\end{aligned}$$

Therefore,

$$\|\hat{X}_{2,3}\|_{\Sigma_1} = O(\alpha^{\frac{3}{2p}-1}),$$

which implies that

$$n(\varepsilon\alpha^{-1},\hat{X}_{2,3}) = O(\alpha^{\frac{3}{2p}}) = o(\alpha^{\frac{2}{p}}), \quad \text{as } \alpha\to\infty.$$

4.4.3 Operator $W_1(D_m - \lambda I)^{-1}W_3$

Applying a similar method, we demonstrate that $n(\varepsilon\alpha^{-1}, W_1(D_m - \lambda I)^{-1}W_3) = o(\alpha^{2/p})$ as $\alpha \to \infty$ for every $\varepsilon > 0$.

The decomposition of the space $L^2(\mathbb{R}^2)$ into the orthogonal sum $L^2(\Omega_1(\alpha)) \oplus L^2(\Omega_2(\alpha)) \oplus L^2(\Omega_3(\alpha))$ leads to the decomposition of the operator $X(\lambda)$ formally displayed by the matrix

$$X(\lambda) = \begin{pmatrix} W_1(D_m - \lambda I)^{-1}W_1 & W_1(D_m - \lambda I)^{-1}W_2 & W_1(D_m - \lambda I)^{-1}W_3 \\ \\ W_2(D_m - \lambda I)^{-1}W_1 & W_2(D_m - \lambda I)^{-1}W_2 & W_2(D_m - \lambda I)^{-1}W_3 \\ \\ W_3(D_m - \lambda I)^{-1}W_1 & W_3(D_m - \lambda I)^{-1}W_2 & W_3(D_m - \lambda I)^{-1}W_3 \end{pmatrix}.$$

We have already proved that the off-diagonal elements of this matrix do not contribute to the asymptotics of $N(\lambda, \alpha)$. It remains to compute the contribution of the diagonal elements, whose orthogonal sum is denoted by $X(\lambda)^+$:

$$X_\lambda^+ = \begin{pmatrix} W_1(D_m - \lambda I)^{-1}W_1 & 0 & 0 \\ \\ 0 & W_2(D_m - \lambda I)^{-1}W_2 & 0 \\ \\ 0 & 0 & W_3(D_m - \lambda I)^{-1}W_3 \end{pmatrix}.$$

Namely, according to Proposition 4.5, we need to show that

$$n_+(\tau\varepsilon^{-1}, X(\lambda)^+) \sim \tau^{-2/p}\alpha^{2/p}J(\lambda, m), \quad \text{as } \alpha \to \infty,$$

for every $\tau > 0$.

4.4.4 Operator $W_1(D_m - \lambda I)^{-1}W_1$

We start the investigation of the first self-adjoint operator in (4.14), $W_1(D_m - \lambda I)^{-1}W_1$.

First, recall that, by (4.23), the symbol $F(D_m - \lambda I)^{-1}F^*$ decays as $O(|\xi|^{-2})$ when $|\xi| \to \infty$. Hence, there exists a function $b \in L^\infty(\mathbb{R}^2)$ satisfying

$$F(D_m - \lambda I)^{-1}F^* = \frac{b(\xi)}{1+|\xi|^2},$$

which implies the bound

$$\begin{aligned}\|W_1(D_m - \lambda I)^{-1}W_1\|_{\Sigma_1} &= \|W_1F^*(|\xi|^2+1)^{-1/2}b(\xi)(|\xi|^2+1)^{-1/2}FW_1\|_{\Sigma_1} \\ &\le 2\|b\|_\infty \|W_1F^*(|\xi|^2+1)^{-\frac{1}{2}}\|_{\Sigma_2}\|(|\xi|^2+1)^{-\frac{1}{2}}FW_1\|_{\Sigma_2}. \quad (4.34)\end{aligned}$$

