NEW FORMULAS FOR THE JONES POLYNOMIAL OF A RATIONAL LINK

YUANAN DIAO AND GÁBOR HETYEI

Abstract. We derive new formulas for the Jones polynomial and the Kauffman bracket polynomial of a rational link represented by a standard diagram that is not necessarily alternating. These formulas generalize the results of Qazaqzeh, Yasein, and Abu-Qamar for the Tutte polynomial of the Tait graph of an alternating diagram of a rational link, as well as the matrix formulas of Lawrence and Rosenstein for the Jones polynomial of a rational link. Our approach uses the colored version of Brylawski's tensor product formula for Tutte polynomials of colored graphs, due to Diao, Hetyei, and Hinson. Furthermore, generalizing the formulas of Qazaqzeh, Yasein, and Abu-Qamar, we present a finite automaton that computes the crossing signs, thereby enabling the calculation of the writhe of a standard diagram of a rational link.

Introduction

There is a sizable literature devoted to the study of the Jones polynomials of oriented rational links, relying on at least three fundamentally different approaches. The first approach uses the fact that the Jones polynomial may be obtained by substitution into a more general invariant, such as the HOMFLY polynomial. Results on the HOMFLY polynomials of oriented rational links abound [8, 11, 24]. The second approach is to use the skein relations of the Jones polynomial and related polynomials directly. This method is used in [15] to prove that infinitely many rational knots have the same Jones polynomial (see also [29]). The third approach is to compute the Kauffman bracket polynomial [21, 22, 23, 27], from which the Jones polynomial may be obtained after computing the writhe of the oriented rational link diagram. Such computations may be recursive [22, 23] or may rely on the fact that the Kauffman bracket polynomial of an alternating link diagram is closely related to the Tutte polynomial of its Tait graph [27].

The present work generalizes the approach of Qazaqzeh, Yasein, and Abu-Qamar [27] to rational link diagrams that need not be alternating and in which the number of twist boxes is not necessarily odd. Our formulas differ even when specialized to the case covered in [27, Theorem 3.4], which concerns the Tutte polynomial of the Tait graph of an alternating link with an odd number of twist boxes. Computing the signed Tutte polynomial using our results requires a summation of F_n similar products, where F_n denotes the nth Fibonacci number, which is asymptotically equal to ϕ^n, where $\phi \approx 1.618$ is the golden ratio. Using our formulas is thus comparable in efficiency to computing the HOMFLY polynomial via the classical result of Lickorish and Millett [24], which, however, requires representing a rational link so that each twist box contains an even number of crossings. As noted in the concluding remarks, a recent translation of the Lickorish–Millett result to alternating rational link diagrams was published in [8], but that formula cannot be generalized to arbitrary non-alternating link diagrams.

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The key result enabling the derivation of our formulas is the generalization of Brylawski's tensor product formula to colored graphs, established in [9]. A tensor product of a graph G with a pointed graph $\widehat{N}$ (containing a distinguished edge e) is obtained by replacing each edge of G with a copy of $\widehat{N} - e$ in such a way that the original edge of G is identified with the deleted distinguished edge e. In the colored setting, different edges may be replaced with different pointed graphs. In our work we apply this construction to the case where the substituted graphs are augmented paths and their duals. The inverse operations were called series-parallel reductions in the work of Traldi [30]. Generalizing the observations made in [27] from the non-colored setting, in Section 2 we show that the Tait graph of any unoriented rational link diagram in standard form is a colored tensor product of a core graph with augmented path graphs and their duals. Section 3 contains the computation of the relevant colored Tutte invariants of the augmented path graphs and their duals, while the colored Tutte polynomials of the core graphs are computed in Section 4. In the important special case where division by our colored variables is allowed, we present two variants in Section 5. The first variant connects our colored Tutte polynomial computations to ordinary continued fractions and leads to an exact evaluation of the Jones polynomial at $t = -1$ in Section 7. The second reformulation generalizes the main formula of Lawrence and Rosenstein [21, Theorem 4.2]. This is a matrix formula in which each matrix factor is associated with a single twist box, and the size and twisting sign of that twist box determine the factor. In our generalization, each matrix factor is associated with a pointed graph replacing a single edge of the core graph, and the core graph alone determines the corresponding matrix factor.

Our main formulas for the Kauffman bracket of an unoriented rational link diagram appear in Section 6. These are obtained by replacing our colored variables with the appropriate power of the variable $A = t^{-1/4}$, where t is the variable of the Jones polynomial. When presenting our variants of the Lawrence–Rosenstein formulas in Section 8, we use the fact that $d = -A^2 - A^{-2}$ is nonzero. This makes the substitution $t = -1$ into the Jones polynomial a particularly interesting special case, since this substitution is essentially equivalent to setting $d = 0$. The absolute value of the Jones polynomial evaluated at $t = -1$ is well known. Using our results, we derive in Section 7 a formula for the exact value of this substitution. Variants of the formulas in [21] are derived in Section 8. In particular, we provide a combinatorial formula that facilitates computation of the Jones polynomial.

Our general formulas also apply in the special case where the rational link diagram is alternating and the ordinary Tutte polynomial of its (unsigned) Tait graph may be used to compute its Kauffman bracket. In Section 9 we provide direct Tutte polynomial formulas for this important case. Our results not only generalize those in [27], but also suggest a different method for computing the Tutte polynomial even in the special case of an odd number of twist boxes considered in [27]. We further obtain a variant of the Lawrence–Rosenstein formulas in this setting: a matrix product formula in which each matrix depends only on the size of the corresponding twist box.

To compute the Jones polynomial from the Kauffman bracket, we must also determine the writhe. One method is presented in [27]. In Section 10 we introduce another approach, inspired by the finite automaton method used in [8]. Our approach applies not only to alternating links: it generalizes the key formulas stated in [27] and provides complete proofs in cases that were only stated in [27] to avoid lengthy but analogous verifications.

In the final Section 11, we outline a possible alternative approach to computing the Jones polynomial of an alternating link, based on the results of [8], and present directions for future research.

1. Preliminaries

1.1. **Rational links.** We will use the notation and terminology of [8]. An unoriented rational link may be presented by a diagram of the form shown in Figure 1; such a diagram is called a *4-plat*. See Figure 3 for a concrete example. An unoriented rational link can be encoded by a continued fraction $p/q = [0, a_1, a_2, \ldots, a_n]$ where $a_1 \cdots a_n \neq 0$, namely

$$\frac{p}{q} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

The integers $|a_1|, \ldots, |a_n|$ record the numbers of consecutive half-turn twists in the twist boxes $B_1, \ldots, B_n$, following the *twist sign* convention in Figure 1.

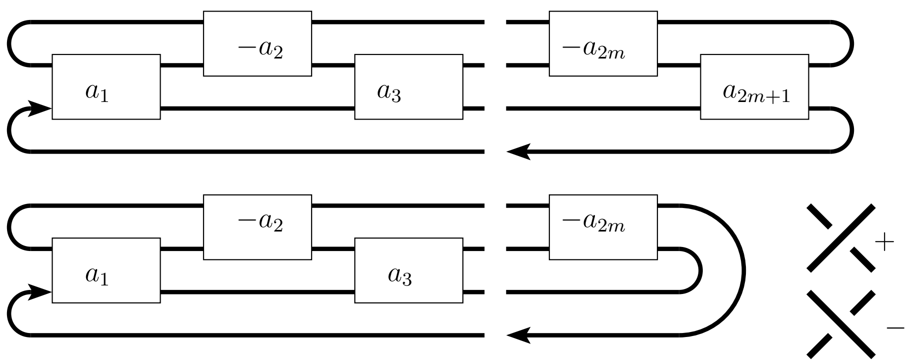

Figure 1. The twist sign convention used to define the standard form of an unoriented rational link; the role of the arrow will be explained in Section 10.

It is an immediate consequence of the definitions that the twist sign $\operatorname{sign}_t(a_i)$ of each of the $|a_i|$ crossings in twist box B_i is

$$\operatorname{sign}_t(a_i) = (-1)^{i-1} \operatorname{sign}(a_i). \tag{1.1}$$

We call such a diagram a *standard diagram* of the unoriented rational link. The left end of the diagram is fixed, while the closing on the right depends on the parity of n, as indicated in Figure 1. A rational link diagram in standard form is not necessarily alternating. Indeed, $p/q = [0, a_1, a_2, \ldots, a_n]$ encodes an alternating link diagram if and only if the continued fraction expansion is in *non-alternating denominator form*, i.e., all a_i have the same sign. In this special case, (1.1) simplifies to

$$\operatorname{sign}_t(a_i) = (-1)^{i-1} \operatorname{sign}_t(a_1). \tag{1.2}$$

It is known that every rational link admits an alternating diagram; see [4, 6].

Remark 1.1. In some literature, the rational numbers used to classify rational links are the reciprocals of those used in this paper. That is, one uses q/p with a continued fraction decomposition of the form

$$\frac{q}{p} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

In this paper we follow the convention of Duzhin and Shkolnikov [11].

The Jones polynomial is an invariant of *oriented* links. Thus we will eventually impose an orientation on the diagram, and crossings will acquire a *crossing sign* as in the convention shown on the right-hand side of Figure 2 in Subsection 1.3. If the rational link has one component, then the orientation of the strand in the top-left corner of a standard diagram is determined by the link and the chosen diagram (and is not free to choose). If the rational link has two components, then the top-left strand and the bottom strand belong to different components, so there are two choices for the orientation of the top-left strand: left-to-right or right-to-left. These two choices typically yield two distinct oriented rational links.

1.2. **Colored Tutte polynomials.** A classification of Tutte invariants of colored graphs was first developed by Zaslavsky [31] and later generalized by Bollobás and Riordan [1]. A unified treatment of these approaches appears in the work of Ellis-Monaghan and Traldi [12]. Throughout this paper we follow the notation and terminology of [1].