On the other hand, based on Corollary 4.1, we obtain

$$\begin{aligned}&\|W_1F^*(|\xi|^2+1)^{-1/2}\|_{\Sigma_2}^2 = \|W_1F^*(|\xi|^2+1)^{-1/2}(|\xi|^2+1)^{-1/2}FW_1\|_{\Sigma_1} \\ &= \|W_1F^*(|\xi|^2+1)^{-1}FW_1\|_{\Sigma_1} = \|W_1(-\Delta+I)^{-1}W_1\|_{\Sigma_1} \le C_0[W_1^2]_1. \quad (4.35)\end{aligned}$$

Therefore,

$$\begin{aligned}&\|W_1(D_m - \lambda I)^{-1}W_1\|_{\Sigma_1} \le 2C_0\|b\|_\infty[W_1^2]_1 = 2C_0\|b\|_\infty \sum_{n\in\mathbb{Z}^d} \|W_1^2\|_{L^q(\mathbb{Q}_n)} \\ &\le C_{15} \sum_{n\in\mathbb{Z}^d, |n|<\varepsilon_1\alpha^{1/p}} \frac{1}{|n|^p+1} \le C_{16}\int_{|x|<\varepsilon_1\alpha^{1/p}} \frac{dx}{|x|^p+1} \\ &\asymp C_{16}\int_0^{2\pi}\int_{r<\varepsilon_1\alpha^{1/p}} r^{1-p}\,dr\,d\theta = 2\pi C_{16}\left[\frac{r^{2-p}}{2-p}\right]\Big|_{r=0}^{r=\varepsilon_1\alpha^{1/p}} = C_{17}\varepsilon_1^{2-p}\alpha^{\frac{2}{p}-1},\end{aligned}$$

where C_{15}, C_{16} and C_{17} are non-negative constants. This suggests that

$$n(\tau\alpha^{-1}, W_1(D_m - \lambda I)^{-1}W_1) \le C_{17}\varepsilon_1^{2-p}\alpha^{2/p}\tau^{-1}, \qquad \forall\tau > 0. \quad (4.36)$$

This concludes the analysis of the operator $W_1(D_m - \lambda I)^{-1}W_1$.

4.4.5 Operator $W_3(D_m - \lambda I)^{-1}W_3$

Let us show that

$$\|W_3(D_m - \lambda I)^{-1}W_3\| \le \tau\alpha^{-1}, \tag{4.37}$$

if ε_2 is sufficiently large. This would imply that

$$n(\tau\alpha^{-1}, W_3(D_m - \lambda I)^{-1}W_3) = 0. \tag{4.38}$$

Note that there is a constant $C \ge 0$ such that $|W_3(x)|$ is bounded from above by $C|x|^{-p/2}$. Consequently,

$$\|W_3(D_m - \lambda I)^{-1}W_3\| \le \tfrac{1}{\mathrm{dist}(\lambda,\sigma(D_m))}\|W_3^2\|_\infty = C_\lambda \varepsilon_2^{-p}\alpha^{-1},$$

with the constant

$$C_\lambda = \frac{C}{\mathrm{dist}(\lambda, \sigma(D_m))}.$$

Hence, if we choose $\varepsilon_2 > (C_\lambda/\tau)^{1/p}$, then (4.37) will hold. The analysis of the operator $W_3(D_m - \lambda I)^{-1}W_3$ is thereby completed.

4.4.6 Operator $W_2(D_m - \lambda I)^{-1}W_2$

We now consider the only remaining self-adjoint operator in (4.14), $W_2(D_m - \lambda I)^{-1}W_2$. To get started, we introduce the notation $\tilde{\Omega}_2 = \{x \in \mathbb{R}^2 : \varepsilon_1 < |x| < \varepsilon_2\}$. Thus, $\Omega_2(\alpha) = \alpha^{1/p}\tilde{\Omega}_2$, which means that one set is obtained from the other by scaling. Then we decompose $\tilde{\Omega}_2$ into finitely many sets denoted by $\{Q_j\}_{j=1}^{l}$, where Q_j is a square for $1 \le j \le l-1$ and $Q_l = \tilde{\Omega}_2 \setminus \cup_{j=1}^{l-1} Q_j$. Evidently, each square Q_j can be expressed in the form

$$\delta([0,1)^2 + n) \tag{4.39}$$

for some $\delta > 0$ and $n \in \mathbb{Z}^2$ when $1 \leq j \leq l-1$. Moreover, $\Omega_2(\alpha) = \alpha^{1/p}\tilde{\Omega}_2 = \cup_{j=1}^{l}\alpha^{1/p}Q_j$. Employing the same methodology as in the proof of Proposition 4.5 and the relation (4.16), we can show that

$$\begin{aligned} & n_+(\tau\alpha^{-1}, W_2(D_m - \lambda I)^{-1}W_2) \\ & \sim \sum_{j=1}^{l} n_+(\tau\alpha^{-1}, \chi_j W(D_m - \lambda I)^{-1}W\chi_j) + o(\alpha^{2/p}), \quad \text{as } \alpha \to \infty, \end{aligned} \tag{4.40}$$

where χ_j represents the characteristic function of set $\alpha^{1/p}Q_j$ for $1 \leq j \leq l$. In order to compute the asymptotics of $n_+(\tau\alpha^{-1}, W_2(D_m - \lambda I)^{-1}W_2)$, it suffices to analyze each individual term on the right-hand side of (4.40).

Let Q be a square of the form (4.39), let ϕ_β be the characteristic function of the set βQ. We want to investigate the behavior of the quantity $n_+(\tau, \phi_\beta(D_m - \lambda I)^{-1}\phi_\beta)$ when the value of β approaches infinity and $\tau > 0$ is fixed. We intend to calculate the value of the limit $\lim_{\beta\to\infty}\beta^{-2}n_+(\tau, \phi_\beta(D_m - \lambda I)^{-1}\phi_\beta)$.

Note that for each self-adjoint operator A, we have $\chi_{(\tau,\infty)}(A) = E_A(\tau,\infty)$. Thus,

$$n_+(\tau, A) = \operatorname{tr}(E_A(\tau,\infty)) = \operatorname{tr}(\chi_{(\tau,\infty)}(A)),$$

if A is compact and self-adjoint.

We are going to prove the following important proposition.

Proposition 4.6. *For every fixed $\tau > 0$,*

$$n_+\big(\tau, \phi_\beta(D_m - I)^{-1}\phi_\beta\big) \sim (4\pi)^{-1}\beta^2\Big((\tau^{-1} + \lambda)_+^2 - m^2\Big)_+^{1/2}\operatorname{Area} Q, \quad \textit{as } \beta \to \infty,$$

where $x_+ = \max(0, x)$ *for all* $x \in \mathbb{R}$.

Proof. First, note that

$$\pi\Big((\tau^{-1} + \lambda)_+^2 - m^2\Big)_+^{1/2} = \text{area}\,\{\xi \in \mathbb{R}^2 : \quad \big(\sqrt{|\xi|^4 + m^2} - \lambda\big)^{-1} > \tau\}.$$

Based on the fact that $\pm\sqrt{|\xi|^4 + m^2}$ are eigenvalues of the symbol

$$\hat{D}_m(\xi) := \begin{pmatrix} m & (\xi_2 + i\xi_1)^2 \\ (\xi_2 - i\xi_1)^2 & -m \end{pmatrix}$$

we find

$$\text{tr}\,\Phi\big(\phi_\beta (D_m - I)^{-1} \phi_\beta\big) \sim \text{tr}\,\Big[\phi_\beta \Phi\big((D_m - I)^{-1}\big)\phi_\beta\Big], \quad \text{as } \beta \to \infty,$$

for Φ being the characteristic function of the interval (τ, ∞). It is clear that such a function Φ could be squeezed between two comparable functions of the form

$$\Phi_\varepsilon(s) = \begin{cases} 0, & \text{if } s < \tau \\ (s - \tau)/\varepsilon; & \text{if } \tau \le s \le \tau + \varepsilon; \\ 1, & \text{if } s > \tau + \varepsilon. \end{cases}$$