Let G be a connected graph with n edges, bijectively labeled by $\{1, 2, \ldots, n\}$. Assume that each edge is assigned a color λ from a color set Λ. For each $\lambda \in \Lambda$ we introduce variables $X_\lambda,\ x_\lambda,\ Y_\lambda,\ y_\lambda$ and consider the polynomial ring

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[X_\lambda, x_\lambda, Y_\lambda, y_\lambda \ :\ \lambda \in \Lambda]\,.$$

Let T be a spanning tree of G. An edge $e \in T$ is called *internally active* if for every edge $f \neq e$ in G such that $(T - e) \cup f$ is a spanning tree, the label of e is smaller than the label of f. Otherwise e is *internally inactive*. We assign weight X_λ (respectively x_λ) to each internally active (respectively inactive) edge of color λ. An edge $f \in G - T$ is *externally active* if f has the smallest label among the edges in the unique cycle contained in $T \cup f$. Otherwise f is *externally inactive*. We assign weight Y_λ (respectively y_λ) to each externally active (respectively inactive) edge of color λ.

The weight $w(T)$ of a spanning tree T is the product of the weights of its edges. The *colored Tutte polynomial* $T(G)$ is defined as

$$T(G) = \sum_T w(T),$$

where the sum is taken over all spanning trees of G.

Theorem 1.2 (Bollobás–Riordan)**.** *Let I be an ideal of $\mathbb{Z}[\Lambda]$. The image of $T(G)$ in $\mathbb{Z}[\Lambda]/I$ is independent of the edge labeling of G if and only if, for all $\lambda, \mu, \nu \in \Lambda$, the differences*

$$\det\begin{bmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{bmatrix} - \det\begin{bmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{bmatrix},$$

$$Y_\nu \det\begin{bmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{bmatrix} - Y_\nu \det\begin{bmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{bmatrix},$$

and

$$X_\nu \det\begin{bmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{bmatrix} - X_\nu \det\begin{bmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{bmatrix}$$

belong to I.

As shown in [1, Lemma 5], if I satisfies the conditions of Theorem 1.2, then $T(G)$ may be computed in $\mathbb{Z}[\Lambda]/I$ using the following deletion–contraction recurrence. Let e be an edge of G of color λ. Then

$$T(G) = \begin{cases} X_\lambda\, T(G/e) & \text{if } e \text{ is a bridge,} \\ Y_\lambda\, T(G-e) & \text{if } e \text{ is a loop,} \\ x_\lambda\, T(G/e) + y_\lambda\, T(G-e) & \text{otherwise.} \end{cases} \tag{1.3}$$

Following [10], we regard the colored Tutte polynomial as an element of $\mathbb{Z}[\Lambda]/P$, where P is a prime ideal containing the ideal I_1 generated by the polynomials

$$\det\begin{bmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{bmatrix} - \det\begin{bmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{bmatrix} \text{ and } \det\begin{bmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{bmatrix} - \det\begin{bmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{bmatrix}.$$

As observed in [10], I_1 is a prime homogeneous ideal containing the ideal I_0 generated by the differences in Theorem 1.2. Since I_1 is generated by homogeneous polynomials of degree 2, the variables $X_\lambda, x_\lambda, Y_\lambda, y_\lambda$ represent nonzero congruence classes modulo I_1. Assuming the same holds for P, then in the field of fractions K of $\mathbb{Z}[\Lambda]/P$, the relations

$$\det\begin{bmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{bmatrix} - \det\begin{bmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{bmatrix} = 0, \text{ and } \det\begin{bmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{bmatrix} - \det\begin{bmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{bmatrix} = 0$$

are equivalent to

$$\frac{X_\lambda - x_\lambda}{y_\lambda} = \frac{X_\mu - x_\mu}{y_\mu} \text{ and} \tag{1.4}$$

$$\frac{Y_\lambda - y_\lambda}{x_\lambda} = \frac{Y_\mu - y_\mu}{x_\mu}, \tag{1.5}$$

for all $\lambda, \mu \in \Lambda$. Hence there exist $u, v \in K$ such that

$$\frac{X_\lambda - x_\lambda}{y_\lambda} = u, \tag{1.6}$$

$$\frac{Y_\lambda - y_\lambda}{x_\lambda} = v, \tag{1.7}$$

for all $\lambda \in \Lambda$. Equivalently,

$$X_\lambda = x_\lambda + y_\lambda u, \tag{1.8}$$

$$Y_\lambda = x_\lambda v + y_\lambda. \tag{1.9}$$

Thus the parameters X_λ and Y_λ may be expressed in terms of x_λ, y_λ, u, and v, reducing the number of independent parameters from $4|\Lambda|$ to $2|\Lambda| + 2$. In Zaslavsky's terminology [31], such Tutte polynomials are called *normal functions*. His notation is recovered by replacing X_λ with x_e, Y_λ with y_e, x_λ with b_e, and y_λ with a_e.

The main tool in our proofs will be the colored generalization of Brylawski's tensor product formula [10].

Definition 1.3. *A* pointed colored graph $\widehat{N}$ *is an undirected graph together with a distinguished edge e that is neither a loop nor a bridge. All edges of $\widehat{N} - e$ are colored by elements of Λ.*

Next we recall the definition of the colored versions of the *pointed Tutte polynomials* $T_C(\widehat{N},e)$ *and* $T_L(\widehat{N},e)$. The original non-colored version of these polynomials was first introduced by Brylawski [2].

Definition 1.4. *Let $\widehat{N}$ be a pointed colored graph with distinguished edge e. The polynomial $T_L(\widehat{N},e)$ is obtained from the usual colored Tutte polynomial of $\widehat{N}-e$, except that internally active edges in cycles closed by e are treated as internally inactive. Similarly, $T_C(\widehat{N},e)$ is obtained from the usual colored Tutte polynomial of $\widehat{N}/e$, except that externally active edges that would close a cycle containing e are treated as externally inactive.*

It is shown in [10, Theorem 2] that $T_C(\widehat{N},e)$ and $T_L(\widehat{N},e)$ satisfy

$$x_\lambda\big(T(\widehat{N}/e)-T_C(\widehat{N},e)\big)=(Y_\lambda-y_\lambda)T_L(\widehat{N},e), \tag{1.10}$$

$$y_\lambda\big(T(\widehat{N}-e)-T_L(\widehat{N},e)\big)=(X_\lambda-x_\lambda)T_C(\widehat{N},e). \tag{1.11}$$

Definition 1.5. *Let M be a colored graph and $\widehat{N}$ a pointed colored graph with distinguished edge e. Fix $\lambda\in\Lambda$. The λ-*tensor product *$M\otimes_\lambda\widehat{N}$ is the colored graph obtained by replacing each λ-colored edge of M with a copy of $\widehat{N}-e$ and identifying the pointed edge e with the replaced edge.*

Theorem 1.6 (Diao–Hetyei–Hinson [10])**.** *The polynomial $T(M\otimes_\lambda\widehat{N})$ is obtained from $T(M)$ by the substitutions*

$$X_\lambda\mapsto T(\widehat{N}-e),\quad x_\lambda\mapsto T_L(\widehat{N},e),\quad Y_\lambda\mapsto T(\widehat{N}/e),\quad y_\lambda\mapsto T_C(\widehat{N},e),$$

while leaving all variables of color $\mu\neq\lambda$ unchanged.

1.3. **Tait graph and the Jones polynomial.** For a given link diagram D with a checkerboard shading, a crossing can be assigned a + or a − sign relative to this shading as shown on the left side of Figure 2. We shall call this sign the *shading sign* of the crossing as it is relative to the chosen checkerboard shading. The shading sign is not to be confused with the crossing sign with respect to the orientation of the link which is used in the definition of the writhe of D as shown on the right side of Figure 2.

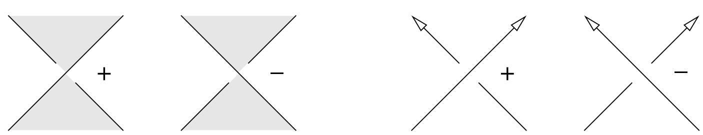

FIGURE 2. Left: the shading sign with respect to the checkerboard shading; Right: the crossing sign with respect to the orientation of the link.

The Tait graph of a link diagram is obtained from a checkerboard shading of the diagram such that each dark region corresponds to a vertex and a crossing between two adjacent dark regions corresponds to an edge connecting the two corresponding vertices. A Tait graph is a signed graph with the sign of each edge being the shading sign of the crossing corresponding to the edge. We shall denote the sign of an edge in the Tait graph by ε. We will use Kauffman's results [17, 18] expressing the Jones polynomial of a link diagram in terms of the signed Tutte polynomial. Note that Kauffman uses the weights x_ε and y_ε respectively instead of X_ε and Y_ε respectively; and he uses A_ε and B_ε respectively, instead of x_ε and y_ε respectively. Kauffman's result may be rephrased as follows.

Theorem 1.7. [18] *Let G be the signed Tait graph of a regular link projection D of K as described above, then $T(G)$ equals the Kauffman bracket polynomial $\langle D\rangle$ under the following variable substitutions:*

$$X_\varepsilon \mapsto -A^{-3\varepsilon},\ Y_\varepsilon \mapsto -A^{3\varepsilon},\ x_\varepsilon \mapsto A^{\varepsilon},\ y_\varepsilon \mapsto A^{-\varepsilon}.$$

Furthermore, the Jones polynomial $V_K(t)$ of K can be obtained from

$$V_K(t) = (-A^{-3})^{w(D)}\langle D\rangle \tag{1.12}$$

by setting $A = t^{-\frac{1}{4}}$, where $w(D)$ is the writhe of the projection D.

The kernel of the map given in Theorem 1.7 is a prime ideal, not containing any of the variables $X_\varepsilon, x_\varepsilon, Y_\varepsilon, y_\varepsilon$. Hence, in the quotient ring, the equations (1.8) and (1.9) hold, where both u and v equal to

$$d = \frac{-A^{-3\varepsilon} - A^{\varepsilon}}{A^{-\varepsilon}} = \frac{-A^{3\varepsilon} - A^{-\varepsilon}}{A^{\varepsilon}} = -A^{-2} - A^{2}. \tag{1.13}$$

In the special case of an alternating link diagram all edges of the Tait graph have the same sign, w.l.o.g. we may assume all signs are positive. The substitution rules stated in Theorem 1.7 may be rewritten as

$$A^{-1}X_+ \mapsto -A^{-4},\ AY_+ \mapsto -A^{4},\ A^{-1}x_+ \mapsto 1,\ Ay_+ \mapsto 1.$$

The number of internal edges (active or inactive) is the same for each spanning tree, and the same observation holds for the external edges. After taking out an appropriate factor of A we obtain the following consequence.

Corollary 1.8. *Let K be an alternating link and D an alternating diagram of K. Let G be the Tait graph of D as described above, having v vertices and e edges. Then the Kauffman bracket of K is*

$$\langle D\rangle = A^{2v-e-2}T(G; -A^{-4}, -A^{4}).$$

Here $T(G; X, Y)$ is the Tutte polynomial of the graph G.

2. The Tait graph of an unoriented rational link diagram

In this section we describe the structure of the Tait graph associated to an unoriented rational link diagram. For alternating link diagrams, essentially the same description appears in [27].

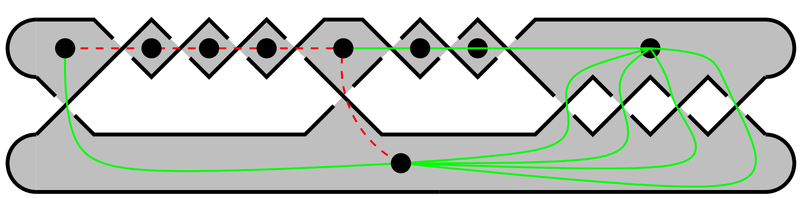

Figure 3. An unoriented rational link diagram with an odd number of twist boxes and its associated Tait graph.

Consider an unoriented rational link diagram D in standard form, as in Figures 3 and 4. (The continued fractions corresponding to these diagrams are $[0, 1, -4, -1, 3, 4]$ and $[0, 1, -4, -1, 3, 3, 1]$, respectively. Both evaluate to $49/40$, and hence the two diagrams represent the same unoriented link.)

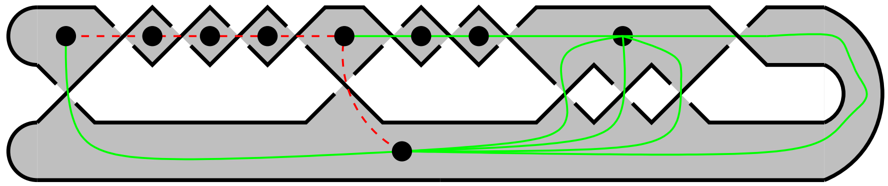

FIGURE 4. An unoriented rational link diagram with an even number of twist boxes and its associated Tait graph.

Choose a checkerboard coloring so that the bounded region adjacent to the lowest strand is black, and let G be the corresponding Tait graph. We call the vertex of G corresponding to this region the *central vertex.*

The structure of G may be described as follows.

First, the remaining vertices of G fall naturally into two classes: those adjacent to the central vertex, which we call *major vertices*, and those not adjacent to it, which we call *minor vertices*.

Second, the edges joining a major vertex to the central vertex correspond precisely to the crossings in the odd-indexed twist boxes.

Third, each minor vertex has degree two and lies on a path joining two major vertices. The edges along such a path correspond to the crossings in an even-indexed twist box.

Finally, the shading sign of a crossing in D agrees with its twist sign when the crossing lies in an odd-indexed twist box, and is opposite to its twist sign when it lies in an even-indexed twist box. Consequently, all edges incident to the central vertex have the same sign, and all edges along a path joining two major vertices also have the same sign.

These observations lead to the following lemma.

Lemma 2.1. *Let D be the standard diagram corresponding to the continued fraction* $[0, a_1, a_2, \ldots, a_n]$, *and let G be its Tait graph. Then the sign of an edge of G corresponding to a crossing in the i-th twist box is equal to* $\operatorname{sign}(a_i)$.

See Figures 3 and 4 for illustrations of this correspondence. Lemma 2.1 immediately implies the following.

Corollary 2.2. *If D is an alternating rational link diagram, then all edges of its Tait graph have the same sign $\varepsilon \in \{+, -\}$.*

These observations motivate the following definition.

Definition 2.3. *Let G be the Tait graph of an unoriented rational link diagram in standard form. The* underlying core graph G_n *is obtained from G by replacing each collection of parallel edges between the central vertex and a major vertex with a single edge, and by replacing each path connecting two consecutive major vertices with a single edge.*

By construction, the number of edges of G_n equals the number of twist boxes of D. We label each edge of G_n by the index of the corresponding twist box. G_{2n+1} and G_{2n} are shown in Figure 5.

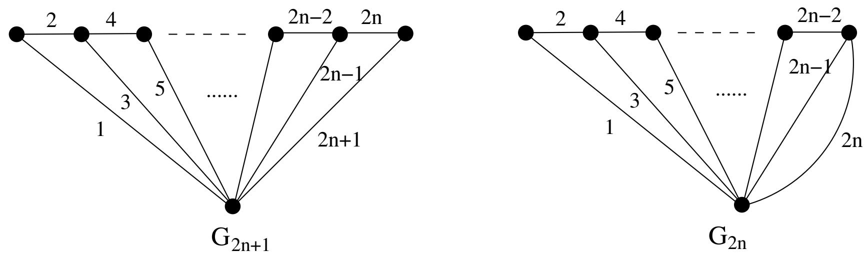

FIGURE 5. The core graphs G_{2n+1} and G_{2n}.

Remark 2.4. *Disregarding edge colors, the operations reducing the Tait graph to its core graph are precisely the* series–parallel reductions *in the sense of Traldi* [30].

We now observe that the Tait graph of an unoriented rational link diagram may be reconstructed from its core graph via colored tensor product operations.

Definition 2.5. *The* augmented path graph $\widehat{P}_m$ *is the pointed graph obtained from a path S_m of length m by joining its endpoints with a distinguished edge e. Dually, the* augmented dual path graph $\widehat{P}^*_m$ *consists of $m+1$ parallel edges between two distinct vertices, one of which is designated as the distinguished edge e.*

Equivalently, $\widehat{P}_m$ is a cycle of length $m+1$ with one distinguished edge. It is planar, and its dual is $\widehat{P}^*_m$.

The following structure theorem is an immediate consequence of the preceding description.

Theorem 2.6. *Let G be the Tait graph of the unoriented rational link diagram corresponding to the continued fraction $[0, a_1, a_2, \dots, a_n]$. Then G admits the decomposition*

$$G = G_n \otimes_1 \widehat{Q}_1 \otimes_2 \widehat{Q}_2 \cdots \otimes_n \widehat{Q}_n,$$

where

$$\widehat{Q}_i = \begin{cases} \widehat{P}_{a_i}, & \text{if } i \text{ is even,} \\ \widehat{P}^*_{a_i}, & \text{if } i \text{ is odd,} \end{cases}$$

and each non-distinguished edge of $\widehat{Q}_i$ is assigned the color $\operatorname{sign}(a_i)$.

As a consequence of Theorem 2.6, Theorem 1.6 provides a method to compute the Kauffman bracket of a rational link from any of its standard unoriented diagrams. It suffices to compute the colored Tutte polynomials of the core graphs in Figure 5, together with the pointed signed Tutte polynomials

$$T_C(\widehat{P}_{a_i}, e), \quad T_L(\widehat{P}_{a_i}, e), \quad T(\widehat{P}_{a_i} - e), \quad T(\widehat{P}_{a_i}/e),$$

and their dual counterparts

$$T_C(\widehat{P}^*_{a_i}, e), \quad T_L(\widehat{P}^*_{a_i}, e), \quad T(\widehat{P}^*_{a_i} - e), \quad T(\widehat{P}^*_{a_i}/e).$$

The pointed Tutte polynomials of the signed augmented path graphs and their duals are computed in Section 3, while the signed Tutte polynomials of the core graphs are computed in Section 4.

3. The Tutte invariants related to the augmented path graphs and their duals

In this section we extend the results of Traldi [30] to the setting of colored Tutte polynomials and derive the corresponding consequences for the Kauffman bracket.

We first compute the (pointed) Tutte polynomials associated to the augmented path graph $\widehat{P}_m$ directly from the activity-based definitions. Label the edges of the path S_m consecutively from 1 to m. Since $\widehat{P}_m - e$ is the path S_m, it has a unique spanning tree, namely S_m itself. All its edges are internally active in $T(\widehat{P}_m - e)$ and internally inactive in $T_L(\widehat{P}_m, e)$. Hence

$$T(\widehat{P}_m - e) = X_\lambda^m, \tag{3.1}$$

and

$$T_L(\widehat{P}_m, e) = x_\lambda^m. \tag{3.2}$$

The graph $\widehat{P}_m/e$ is a cycle of length m. Its spanning trees are obtained by deleting one edge. If the deleted edge has label $i \geq 2$, then the edges with labels less than i are internally active, those with larger labels are internally inactive, and the deleted edge is externally inactive. If the deleted edge is e_1, then all remaining edges are internally inactive while e_1 is externally active. Therefore

$$T(\widehat{P}_m/e) = \sum_{i=2}^{m} X_\lambda^{i-1} y_\lambda x_\lambda^{m-i} + Y_\lambda x_\lambda^{m-1}.$$

Using (1.9), this may be rewritten as

$$T(\widehat{P}_m/e) = \sum_{i=1}^{m} X_\lambda^{i-1} y_\lambda x_\lambda^{m-i} + x_\lambda^m v,$$

that is,

$$T(\widehat{P}_m/e) = y_\lambda x_\lambda^{m-1} [m]_{\frac{X_\lambda}{x_\lambda}} + x_\lambda^m v, \tag{3.3}$$

where we use the q-notation

$$[m]_q = 1 + q + q^2 + \cdots + q^{m-1}.$$

The pointed polynomial $T_C(\widehat{P}_m, e)$ is obtained by replacing Y_λ with y_λ in the preceding argument. Hence

$$T_C(\widehat{P}_m, e) = y_\lambda x_\lambda^{m-1} [m]_{\frac{X_\lambda}{x_\lambda}}. \tag{3.4}$$

We now compute the (pointed) Tutte polynomials associated to the pointed graph $N = \widehat{P}_m^*$, which consists of the distinguished edge e together with m parallel edges. Then $\widehat{P}_m^*/e$ consists of m loops, and hence

$$T(\widehat{P}_m^*/e) = Y_\lambda^m, \tag{3.5}$$

$$T_C(\widehat{P}_m^*, e) = y_\lambda^m. \tag{3.6}$$

To compute $T(\widehat{P}_m^* - e)$ and $T_L(\widehat{P}_m^*, e)$, label the m parallel edges of $\widehat{P}_m^* - e$ from 1 to m. A spanning tree consists of a single edge, say the edge of label i; all edges with larger labels are externally inactive.