Namely,

$$\Phi_\varepsilon(s) \le \Phi(s) \le \Phi_\varepsilon(s + \varepsilon).$$

Therefore, if (4.4.6) holds with Φ replaced by the functions $\Phi_\varepsilon(s)$ and $\Phi_\varepsilon(s+\varepsilon)$, then

$$\frac{\text{Area}\, Q}{4\pi}\Big(((\tau + \varepsilon)^{-1} + \lambda)_+^2 - m^2\Big)_+^{1/2} \le \lim_{\beta \to \infty} \beta^{-2}\text{tr}\,\Phi\big(\phi_\beta (D_m - \lambda I)^{-1}\phi_\beta\big) \le \frac{\text{Area}\, Q}{4\pi}\Big(((\tau - \varepsilon)^{-1} + \lambda)_+^2 - m^2\Big)_+^{1/2}.$$

Since $\varepsilon > 0$ is arbitrary, the latter inequality would result in (4.4.6). Thus, we only have to prove (4.4.6) for functions Φ that are continuous and vanish near the origin.

Every such function Φ may be rewritten as

$$\Phi(s) = s^2\eta(s),$$

where η is a continuous function on the real line $\mathbb{R}$. Notice that, in this situation,

$$\left|\operatorname{tr}\Phi\big(\phi_\beta(D_m - I)^{-1}\phi_\beta\big)\right| \le \|\phi_\beta(D_m - I)^{-1}\phi_\beta\|_{\mathfrak{S}_2}^2\|\eta\|_\infty \le C_{20}\beta^2\|\eta\|_\infty,$$

for some constant $C_{20} \ge 0$. By $\Phi((D_m - I)^{-1}) = (D_m - I)^{-1}\eta((D_m - I)^{-1})(D_m - I)^{-1}$, we derive that

$$\left|\operatorname{tr}\phi_\beta\Phi\big((D_m - I)^{-1}\big)\phi_\beta\right| \le \|\phi_\beta(D_m - I)^{-1}\|_{\mathfrak{S}_2}\|(D_m - I)^{-1}\phi_\beta\|_{\mathfrak{S}_2}\|\eta\|_\infty \le C_{20}\beta^2\|\eta\|_\infty.$$

Consequently, both sides of (4.4.6) may be estimated by $C_{20}\beta^2\|\eta\|_\infty$, allowing one to assume that η is a polynomial.

Indeed, functions of a given self-adjoint operator only need to be defined on the spectrum of the operator. On the other hand, the spectrum of operator $(D_m - \lambda I)^{-1}$ is contained in $[-L, L]$, where $L = 1/(m - |\lambda|)$. As a result, the functional $\|\eta\|_\infty$ in the last inequality is the L^∞-norm of the function on the compact interval $[-L, L]$. Since on a finite interval, η can be uniformly approximated by polynomials, it suffices to prove (4.4.6) under the assumption that η is a polynomial. Put differently, it is enough to show that (4.4.6) holds for

$$\Phi(s) = s^k, \qquad k \ge 2,$$

by the fact that all polynomials are finite linear combinations of power functions.

Denote $R_{=(D_m-I)^{-1}}$, $\chi_+ = \phi_\beta$ and $\chi_- = 1-\phi_\beta$. We intend to show that

$$\|(\chi_+ R_\lambda \chi_+)^k - \chi_+ R_\lambda^k \chi_+\|_{\mathfrak{S}_1} = o(\beta^2), \qquad \text{as} \quad \beta \to \infty.$$

For that reason, we represent $\chi_+ R_{^k \chi_+}$ as

$$\chi_+ R_\lambda^k \chi_+ = (\chi_+ R_\lambda \chi_+)^k + \sum_{j=0}^{k-1} (\chi_+ R_\lambda \chi_+)^j \chi_+ R_\lambda \chi_- R_\lambda^{k-j-1} \chi_+.$$

While the norm of the operator $\chi_+ R_{\chi_-}$ does not approach zero, it is still representable in the form

$$\chi_+ R_\lambda \chi_- = T_1 + T_2, \qquad \text{where } \|T_1\| \to 0, \|T_2\|_{\mathfrak{S}_k} = o(\beta^{2/k}), \quad \text{as } \beta \to \infty.$$

Also, we define T_2 as

$$T_2 = \Theta \chi_+ R_\lambda \chi_- \Theta,$$

where Θ is the operator of multiplication by the characteristic function of the layer

$$\{x \in \mathbb{R}^2 : \quad \operatorname{dist}(x, \beta\partial Q) < \beta^{1/2}\}.$$

Then the area of the support of Θ does not exceed $C\beta^{3/2}$ at least for sufficiently large values of β. Thus,

$$\|T_2\|_{\mathfrak{S}_k} \leq C\beta^{\frac{3}{2k}} = o(\beta^{2/k}), \quad \text{as} \quad \beta \to \infty.$$

In contrast, since the explicit expression for the integral kernel of $(D_m - I)^{-1}$ shows that the latter decays exponentially fast when $|x-y| \to \infty$, we conclude the following estimate for the integral kernel $k(x,y)$ of the operator T_1:

$$|k(x,y)| \leq C\big((1-\Theta(x)) + (1-\Theta(y))\big)e^{-c|x-y|}\chi_+(x)\chi_-(y).$$

Then combining the celebrated Shur estimate

$$\|T_1\| \leq \left(\sup_x \int |k(x,y)|dy \times \sup_y \int |k(x,y)|dx\right)^{1/2}$$

with the observation that x and y for which $k(x,y) \neq 0$ are distance $\beta^{1/2}$ apart, we obtain the inequality

$$\|T_1\| \leq 2\pi C \int_{r>\beta^{1/2}} e^{-cr} r dr,$$

which entails that $\|T_1\| \to 0$ as $\beta \to \infty$.

Therefore, we establish the following result:

$$\begin{aligned}\|(\chi_+R_\lambda\chi_+)^j\chi_+R_\lambda\chi_-R_\lambda^{k-j-1}\chi_+\|_{\mathfrak{S}_1} \leq \|(\chi_+R_\lambda\chi_+)^jT_1R_\lambda^{k-j-1}\chi_+\|_{\mathfrak{S}_1} + \\ \|(\chi_+R_\lambda\chi_+)^jT_2R_\lambda^{k-j-1}\chi_+\|_{\mathfrak{S}_1} \leq \|\chi_+R_\lambda\chi_+\|^j_{\mathfrak{S}_{k-1}}\|T_1\|\|R_\lambda^{k-j-1}\chi_+\|_{\mathfrak{S}_{(k-1)/(k-j-1)}} \\ +\|\chi_+R_\lambda\chi_+\|^j_{\mathfrak{S}_k}\|T_2\|_{\mathfrak{S}_k}\|R_\lambda^{k-j-1}\chi_+\|_{\mathfrak{S}_{k/(k-j-1)}} = o(\beta^2), \qquad \text{as } \beta \to \infty.\end{aligned}$$

Substituting this into (4.4.6), we establish (4.4.6). The proof is complete. □

Observe that Proposition 4.6 implies that

$$n_+(\tau, \chi_j(D_m - \lambda I)^{-1}\chi_j) \sim \frac{\beta^2}{4\pi}[((\lambda+\tau^{-1})^2_+ - m^2)_+]^{1/2}\text{Area } Q_j.$$

Note also that if $\delta > 0$ is sufficiently small, then $\Psi(\theta)|x|^{-p}$ can be well approximated by a constant function on the cube $\delta([0,1)^2+n), \forall n \in \mathbb{Z}^2$. Therefore, we first focus on the case where $V(x)$ is strictly (rather than asymptotically) equal to $\Psi(\theta)|x|^{-p}$, and introduce two points x_j^+ and x_j^- in Q_j such that

$$V(x_j^+) = \max_{x \in Q_j} V(x) \qquad \text{and} \qquad V(x_j^-) = \min_{x \in Q_j} V(x),$$