If $i \geq 2$, then the chosen edge is internally inactive, and the edges with smaller labels are externally active. If $i = 1$, then that edge is internally active.

Proceeding as in the computation of $T(\widehat{P}_m/e)$ and using (1.8), we obtain

$$T(\widehat{P}^*_m - e) = x_\lambda y_\lambda^{m-1}[m]_{\frac{Y_\lambda}{y_\lambda}} + y_\lambda^m u. \tag{3.7}$$

The computation of $T_L(\widehat{P}^*_m, e)$ is dual to that of $T_L(\widehat{P}_m, e)$, yielding

$$T_L(\widehat{P}^*_m, e) = x_\lambda y_\lambda^{m-1}[m]_{\frac{Y_\lambda}{y_\lambda}}. \tag{3.8}$$

Formulas (3.3), (3.4), (3.7), and (3.8) may be simplified using the following identities.

Lemma 3.1. *For each $\lambda \in \Lambda$ and integer $m \geq 1$, the following equalities hold:*

$$y_\lambda x_\lambda^{m-1}[m]_{\frac{X_\lambda}{x_\lambda}} = \begin{cases} m\, y_\lambda x_\lambda^{m-1}, & \text{if } u = 0, \\ \dfrac{X_\lambda^m - x_\lambda^m}{u}, & \text{if } u \neq 0, \end{cases} \tag{3.9}$$

and

$$x_\lambda y_\lambda^{m-1}[m]_{\frac{Y_\lambda}{y_\lambda}} = \begin{cases} m\, x_\lambda y_\lambda^{m-1}, & \text{if } v = 0, \\ \dfrac{Y_\lambda^m - y_\lambda^m}{v}, & \text{if } v \neq 0. \end{cases} \tag{3.10}$$

Proof. We prove (3.9); the proof of (3.10) is analogous.

By (1.8) we have $X_\lambda = x_\lambda + y_\lambda u$. If $u = 0$, then $\frac{X_\lambda}{x_\lambda} = 1$ and $[m]_1 = m$, giving the first case.

Assume now that $u \neq 0$. From the geometric series expansion,

$$y_\lambda x_\lambda^{m-1}[m]_{\frac{X_\lambda}{x_\lambda}} = y_\lambda x_\lambda^{m-1} \frac{(\frac{X_\lambda}{x_\lambda})^m - 1}{\frac{X_\lambda}{x_\lambda} - 1} = y_\lambda \frac{X_\lambda^m - x_\lambda^m}{X_\lambda - x_\lambda}.$$

Substituting $X_\lambda - x_\lambda$ by $y_\lambda u$ then yields the desired result. ∎

4. The colored Tutte polynomial of the core graphs

In this section we derive closed formulas for the colored Tutte polynomials

$$T_n = T(G_n)$$

of the core graphs G_n. Theorem 4.1 provides both our simplest closed form and the most efficient method for computing these polynomials.

In Section 5 we present alternative reformulations under the additional assumption that T_n lies in the field of fractions K of $\mathbb{Z}[\Lambda]/P$, where P is a prime ideal containing I_1 but not containing any of the variables $X_\lambda, x_\lambda, Y_\lambda$, or y_λ.

Applying the deletion–contraction recurrence (1.3) to the edge e_{2n+1} of G_{2n+1}, and observing that e_{2n} becomes a bridge once e_{2n+1} is deleted (and hence must be contracted), we obtain

$$T_{2n+1} = y_{2n+1}X_{2n}T_{2n-1} + x_{2n+1}T_{2n}. \tag{4.1}$$

Similarly, applying (1.3) to e_{2n} in G_{2n}, and observing that e_{2n-1} becomes a loop after e_{2n} is contracted (and hence must be deleted), yields

$$T_{2n} = Y_{2n-1}x_{2n}T_{2n-2} + y_{2n}T_{2n-1}. \tag{4.2}$$

For the initial condition we set

$$T_0 = T(G_0) = 1, \qquad T_1 = T(G_1) = T(\{e_1\}) = X_1,$$

where G_0 is the graph consisting of a single vertex. From the recurrences we obtain

$$T_2 = Y_1x_2 + X_1y_2,$$

$$T_3 = X_1X_2y_3 + Y_1x_2x_3 + X_1y_2x_3,$$

and so forth.

Repeated application of (4.1) and (4.2) shows that T_n is a sum of F_n monomials, where (F_n) is the Fibonacci sequence defined by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Equivalently, F_n is the number of tilings of a $1 \times n$ rectangle by tiles of sizes 1×1 and 2×1.

This combinatorial interpretation leads to the following description of T_n.

Theorem 4.1. *For $n \geq 1$, the polynomial T_n is the total weight of all tilings of a $1 \times n$ rectangle by tiles of size 1×1 and 2×1, with weights assigned as follows:*

(i) A 1×1 tile in position 1 has weight X_1;
(ii) A 1×1 tile in any other odd position $2k+1$ has weight x_{2k+1};
(iii) A 1×1 tile in any even position $2k$ has weight y_{2k};
(iv) A 2×1 tile covering positions $(2k-1, 2k)$ has weight $Y_{2k-1}x_{2k}$;
(v) A 2×1 tile covering positions $(2k, 2k+1)$ has weight $X_{2k}y_{2k+1}$.

Proof. The statement follows directly from the recurrences (4.1) and (4.2) by interpreting each term of the recurrence as the addition of either a 1×1 tile or a 2×1 tile at the end of a tiling. □

Example 4.2. For $n = 4$, $F_4 = 5$ and the five possible tilings are $(1, 2, 3, 4)$, $(1\text{–}2, 3, 4)$, $(1, 2\text{–}3, 4)$, $(1, 2, 3\text{–}4)$ and $(1\text{–}2, 3\text{–}4)$. Their corresponding monomials are $X_1y_2x_3y_4$, $Y_1x_2x_3y_4$, $X_1X_2y_3y_4$, $X_1y_2Y_3x_4$ and $Y_1x_2Y_3x_4$. Therefore

$$T_4 = X_1y_2x_3y_4 + Y_1x_2x_3y_4 + X_1X_2y_3y_4 + X_1y_2Y_3x_4 + Y_1x_2Y_3x_4.$$

Example 4.3. For $n = 11$ we have $F_{11} = 144$, so the polynomial T_{11} contains 144 monomials. For example, the tiling $(1, 2, 3, 4\text{–}5, 6\text{–}7, 8, 9\text{–}10, 11)$ has weight $X_1y_2x_3X_4y_5X_6y_7y_8Y_9x_{10}x_{11}$, and hence contributes this monomial to T_{11}.

We now derive a closed form expression for the sequence $(T_n)_{n\geq 0}$. From (4.1) and (4.2), we may write, for $k \geq 2$,

$$T_k = u_kT_{k-2} + v_kT_{k-1}, \tag{4.3}$$

with initial conditions

$$T_0 = 1, \qquad T_1 = X_1,$$

where

$$u_k = \begin{cases} X_{k-1}y_k, & \text{if } k \text{ is odd,} \\ Y_{k-1}x_k, & \text{if } k \text{ is even,} \end{cases} \tag{4.4}$$

and

$$v_k = \begin{cases} x_k, & \text{if } k \text{ is odd,} \\ y_k, & \text{if } k \text{ is even.} \end{cases} \tag{4.5}$$

Equation (4.3) may be rewritten in matrix form as

$$\begin{bmatrix} T_n \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} v_n & u_n \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_{n-1} & u_{n-1} \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} v_2 & u_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ 1 \end{bmatrix} \tag{4.6}$$

for all $n \geq 2$.

In many applications, including the computation of the Kauffman bracket, the Tutte invariants are considered in the field of fractions K of $\mathbb{Z}[\Lambda]/P$, where P is a prime ideal containing I_1 but none of the variables $X_\lambda, x_\lambda, Y_\lambda, y_\lambda$. Under this assumption, equation (1.8) gives

$$X_1 = x_1 + y_1 u,$$

and hence

$$\begin{bmatrix} X_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix}.$$

Setting

$$u_1 = y_1, \qquad v_1 = x_1, \tag{4.7}$$

we may rewrite (4.6) in the uniform form

$$\begin{bmatrix} T_n \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} v_n & u_n \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_{n-1} & u_{n-1} \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} v_1 & u_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix}, \tag{4.8}$$

which holds for all $n \geq 1$.

Remark 4.4. *The choices in* (4.7) *are consistent with* (4.5) *and, after formally setting* $X_0 = 1$*, also with* (4.4)*. With the convention* $T_{-1} = u$*, the recurrence* (4.3) *extends to* $k = 1$*, yielding*

$$T_1 = y_1 T_{-1} + x_1 T_0 = u_1 T_{-1} + v_1 T_0.$$

Taking ratios in (4.8) yields the generalized continued fraction representation

$$\frac{T_n}{T_{n-1}} = v_n + \cfrac{u_n}{v_{n-1} + \cfrac{u_{n-1}}{\ddots + \cfrac{u_2}{v_1 + u_1 u}}}. \tag{4.9}$$

5. Two reformulations of our results on the core graphs

In this section we assume that our Tutte invariants lie in the field of fractions K of $\mathbb{Z}[\Lambda]/P$, where P is a prime ideal containing I_1 but none of the variables $X_\lambda, x_\lambda, Y_\lambda, y_\lambda$. Under this assumption we present two reformulations of the results of Section 4. The first reformulation converts (4.9) into an ordinary continued fraction.

Definition 5.1. *Let $(p_i)_{i\geq 0}$ be a sequence of parameters. For $m \geq 0$, the* generalized semifactorials induced by (p_i) *are defined by*

$$p_{2m}!! = \prod_{j=0}^{m} p_{2j}, \qquad p_{2m+1}!! = \prod_{j=0}^{m} p_{2j+1}.$$

If $p_0 = 1$ and $p_i = i$ for $i > 0$, this reduces to the usual definition of semifactorials.

By setting $u_0 = 1$ and $u_{-1} = 1$, we can extend the definition of $u_n!!$ to include the case $n = 0$ and $n = -1$, namely $u_0!! = u_{-1}!! = 1$. This allows us to define

$$S_n = \frac{T_n}{u_n!!} \tag{5.1}$$

for $n \geq -1$.