respectively. Let $x \in \alpha^{1/p} Q_j$, then $\alpha V(x) = \alpha \Psi(\theta)|x|^{-p} = \Psi(\theta)|x\alpha^{-1/p}|^{-p}$. Thus, $V(x_j^-) \leq \alpha V(x) \leq V(x_j^+), \forall x \in \alpha^{1/p} Q_j$. Taking the square root on all sides of this inequality, we obtain $W(x_j^-) \leq \sqrt{\alpha} W(x) \leq W(x_j^+)$. Using the monotonicity of $n_+(\tau\alpha^{-1}, \chi_j W(D_m - \lambda I)^{-1} W \chi_j)$ with respect to W, we get

$$\begin{aligned}
&n_+(\tau\alpha^{-1}, \chi_j W(D_m - \lambda I)^{-1} W \chi_j) = n_+(\tau, \sqrt{\alpha}\chi_j W(D_m - \lambda I)^{-1} W \chi_j \sqrt{\alpha}) \\
&\leq n_+(\tau, W(x_j^+)\chi_j (D_m - \lambda I)^{-1} \chi_j W(x_j^+)) = n_+(\tau, V(x_j^+)\chi_j (D_m - \lambda I)^{-1} \chi_j) \\
&= n_+\Big(\frac{\tau}{V(x_j^+)}, \chi_j (D_m - \lambda I)^{-1} \chi_j\Big) \sim \frac{\alpha^{2/p}}{4\pi} [((\lambda + \tau^{-1} V(x_j^+))_+^2 - m^2)_+]^{1/2} \text{Area } Q_j.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\limsup_{\alpha \to \infty} \alpha^{-2/p} N_\tau(\lambda, \alpha) &= \limsup_{\alpha \to \infty} \alpha^{-2/p} \sum_{j=1}^{l} n_+(\tau\alpha^{-1}, \chi_j W(D_m - \lambda I)^{-1} W \chi_j) \\
&\leq \frac{1}{4\pi} \sum_{j=1}^{l} [((\lambda + \tau^{-1} V(x_j^+))_+^2 - m^2)_+]^{1/2} \text{Area } Q_j,
\end{aligned}$$

where $N_\tau(\lambda, \alpha) = n_+(\tau\alpha^{-1}, W_2(D_m - \lambda I)^{-1} W_2)$. This leads to the following estimate:

$$\limsup_{\alpha \to \infty} \alpha^{-2/p} N_\tau(\lambda, \alpha) \leq \frac{1}{4\pi} \int_{\varepsilon_1 < |x| < \varepsilon_2} [((\lambda + \tau^{-1} \Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2} \, dx. \quad (4.41)$$

Similarly,

$$\begin{aligned}
\liminf_{\alpha \to \infty} \alpha^{-2/p} N_\tau(\lambda, \alpha) &= \liminf_{\alpha \to \infty} \alpha^{-2/p} \sum_{j=1}^{l} n_+(\tau\alpha^{-1}, \chi_j W(D_m - \lambda I)^{-1} W \chi_j) \\
&\geq \frac{1}{4\pi} \int_{\varepsilon_1 < |x| < \varepsilon_2} [((\lambda + \tau^{-1} \Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2} \, dx. \quad (4.42)
\end{aligned}$$

Synthesizing inequalities (4.41) and (4.42), we obtain

$$\lim_{\alpha\to\infty} \alpha^{-2/p} N(\lambda,\alpha) = \frac{1}{4\pi}\int_{\varepsilon_1<|x|<\varepsilon_2} [((\lambda+\tau^{-1}\Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\,dx. \tag{4.43}$$

This completes the analysis of the operator $W_2(D_m-\lambda I)^{-1}W_2$.

The end of the proof of Theorem 4.2. By (4.36), (4.38) and (4.43), we deduce that

$$\begin{aligned}&\limsup_{\alpha\to\infty} \alpha^{-2/p} N(\lambda,\alpha)\\ &\le C_{18}\varepsilon_1^{2-p}\tau^{-1} + \frac{1}{4\pi}\int_{\varepsilon_1<|x|<\varepsilon_2} [((\lambda+\tau^{-1}\Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\,dx,\end{aligned} \tag{4.44}$$

for sufficiently large ε_2 and any $\tau\in(0,1)$. Furthermore, if

$$\lambda+\tau^{-1}\|\Psi\|_\infty\varepsilon_2^{-p} < m,$$

or equivalently,

$$\varepsilon_2 > \left(\frac{\tau^{-1}\|\Psi\|_\infty}{m-\lambda}\right)^{1/p},$$

then

$$\int_{|x|>\varepsilon_2} [((\lambda+\tau^{-1}\Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\,dx = 0.$$

Taking the limit as $\varepsilon_1\to 0$, we get

$$\begin{aligned}\limsup_{\alpha\to\infty} \alpha^{-2/p} N(\lambda,\alpha) &\le \frac{1}{4\pi}\int_{\mathbb{R}^2} [((\lambda+\tau^{-1}\Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\,dx\\ &= \frac{1}{4\pi\tau^{2/p}}\int_{\mathbb{R}^2} [((\lambda+\Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\,dx.\end{aligned}$$

Note that the integral on the right-hand side converges for $0<p<2$. Taking the limit as $\tau \to 1$, we obtain

$$\limsup_{\alpha\to\infty} \alpha^{-2/p} N(\lambda,\alpha) \leq \frac{1}{4\pi}\int_{\mathbb{R}^2} [((\lambda+\Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\, dx. \tag{4.45}$$

Also, for any $\tau > 1$,

$$\liminf_{\alpha\to\infty} \alpha^{-2/p} N(\lambda,\alpha) \geq \liminf_{\alpha\to\infty} \alpha^{-2/p} N_\tau(\lambda,\alpha).$$

Repeating the same steps as for the upper limit, we infer that

$$\liminf_{\alpha\to\infty} \alpha^{-2/p} N(\lambda,\alpha) \geq \frac{1}{4\pi}\int_{\mathbb{R}^2} [((\lambda+\Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\, dx. \tag{4.46}$$

Combining (4.45) and (4.46), we derive the following formula

$$N(\lambda,\alpha) \sim \frac{\alpha^{2/p}}{4\pi}\int_{\mathbb{R}^2} [((\lambda+\Psi(\theta)|x|^{-p})_+^2 - m^2)_+]^{1/2}\, dx, \quad \text{as } \alpha\to\infty,$$

which finalizes the proof of Theorem 4.2. ■

CHAPTER 5: FUTURE WORK

During the process of writing this dissertation, a couple of conjectures came to our mind, mainly due to that some of the assumptions are rather idealistic and therefore may fail to reflect a perfectly realistic framework.

An excellent topic for subsequent research is the emergence of eigenvalues in the spectral gaps of operators subjected to non-sign-definite perturbations. Specifically, when a periodic operator is perturbed by a localized potential, eigenvalues can appear inside these gaps. The well-understood case involves perturbations with a definite sign (e.g., positive). However, many physical systems, such as those with varying magnetic fields, involve perturbations that change sign (non-sign-definite). The main difficulty of working with non-sign-definite perturbations is that the motion of the eigenvalues is no longer monotone with the growth of α. In particular, they might move in both directions: from the left to the right and from the right to the left. Therefore, it is even difficult to define the function $N(\lambda, \alpha)$.

Concerning the expression (4.2) for the graphene operator, we can consider more general cases, where V is a matrix-valued function of the form

$$V = \begin{pmatrix} V_1 & V_{1,2} \\ \overline{V}_{1,2} & V_2 \end{pmatrix}, \tag{5.1}$$

with real-valued V_1 and V_2 and complex-valued $V_{1,2}$. In that case, we would like to study the impact of the four components of the matrix (5.1) on the asymptotic behavior of $N(\lambda, \alpha)$. We suspect that the contribution of the diagonal elements, V_1 and V_2, to the asymptotics of $N(\lambda, \alpha)$, is more significant than that of the off-diagonal elements, namely, $V_{1,2}$ and $\overline{V}_{1,2}$.