Using (4.3) and the definition of u_0 and u_{-1}, for any $j \geq 0$, we compute

$$\begin{aligned} S_{2j+1} = \frac{T_{2j+1}}{u_{2j+1}!!} &= \frac{u_{2j+1}T_{2j-1} + v_{2j+1}T_{2j}}{u_{2j+1}!!} \\ &= S_{2j-1} + \frac{v_{2j+1}u_{2j}!!}{u_{2j+1}!!} S_{2j}, \end{aligned}$$

and

$$\begin{aligned} S_{2j+2} = \frac{T_{2j+2}}{u_{2j+2}!!} &= \frac{u_{2j+2}T_{2j} + v_{2j+2}T_{2j+1}}{u_{2j+2}!!} \\ &= S_{2j} + \frac{v_{2j+2}u_{2j+1}!!}{u_{2j+2}!!} S_{2j+1}. \end{aligned}$$

These relations may be summarized as follows.

Proposition 5.2. *The sequence (S_n) satisfies*

$$S_n = S_{n-2} + \frac{v_n\, u_{n-1}!!}{u_n!!}\, S_{n-1}, \qquad n \geq 1,$$

with initial conditions

$$S_{-1} = u, \qquad S_0 = 1.$$

We now obtain the first reformulation of the closed form for T_n.

Theorem 5.3. *Under the assumption that the Tutte invariants lie in the field of fractions K of $\mathbb{Z}[\Lambda]/P$, where P is a prime ideal containing I_1 but none of the variables $X_\lambda, x_\lambda, Y_\lambda, y_\lambda$, we have*

$$T_n = u_n!!\, S_n,$$

where

$$\begin{bmatrix} S_n \\ S_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{v_n\, u_{n-1}!!}{u_n!!} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{v_{n-1}\, u_{n-2}!!}{u_{n-1}!!} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} \frac{v_1 u_0!!}{u_1!!} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix}, \qquad n \geq 0.$$

Here (u_k) *and* (v_k) *are given by* (4.4), (4.5), *and* (4.7).

Taking ratios yields the ordinary continued fraction

$$\frac{S_n}{S_{n-1}} = \frac{v_n\, u_{n-1}!!}{u_n!!} + \cfrac{1}{\frac{v_{n-1}\, u_{n-2}!!}{u_{n-1}!!} + \cfrac{1}{\ddots + \cfrac{1}{\frac{v_1 u_0!!}{u_1!!} + \cfrac{1}{u}}}}. \tag{5.2}$$

Our second reformulation is inspired by [21, Theorem 4.2] and generalizes it to the present setting. The goal is to replace (4.8) by a matrix product in which the k-th matrix (counting from the right) involves only the variables indexed by k, namely X_k, x_k, Y_k, and y_k.

Extracting the common factor

$$\prod_{1 \leq j \leq \lfloor n/2 \rfloor} X_{2j} \cdot \prod_{1 \leq j \leq \lfloor (n+1)/2 \rfloor} Y_{2j-1}$$

from the expression for T_n in Theorem 4.1 yields the following reformulation.

Corollary 5.4. *The polynomial* T_n *equals*

$$\prod_{1 \leq j \leq \lfloor n/2 \rfloor} X_{2j} \cdot \prod_{1 \leq j \leq \lfloor (n+1)/2 \rfloor} Y_{2j-1}$$

times the total weight of all tilings of a $1 \times n$ *rectangle by* 1×1 *and* 2×1 *tiles, with weights assigned as follows:*

(i) A 1×1 *tile in position* 1 *has weight* $\frac{X_1}{Y_1}$*;*
(ii) A 1×1 *tile in any other odd position* $2k+1$ *has weight* $\frac{x_{2k+1}}{Y_{2k+1}}$*;*
(iii) A 1×1 *tile in any even position* $2k$ *has weight* $\frac{y_{2k}}{X_{2k}}$*;*
(iv) A 2×1 *tile covering* $(2k-1, 2k)$ *has weight* $\frac{x_{2k}}{X_{2k}}$*;*
(v) A 2×1 *tile covering* $(2k, 2k+1)$ *has weight* $\frac{y_{2k+1}}{Y_{2k+1}}$*.*

Indeed, the prefactor

$$\prod_{1 \leq j \leq \lfloor n/2 \rfloor} X_{2j} \cdot \prod_{1 \leq j \leq \lfloor (n+1)/2 \rfloor} Y_{2j-1}$$

contains exactly one variable indexed by k for each $1 \leq k \leq n$. Thus the contribution of each tile in Theorem 4.1 may be divided by the variable whose index matches the position covered by that tile.

In analogy with (4.6) and (4.8), we obtain the following matrix formulation.

Theorem 5.5. *Define*

$$R_n = \frac{T_n}{\prod_{1 \leq j \leq \lfloor n/2 \rfloor} X_{2j} \cdot \prod_{1 \leq j \leq \lfloor (n+1)/2 \rfloor} Y_{2j-1}}.$$

Then

$$R_n = \begin{bmatrix}1 & 0\end{bmatrix} M_n M_{n-1} \cdots M_1 \begin{bmatrix}1\\ u\end{bmatrix},$$

where

$$M_k = \begin{cases} \begin{bmatrix} \frac{y_k}{X_k} & \frac{x_k}{X_k} \\ 1 & 0 \end{bmatrix}, & \textit{if } k \textit{ is even,} \\ \begin{bmatrix} \frac{x_k}{Y_k} & \frac{y_k}{Y_k} \\ 1 & 0 \end{bmatrix}, & \textit{if } k \textit{ is odd.} \end{cases}$$

In particular, each matrix M_k depends only on the variables indexed by k, reflecting the local contribution of the k-th twist box.

6. Formulas for the Kauffman bracket

In this section we combine Theorems 1.6 and 2.6 with the results of Sections 3, 4, and 5 to obtain explicit formulas for the Kauffman bracket of a rational link diagram in standard form.

For brevity, we write

$$K_n = K\big([0, a_1, a_2, \dots, a_n]\big)$$

for the Kauffman bracket of the unoriented rational link diagram encoded by the continued fraction $[0, a_1, a_2, \dots, a_n]$. Let

$$\varepsilon_i = \mathrm{sign}(a_i).$$

By Lemma 2.1, ε_i is precisely the sign of the edge in the Tait graph corresponding to crossings in the i-th twist box.

Setting $\lambda = \varepsilon \in \{+, -\}$ and applying Theorem 1.7 to (3.1)–(3.8), we obtain the following correspondence rules for computing the Kauffman bracket:

$$T(\widehat{P}_m - e) \longmapsto (-A)^{-3m\varepsilon}, \tag{6.1}$$

$$T_L(\widehat{P}_m, e) \longmapsto A^{m\varepsilon}, \tag{6.2}$$

$$T(\widehat{P}_m/e) \longmapsto A^{(m-2)\varepsilon}[m]_{-A^{-4\varepsilon}} + A^{m\varepsilon} d, \tag{6.3}$$

$$T_C(\widehat{P}_m, e) \longmapsto A^{(m-2)\varepsilon}[m]_{-A^{-4\varepsilon}}, \tag{6.4}$$

$$T(\widehat{P}^*_m/e) \longmapsto (-A)^{3m\varepsilon}, \tag{6.5}$$

$$T_C(\widehat{P}^*_m, e) \longmapsto A^{-m\varepsilon}, \tag{6.6}$$

$$T(\widehat{P}^*_m - e) \longmapsto A^{-(m-2)\varepsilon}[m]_{-A^{4\varepsilon}} + A^{-m\varepsilon} d, \tag{6.7}$$

$$T_L(\widehat{P}^*_m, e) \longmapsto A^{-(m-2)\varepsilon}[m]_{-A^{4\varepsilon}}. \tag{6.8}$$

Here $d = -A^{-2} - A^2$ as in (1.13).

Since replacing ε by $-\varepsilon$ simply interchanges the two possible edge signs, the two cases in Lemma 3.1 lead to the same structural consequence. Thus by Theorem 1.7, we obtain the following specialization.

Corollary 6.1.

$$A^{(2-m)\varepsilon}[m]_{-A^{4\varepsilon}} = \begin{cases} m\,A^{(2-m)\varepsilon}, & \text{if } d=0, \\ \dfrac{(-A)^{3m\varepsilon} - A^{-m\varepsilon}}{d}, & \text{otherwise.} \end{cases}$$

Note that $d = 0$ is equivalent to $A^{-4} = -1$. In the computation of the Jones polynomial, this occurs precisely when $t = -1$.

For even indices $2j$, we have

$$\widehat{Q}_{2j} = \widehat{P}_{|a_{2j}|}, \qquad \varepsilon_{2j} = \operatorname{sign}(a_{2j}),$$

so the substitutions required to obtain K_n from T_n are

$$X_{2j} \mapsto T(\widehat{Q}_{2j} - e) \mapsto (-A)^{-3a_{2j}}, \tag{6.9}$$

$$Y_{2j} \mapsto T(\widehat{Q}_{2j}/e) \mapsto A^{a_{2j}-2\varepsilon_{2j}}[|a_{2j}|]_{-A^{-4\varepsilon_{2j}}} + A^{a_{2j}} \cdot d, \tag{6.10}$$

$$x_{2j} \mapsto T_L(\widehat{Q}_{2j}, e) \mapsto A^{a_{2j}}, \tag{6.11}$$

$$y_{2j} \mapsto T_C(\widehat{Q}_{2j}, e) \mapsto A^{a_{2j}-2\varepsilon_{2j}}[|a_{2j}|]_{-A^{-4\varepsilon_{2j}}}, \tag{6.12}$$

(Observe that Y_{2j} does not appear explicitly in the final expression for K_n.)