Also for the graphene operator, we have the following challenging problem. Assume that V satisfies (4.3). What is the asymptotics of the number of eigenvalues situated between λ_1 and λ_2 that both belong to the gap $(-m, m)$? We remind the reader that, in this case,

$$N(\lambda, \alpha) \sim \frac{\alpha}{4\pi} \int_{\mathbb{R}^2} V\, dx, \quad \text{as } \alpha \to \infty.$$

Since the asymptotic coefficient on the right-hand side does not depend on λ, we can only infer that

$$N(\lambda_2, \alpha) - N(\lambda_1, \alpha) = o(\alpha), \quad \text{as } \alpha \to \infty.$$

Note that the difference on the left-hand side coincides with the number of eigenvalues inside the interval $[\lambda_1, \lambda_2)$. Thus, all we can get from this formula is that the number of eigenvalues in $[\lambda_1, \lambda_2)$ equals $o(\alpha)$ as $\alpha \to \infty$, while we would like to obtain a more precise asymptotic formula.

The above conjectures point out clear directions for our future research. Also, we may study another popular subject – Schrödinger operators on quantum graphs.

REFERENCES

[1] S. Alama, P. Deift, and R. Hempel: *Eigenvalue branches of the Schrödinger operator* $H - \lambda W$ *in a gap of* $\sigma(H)$, Comm. Math. Phys. **121** (1989), no. 2, 291-321.

[2] M. Birman: *Discrete spectrum in gaps of a continuous one for perturbations with large coupling constants*, Adv. Soviet Math. **7** (1991), 57–73.

[3] M. Birman: *Discrete spectrum in the gaps of a perturbed periodic Schrödinger operator. II. Nonregular perturbations*, St. Petersburg Math. J. **9** (1998), no. 6, 1073–1096.

[4] M. Birman, and M. Solomyak: *Asymptotics of the spectrum of pseudo-differential operators with anisotropic-homogeneous symbols*, Vest. Len. Univ. Math. **10** (1982), 237–247.

[5] M. Cwikel: *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. Math. (2) **106** (1977), no. 1, 93–100.

[6] P. Deift and R. Hempel: *On the existence of eigenvalues of the Schrödinger operator* $H - \lambda W$ *in a gap of* $\sigma(H)$, Comm. Math. Phys. **103** (1986), 461–490.

[7] F. Gesztesy, D. Gurarie, H. Holden, M. Klaus, L. Sadun, B. Simon, and P. Vogl: *Trapping and cascading of eigenvalues in the large coupling constant limit*, Comm. Math. Phys. **118** (1988), 597–634.

[8] F. Gesztesy and B. Simon: *On a Theorem of Deift and Hempel*, Comm. Math. Phys. **116** (1988), 503–505.

[9] R. Hempel: *On the asymptotic distribution of the eigenvalue branches of the Schrödinger operator* $H \pm \lambda W$ *in a spectral gap of* H, J. Reine Angew. Math. **399** (1989), 38–59.

[10] R. Hempel: *Eigenvalues in gaps and decoupling by Neumann boundary conditions*, J. Math. Anal. Appl. **169** (1992), no. 1, 229–259.

[11] M. Klaus: *Some applications of the Birman-Schwinger principle*, Helv. Phys. Acta **55** (1982), 49–68.

[12] E. Lieb: *The number of bound states of one-body Schrödinger operators and the Weyl problem*, in: Geometry of the Laplace Operator (Proc. Sympos. Pure Math., Vol. 36, 1979), pp. 241–252.

[13] S. Molchanov and B. Vainberg: *On negative eigenvalues of low-dimensional Schrödinger operators*, arXiv:1105.0937 (2011).

[14] Raikov, G. D: *Eigenvalue asymptotics for the Schrödinger operator with perturbed periodic potential*. Inventiones mathematicae **110.1** (1992): 75–93.

[15] O. Safronov: *The discrete spectrum in gaps of the continuous spectrum for indefinite-sign perturbations with a large coupling constant*, Algebra i Analiz **8** (1996), no. 2, 162–194.

[16] O. Safronov, A. Laptev, and F. Ferrulli: *Eigenvalues of the bilayer graphene operator with a complex valued potential*. Analysis and Mathematical Physics **9** (2019): 1535–1546.