Similarly, for odd indices $2j+1$,

$$\widehat{Q}_{2j+1} = \widehat{P}^*_{|a_{2j+1}|}, \qquad \varepsilon_{2j+1} = \operatorname{sign}(a_{2j+1}),$$

and we obtain

$$X_{2j+1} \longmapsto A^{-a_{2j+1}+2\varepsilon_{2j+1}}[|a_{2j+1}|]_{-A^{4\varepsilon_{2j+1}}} + A^{-a_{2j+1}}d, \tag{6.13}$$

$$Y_{2j+1} \longmapsto (-A)^{3a_{2j+1}}, \tag{6.14}$$

$$x_{2j+1} \longmapsto A^{-a_{2j+1}+2\varepsilon_{2j+1}}[|a_{2j+1}|]_{-A^{4\varepsilon_{2j+1}}}, \tag{6.15}$$

$$y_{2j+1} \longmapsto A^{-a_{2j+1}}. \tag{6.16}$$

Finally, X_{2j+1} appears in the formulation of K_n only for $j = 0$. In that case,

$$X_1 \longmapsto A^{-a_1+2\varepsilon_1}[|a_1|]_{-A^{4\varepsilon_1}} + A^{-a_1}(-A^2 - A^{-2}). \tag{6.17}$$

Applying the substitution rules above to (4.4), we obtain

$$u_{2j+1} \longmapsto X_{2j}y_{2j+1} \longmapsto (-A)^{-3a_{2j}}A^{-a_{2j+1}} = (-1)^{a_{2j}}A^{-3a_{2j}-a_{2j+1}},$$
$$u_{2j} \longmapsto Y_{2j-1}x_{2j} \longmapsto (-A)^{3a_{2j-1}}A^{a_{2j}} = (-1)^{a_{2j-1}}A^{3a_{2j-1}+a_{2j}}.$$

These two cases may be summarized as

$$u_k \longmapsto (-1)^{a_{k-1}}A^{(-1)^k(3a_{k-1}+a_k)}. \tag{6.18}$$

Similarly, applying the substitution rules to (4.5) yields

$$v_{2j+1} \longmapsto x_{2j+1} \longmapsto A^{-a_{2j+1}+2\varepsilon_{2j+1}}[|a_{2j+1}|]_{-A^{4\varepsilon_{2j+1}}},$$
$$v_{2j} \longmapsto y_{2j} \longmapsto A^{a_{2j}-2\varepsilon_{2j}}[|a_{2j}|]_{-A^{-4\varepsilon_{2j}}}.$$

These may be combined into the uniform expression

$$v_k \longmapsto A^{(-1)^{k-1}(2\varepsilon_k - a_k)}[|a_k|]_{-A^{(-1)^{k-1}4\varepsilon_k}}. \tag{6.19}$$

We are now ready to compute K_n. Combining Theorem 4.1 with (6.17), (6.18), and (6.19), we obtain the following explicit description.

Corollary 6.2. *The Kauffman bracket*

$$K_n = K([0, a_1, a_2, \ldots, a_n])$$

is the total weight of all tilings of a path of length n by 1×1 and 2×1 tiles, with weights assigned as follows:

(1) A 1×1 tile in position 1 has weight

$$A^{-a_1+2\varepsilon_1}[|a_1|]_{-A^{4\varepsilon_1}} + A^{-a_1}d.$$

(2) A 1×1 tile in position $k > 1$ has weight

$$A^{(-1)^{k-1}(2\varepsilon_k - a_k)}[|a_k|]_{-A^{(-1)^{k-1}4\varepsilon_k}}.$$

(3) A 2×1 tile covering $(k, k+1)$ has weight

$$(-1)^{a_k} A^{(-1)^{k+1}(3a_k + a_{k+1})}.$$

An equivalent expression is obtained by substituting (6.18) and (6.19) directly into the matrix formulation (4.8). The verification is straightforward and is omitted.

Example 6.3. Let $D = [0, 2, 1, 3, 3]$ be the two-component link, with the components oriented so that $w(D) = -5$. There are five tilings of the 1×4 strip:

$$\begin{aligned}
&(1 \times 1, 1 \times 1, 1 \times 1, 1 \times 1),\\
&(1 \times 1, 1 \times 1, 2 \times 1),\\
&(1 \times 1, 2 \times 1, 1 \times 1),\\
&(2 \times 1, 1 \times 1, 1 \times 1),\\
&(2 \times 1, 2 \times 1).
\end{aligned}$$

Their combined weights are

$$\begin{aligned}
&- A^{-5}(1 + A^8)(1 - A^4 + A^8)(1 - A^{-4} + A^{-8}),\\
&A^7(1 + A^8),\\
&A^{-9}(1 + A^8)(1 - A^{-4} + A^{-8}),\\
&A^7(1 - A^4 + A^8)(1 - A^{-4} + A^{-8}),\\
&- A^{-19},
\end{aligned}$$

respectively. Summing these contributions, multiplying by $(-A^3)^{-w(D)} = -A^{15}$, and substituting $A^{-4} = t$, we obtain

$$J(D) = t^{-\frac{1}{2}}\big(-t + 2 - 4t^{-1} + 5t^{-2} - 6t^{-3} + 6t^{-4} - 6t^{-5} + 3t^{-6} - 2t^{-7} + t^{-8}\big).$$

We next compute the substitutions into $u_k!!$ in order to apply Theorem 5.3. Recall that $u_1 = y_1 \mapsto A^{-a_1}$. Using (6.18), we obtain

$$\begin{aligned} u_{2j+1}!! &\longmapsto \prod_{k=0}^{j} (-1)^{a_{2k}} A^{-(3a_{2k}+a_{2k+1})} \\ &= (-A^{-2})^{a_2+a_4+\cdots+a_{2j}} A^{-(a_1+\cdots+a_{2j+1})}, \end{aligned} \tag{6.20}$$

$$\begin{aligned} u_{2j+2}!! &\longmapsto \prod_{k=0}^{j} (-1)^{a_{2k+1}} A^{3a_{2k+1}+a_{2k+2}} \\ &= (-A^{2})^{a_1+a_3+\cdots+a_{2j+1}} A^{a_1+\cdots+a_{2j+2}}. \end{aligned} \tag{6.21}$$

Taking quotients gives

$$\frac{u_{2j+1}!!}{u_{2j+2}!!} \longmapsto (-A^{-4})^{a_1+\cdots+a_{2j+1}} A^{-a_{2j+2}},$$

and, similarly,

$$\frac{u_{2j}!!}{u_{2j+1}!!} \longmapsto (-A^{4})^{a_1+\cdots+a_{2j}} A^{a_{2j+1}}.$$

Using (6.19), we obtain

$$\frac{v_{2j+2}u_{2j+1}!!}{u_{2j+2}!!} \longmapsto A^{-2\varepsilon_{2j+2}} [|a_{2j+2}|]_{-A^{-4\varepsilon_{2j+2}}} (-A^{-4})^{a_1+\cdots+a_{2j+1}}, \tag{6.22}$$

$$\frac{v_{2j+1}u_{2j}!!}{u_{2j+1}!!} \longmapsto A^{2\varepsilon_{2j+1}} [|a_{2j+1}|]_{-A^{4\varepsilon_{2j+1}}} (-A^{4})^{a_1+\cdots+a_{2j}}. \tag{6.23}$$

These may be summarized in the uniform form

$$\frac{v_{k+1}u_{k}!!}{u_{k+1}!!} \longmapsto A^{2(-1)^k\varepsilon_{k+1}} [|a_{k+1}|]_{-A^{4(-1)^k\varepsilon_{k+1}}} (-A^{4(-1)^k})^{a_1+\cdots+a_{k}}. \tag{6.24}$$

Similarly, (6.20) and (6.21) may be summarized as

$$u_{k+1}!! \longmapsto (-A^{2(-1)^{k+1}})^{a_k+a_{k-2}+\cdots} A^{(-1)^{k+1}(a_1+\cdots+a_{k+1})}. \tag{6.25}$$

Definition 6.4. *Define*

$$\omega_k(a_1,\ldots,a_k) = (-A^{2(-1)^k})^{a_{k-1}+a_{k-3}+\cdots} A^{(-1)^k(a_1+\cdots+a_k)},$$

and

$$\gamma_k(a_1,\ldots,a_k) = A^{2(-1)^{k-1}\varepsilon_k} [|a_k|]_{-A^{4(-1)^{k-1}\varepsilon_k}} (-A^{4(-1)^{k-1}})^{a_1+\cdots+a_{k-1}}.$$

Substituting into Theorem 5.3 yields the following.

Corollary 6.5. *The Kauffman bracket*

$$K_n = K([0,a_1,\ldots,a_n])$$

is given by

$$K_n = \omega_n(a_1,\ldots,a_n)\,\kappa_n,$$

where κ_n *satisfies*

$$\begin{bmatrix}\kappa_n\\ \kappa_{n-1}\end{bmatrix} = \begin{bmatrix}\gamma_n & 1\\ 1 & 0\end{bmatrix}\cdots\begin{bmatrix}\gamma_1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ d\end{bmatrix}.$$

Here $d = -A^2 - A^{-2}$ as in (1.13).

Remark 6.6. Using (1.1) we may express $\gamma_k(a_1, \ldots, a_k)$ as

$$\gamma_k(a_1, \ldots, a_k) = A^{2\,\mathrm{sign}_t(a_k)}[|a_k|]_{-A^{4\,\mathrm{sign}_t(a_k)}}(-A^{4(-1)^{k-1}})^{a_1+a_2+\cdots+a_{k-1}}.$$

Finally, substituting (6.9)–(6.16) into Theorem 5.5 gives a fully explicit matrix product.

Theorem 6.7. *The Kauffman bracket satisfies*

$$K_n = (-A)^{3\sum_{k=1}^{n}(-1)^{k-1}a_k}\ \rho_n,$$

where

$$\rho_n = \begin{bmatrix}1 & 0\end{bmatrix} M_n \cdots M_1 \begin{bmatrix}1\\ d\end{bmatrix},$$

with

$$M_k = \begin{bmatrix}-A^{2(-1)^k\varepsilon_k}[|a_k|]_{-A^{4(-1)^k\varepsilon_k}} & (-A^{4(-1)^k})^{a_k}\\ 1 & 0\end{bmatrix}. \tag{6.26}$$

7. Substitutions into the Kauffman bracket and the Jones polynomial

Corollary 6.1 shows that the formulas of Section 6 simplify considerably once a specific substitution is fixed and we determine whether $d = 0$ holds. We treat the two possible cases separately.

We first consider the special case $d = 0$. When evaluating the Jones polynomial, this corresponds to the substitution $t = -1$. In this situation it is convenient to assume that the rational link is represented by an alternating diagram.

Theorem 7.1. *Let D be a rational link diagram encoded by the continued fraction*

$$[0, a_1, a_2, \ldots, a_n] = \frac{p}{q},$$

where $a_k > 0$ for all k, and $p, q > 0$ are relatively prime. Then

$$V_L(-1) = (-1)^{\binom{n}{2}} i^n \left(\frac{1+i}{\sqrt{2}}\right)^{w(D)+\sum_{k=1}^{n}(-1)^k a_k} q.$$

Proof. We evaluate the Kauffman bracket using Corollary 6.5 at the primitive eighth root of unity

$$A = \frac{1+i}{\sqrt{2}} = e^{i\frac{\pi}{4}}.$$

For this substitution we have

$$A^2 = i, \qquad A^4 = -1, \qquad -A^2 - A^{-2} = 0.$$

For an alternating rational diagram with $a_k > 0$, the twisting sign of the kth twist box is

$$\mathrm{sign}_t(a_k) = (-1)^{k-1}.$$

By Remark 6.6,

$$\gamma_k(a_1,\dots,a_k) = i^{(-1)^{k-1}} a_k = (-1)^{k-1} i\, a_k.$$

Hence Corollary 6.5 becomes

$$\begin{bmatrix} \kappa_n \\ \kappa_{n-1} \end{bmatrix} = \begin{bmatrix} (-1)^{n-1} i\, a_n & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} i\, a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore κ_n/κ_{n-1} equals the continued fraction

$$[(-1)^{n-1} i\, a_n, (-1)^{n-2} i\, a_{n-1}, \dots, -i\, a_2, i\, a_1].$$

By the Euler–Minding formulas (see [14, formula (1.3)]), the numerator κ_n is given by

$$\kappa_n = \sum_{S\subseteq_e\{1,\dots,n\}} \prod_{k\notin S} (-1)^{k-1} i\, a_k = \sum_{S\subseteq_e\{1,\dots,n\}} \frac{(-1)^{\binom{n}{2}} i^n a_1\cdots a_n}{\prod_{k\in S}((-1)^{k-1} i a_k)},$$

where $S \subseteq_e$ denotes the even subsets, i.e. unions of disjoint intervals of even cardinality. Since each subset S can be partitioned into disjoint subsets of the form $\{j, j+1\}$, and the terms in $\prod_{k\in S}((-1)^{k-1} i a_k$ corresponding to j and $j+1$ can be simplified as

$$(-1)^{j-1} i a_j \cdot (-1)^j i a_{j+1} = a_j a_{j+1},$$

we have

$$\sum_{S\subseteq_e\{1,\dots,n\}} \frac{(-1)^{\binom{n}{2}} i^n a_1\cdots a_n}{\prod_{k\in S}((-1)^{k-1} i a_k)} = (-1)^{\binom{n}{2}} i^n \sum_{S\subseteq_e\{1,\dots,n\}} \frac{a_1\cdots a_n}{\prod_{k\in S} a_k}.$$

The final summation equals the numerator of the continued fraction $[a_1, a_2, \dots, a_n]$, which is the denominator q of $[0, a_1, \dots, a_n]$. Thus

$$\kappa_n = (-1)^{\binom{n}{2}} i^n q.$$

It remains to evaluate ω_n at $A = \frac{1+i}{\sqrt{2}}$. Using $A^2 = i$ and $-A^2 - A^{-2} = 0$, a direct calculation gives

$$\omega_n(a_1,\dots,a_n) = \left(\frac{1+i}{\sqrt{2}}\right)^{\sum_{k=1}^n (-1)^k a_k}.$$

Therefore

$$K_n = (-1)^{\binom{n}{2}} i^n \left(\frac{1+i}{\sqrt{2}}\right)^{\sum_{k=1}^n (-1)^k a_k} q.$$

Finally, since $A^4 = -1$, we have $-A^{-3} = A$, hence

$$(-A^{-3})^{w(D)} = A^{w(D)} = \left(\frac{1+i}{\sqrt{2}}\right)^{w(D)}.$$

Multiplying by this factor yields the stated formula for $V_L(-1)$. □

Remark 7.2. According to Murasugi [25, Proposition 11.2.7], every link L satisfies

$$V_L(-1) = (-1)^{\alpha(L)-1} \Delta_L(-1),$$

where $\alpha(L)$ is the number of components and Δ_L is the Alexander polynomial. By [25, Proposition 6.1.5], for a knot K, $|\Delta_K(-1)|$ equals the determinant of K. Cromwell [6, Theorem 8.7.7] shows that a rational link encoded by p/q has determinant $|q|$ (with the convention that Cromwell's numerator and denominator are interchanged relative to ours). Theorem 7.1 is therefore consistent with these classical results and determines the precise phase factor relating $V_L(-1)$ and q.

Henceforth we assume that $d \neq 0$. In this case, Corollary 6.1 may be restated as

$$A^{2\varepsilon}[m]_{-A^{4\varepsilon}} = \frac{(-1)^m A^{4m\varepsilon} - 1}{d}. \tag{7.1}$$

Using (7.1), Corollary 6.2 takes the following form.

Corollary 7.3. *Assume $d \neq 0$. Then the Kauffman bracket*

$$K_n = K([0, a_1, a_2, \dots, a_n])$$

is the total weight of all tilings of a unit width path of length n by 1×1 and 2×1 tiles, with weights assigned as follows:

(1) A 1×1 tile in position 1 has weight

$$\frac{(-A)^{3a_1} - A^{-a_1}}{d} + A^{-a_1} d;$$

(2) A 1×1 tile in position $k > 1$ has weight

$$\frac{(-A)^{3(-1)^{k-1}a_k} - A^{(-1)^k a_k}}{d};$$

(3) A 2×1 tile covering $(k, k+1)$ has weight

$$(-1)^{a_k} A^{(-1)^{k+1}(3a_k + a_{k+1})}.$$

Indeed, for $k \geq 1$, equation (7.1) implies

$$A^{(-1)^{k-1}(2\varepsilon_k - a_k)}[|a_k|]_{-A^{(-1)^{k-1}4\varepsilon_k}} = A^{(-1)^k a_k} \frac{(-1)^{a_k} A^{4(-1)^{k-1}a_k} - 1}{d},$$

which is exactly the replacement used for the 1×1 tile weights.

Using (7.1), the parameters $\gamma_k(a_1, \dots, a_k)$ may be simplified as

$$\begin{aligned} \gamma_k(a_1, \dots, a_k) &= A^{2(-1)^{k-1}\varepsilon_k}[|a_k|]_{-A^{4(-1)^{k-1}\varepsilon_k}} (-A^{4(-1)^{k-1}})^{a_1 + \dots + a_{k-1}} \\ &= \frac{(-1)^{a_k} A^{4(-1)^{k-1}a_k} - 1}{d} (-A^{4(-1)^{k-1}})^{a_1 + \dots + a_{k-1}}. \end{aligned} \tag{7.2}$$

Keeping in mind $t = A^{-4}$, equation (7.2) may be rewritten as

$$\gamma_k(a_1, \dots, a_k) = \frac{(-t)^{(-1)^k(a_1 + \dots + a_k)} - (-t)^{(-1)^k(a_1 + \dots + a_{k-1})}}{d}. \tag{7.3}$$

This yields a more compact expression for κ_n in Corollary 6.5.

Using (7.1), the substitution rule (6.26) may be rewritten as

$$M_k \longmapsto \begin{bmatrix} \dfrac{1-(-A^4)^{(-1)^k a_k}}{d} & (-A^4)^{(-1)^k a_k} \\ 1 & 0 \end{bmatrix}.$$

Substituting $t = A^{-4}$ (so that $-A^4 = -t^{-1}$) and simplifying gives the following consequence of Theorem 6.7.

Corollary 7.4. *Assume $d \neq 0$. Then*

$$K_n = (-A)^{3\sum_{k=1}^{n}(-1)^{k-1}a_k}\,\rho_n,$$

where

$$\rho_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{1-(-t)^{(-1)^{n-1}a_n}}{d} & (-t)^{(-1)^{n-1}a_n} \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} \dfrac{1-(-t)^{a_1}}{d} & (-t)^{a_1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ d \end{bmatrix}.$$

Exploring substitutions of roots of unity into the Jones polynomial is a subject of intensive ongoing research, see the third remark on page 383 in [26]. In this exploration the following result may be helpful.

Theorem 7.5. *Let $\delta > 1$ be an integer such that $\delta \mid (|a_1|-1)$ and $\delta \mid a_i$ for $i = 2, \ldots, n$. Let ρ be a δth root of unity. Then the Jones polynomial of the rational link encoded by $[0, a_1, a_2, \ldots, a_n]$, evaluated at $t = -\rho$, is a (4δ)th root of unity.*

Proof. If $\delta \mid m$, then for any δth root of unity ρ we have

$$[m]_q = \frac{q^m - 1}{q - 1} = 0 \quad \text{at } q = \rho,$$

and likewise at $q = \rho^{-1}$.

Now substitute $(t =)A^{-4} = -\rho$. Since $\delta \mid a_k$ for $k \geq 2$, we have

$$[|a_k|]_{-A^{\pm 4}} = 0 \quad \text{for } k \geq 2.$$

On the other hand, $\delta \mid (|a_1| - 1)$ implies

$$[|a_1|]_{-A^{\pm 4}} = 1.$$

Under this substitution, in the matrix product of Corollary 6.5 we therefore obtain

$$\gamma_k = 0 \quad \text{for } k \geq 2, \qquad \gamma_1 = A^2.$$

Hence

$$\begin{bmatrix} \kappa_n \\ \kappa_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} A^2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ d \end{bmatrix}.$$

Since

$$\begin{bmatrix} A^2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ d \end{bmatrix} = \begin{bmatrix} A^2 + d \\ 1 \end{bmatrix} = \begin{bmatrix} -A^{-2} \\ 1 \end{bmatrix}$$

(using $d = -A^2 - A^{-2}$), we obtain

$$\begin{bmatrix} \kappa_n \\ \kappa_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} -A^{-2} \\ 1 \end{bmatrix}.$$

A straightforward computation now gives

$$\kappa_n = \begin{cases} -A^{-2}, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Equivalently,

$$\kappa_n = (-1)^n A^{(-1)^n - 1}.$$

The remaining factors relating κ_n to $V_L(t)$, namely $(-A^{-3})^{w(D)}$ and ω_n, are products of powers of A. Therefore $V_L(-\rho)$ is a power of $\pm A$. Since $t = A^{-4} = -\rho$ and $\rho^\delta = 1$, it follows that $A^{4\delta} = 1$. Hence $V_L(-\rho)$ is a (4δ)th root of unity. □

8. A combinatorial formula for the Jones polynomial of a rational link

In this section we derive a variant of the Jones polynomial formula of Lawrence and Rosenstein [21, Theorem 4.2] and show that it admits a natural combinatorial reformulation.

Following the notation of Lawrence and Rosenstein, we introduce

$$u = (-t^{-1})^{1/2} = iA^2, \tag{8.1}$$

and define

$$m_k = (-1)^{k-1} a_k. \tag{8.2}$$

By (1.1), the sign of m_k,

$$\operatorname{sign}(m_k) = (-1)^{k-1} \operatorname{sign}(a_k),$$

coincides with the twisting sign of the crossings in the twist box associated with a_k.

Definition 8.1. *Let D be a rational link diagram encoded by*

$$[0, a_1, a_2, \ldots, a_n] = [0, m_1, -m_2, \ldots, (-1)^{n-1} m_n].$$

The Lawrence–Rosenstein polynomial *of D is defined by*

$$R_D(t) = u^{\sum_{k=1}^n m_k} \cdot \frac{d^n\, V_{L(D)}(t)}{(-A)^{-3w(D) + 3\sum_{k=1}^n m_k}}.$$

The polynomial $R_D(t)$ is a Laurent polynomial in $\sqrt{t}$, just like the Jones polynomial, as shown by the following result.

Proposition 8.2. *Let D_n be a rational link diagram encoded by*

$$[0, a_1, \ldots, a_n] = [0, m_1, -m_2, \ldots, (-1)^{n-1} m_n].$$

Then

$$R_D(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u^{m_n} - u^{-m_n} & u^{-m_n} d \\ u^{m_n} d & 0 \end{bmatrix} \cdots \begin{bmatrix} u^{m_1} - u^{-m_1} & u^{-m_1} d \\ u^{m_1} d & 0 \end{bmatrix} \begin{bmatrix} 1 \\ d \end{bmatrix}.$$

Proof. Define

$$\widetilde{R}_D(t) = u^{-\sum_{k=1}^n m_k} R_D(t).$$

We first show that

$$\widetilde{R}_D(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 - (-t)^{m_n} & (-t)^{m_n} d \\ d & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 - (-t)^{m_1} & (-t)^{m_1} d \\ d & 0 \end{bmatrix} \begin{bmatrix} 1 \\ d \end{bmatrix}.$$

When $d \neq 0$, this follows directly from Corollary 7.4, since $\widetilde{R}_D(t) = d^n \rho_n$. The general case then follows due to the fact that both sides of the above are Laurent polynomials of A.

Now multiply both sides by $u^{\sum_{k=1}^n m_k}$. Since

$$(-t)^{m_k} = u^{-2m_k},$$

each matrix factor

$$\begin{bmatrix} 1-u^{-2m_k} & u^{-2m_k}d \\ d & 0 \end{bmatrix}$$

becomes

$$u^{m_k} \begin{bmatrix} 1-u^{-2m_k} & u^{-2m_k}d \\ d & 0 \end{bmatrix} = \begin{bmatrix} u^{m_k}-u^{-m_k} & u^{-m_k}d \\ u^{m_k}d & 0 \end{bmatrix}.$$

Substituting these into the product yields the stated formula. □

Introducing the notation

$$L = \begin{bmatrix} 1 & 0 \\ d & 0 \end{bmatrix}$$

and defining the ⋆–operation by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^\star = \begin{bmatrix} a_{11} & -a_{21} \\ -a_{12} & a_{22} \end{bmatrix},$$

the formula of Proposition 8.2 may be rewritten as

$$R_D(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} (u^{m_n}L - u^{-m_n}L^\star) \cdots (u^{m_1}L - u^{-m_1}L^\star) L \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{8.3}$$

Lemma 8.3. *Let*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the matrices E, L and $L^\star$ satisfy

$$E^2 = E, \quad L^2 = L, \quad (L^\star)^2 = L^\star, \quad L^\star L = (1-d^2)\,E$$

and

$$LE = L, \quad EL = E, \quad L^\star E = E, \quad EL^\star = L^\star.$$

The verification is straightforward.

The polynomial $R_D(t)$ admits a combinatorial expansion that is most naturally expressed in terms of interval decompositions of subsets of $\{1, \ldots, n\}$.

Definition 8.4. *Let S be a set of positive integers. An* interval of S *is a maximal subset of S of the form*

$$\{k, k+1, \ldots, \ell\}$$

for some integers $k \leq \ell$. Equivalently, it is a maximal block of consecutive integers contained in S. We denote the set of intervals of S by $I(S)$.

For example, if

$$S = \{2, 3, 5, 6, 7, 9\},$$

then

$$I(S) = \{[2,3],\ [5,7],\ [9,9]\}.$$

Theorem 8.5. *We have*

$$R_D(t) = \sum_{S\subseteq\{1,\ldots,n\}} \left(\prod_{k\notin S} u^{m_k}\right)\left(\prod_{k\in S}(-u^{-m_k})\right)(u^2+u^{-2}-1)^{|I(S)|}.$$

Proof. Expanding the matrix product in (8.3) produces 2^n terms. For each subset $S\subseteq\{1,\ldots,n\}$, select the factor $-u^{-m_k}L^\star$ whenever $k\in S$, and $u^{m_k}L$ whenever $k\notin S$.

For example, if $n=6$ and $S=\{2,4,5,6\}$ (so $|I(S)|=2$), the corresponding contribution is

$$\begin{bmatrix}1 & 0\end{bmatrix} u^{m_1}L(-u^{-m_2}L^\star)u^{m_3}L(-u^{-m_4}L^\star)(-u^{-m_5}L^\star)(-u^{-m_6}L^\star)L\begin{bmatrix}1\\0\end{bmatrix}.$$

Factoring out the scalar terms gives

$$u^{m_1-m_2+m_3-m_4-m_5-m_6}(-1)^{|S|}\begin{bmatrix}1 & 0\end{bmatrix} LL^\star L(L^\star)^3L\begin{bmatrix}1\\0\end{bmatrix}.$$

By Lemma 8.3,

$$\begin{bmatrix}1 & 0\end{bmatrix} LL^\star L(L^\star)^3L\begin{bmatrix}1\\0\end{bmatrix} = (1-d^2)^2\begin{bmatrix}1 & 0\end{bmatrix} L\begin{bmatrix}1\\0\end{bmatrix} = (1-d^2)^2.$$

In general, each maximal block of consecutive indices in S produces one factor $L^\star L$. Hence for each S the corresponding matrix product reduces to

$$(1-d^2)^{|I(S)|}.$$

Therefore the contribution of S equals

$$\left(\prod_{k\notin S} u^{m_k}\right)\left(\prod_{k\in S}(-u^{-m_k})\right)(1-d^2)^{|I(S)|}.$$

Finally, since

$$1-d^2 = 1-(A^{-2}+A^2)^2 = 1-(iu^{-1}-iu)^2 = u^2+u^{-2}-1,$$

the stated formula follows. □

It is worth noting that

$$u^2+u^{-2}-1 = -\frac{1+t+t^2}{t},$$

so Theorem 8.5 may be restated for $\widetilde{R}_D(t) = u^{-\sum_{k=1}^n m_k}R_D(t)$ as

$$\widetilde{R}_D(t) = \sum_{S\subseteq\{1,\ldots,n\}}(-1)^{|S|+|I(S)|}\prod_{k\in S}(-t)^{m_k}\left(\frac{1+t+t^2}{t}\right)^{|I(S)|}.$$

Example 8.6. For $A=\exp(\pi i/3)$, $t=A^{-4}=\exp(-4\pi i/3)=\exp(2\pi i/3)$ and $1+t+t^2=0$. Hence the only nonzero contribution in Theorem 8.5 comes from $S=\emptyset$. Therefore

$$\widetilde{R}_D(\exp(2\pi i/3)) = 1.$$

Since $(-A)^3 = -\exp(\pi i) = 1$ and $d=-A^2-A^{-2}=1$, it follows that

$$V_D(\exp(2\pi i/3)) = 1.$$

9. The Tutte polynomial of the Tait graph of an alternating rational link diagram

All results of this paper are directly applicable in the special case when the rational link is represented by an alternating diagram. For such diagrams there is, however, another way to compute the Kauffman bracket: one may compute the Tutte polynomial of the Tait graph and then apply Corollary 2.2. In this section we outline this approach and obtain a generalization of [27, Theorem 3.4].

It should be noted that to apply Corollary 2.2 we only need the evaluation of $T(G;X,Y)$ at $Y = X^{-1}$. Under this specialization, many of the formulas below simplify further.

Without loss of generality we may assume that the rational link diagram in standard form is represented by the continued fraction $[0, a_1, a_2, \dots, a_n]$, where $a_1, a_2, \dots, a_n$ are all positive. In this case all edges of the Tait graph have positive sign. The ordinary Tutte polynomial may then be obtained from the signed Tutte polynomial by setting

$$x_+ = 1, \qquad y_+ = 1, \qquad X_+ = X, \qquad Y_+ = Y.$$

Consequently, equations (1.6) and (1.7) become

$$u = X - 1 \qquad \text{and} \qquad v = Y - 1. \tag{9.1}$$

We may still apply Theorems 1.6 and 2.6 to obtain the Tutte polynomial of the Tait graph from the colored Tutte polynomial T_n of the core graph G_n by using suitable specializations of the substitution rules (3.1)–(3.8).

For an even index $2j$ we have $\widehat{Q}_{2j} = \widehat{P}_{a_{2j}}$ and the following substitution rules:

$$X_{2j} \mapsto T(\widehat{Q}_{2j} - e) \mapsto X^{a_{2j}}, \tag{9.2}$$

$$Y_{2j} \mapsto T(\widehat{Q}_{2j}/e) \mapsto [a_{2j}]_X + Y - 1, \tag{9.3}$$

$$x_{2j} \mapsto T_L(\widehat{Q}_{2j}, e) \mapsto 1, \tag{9.4}$$

$$y_{2j} \mapsto T_C(\widehat{Q}_{2j}, e) \mapsto [a_{2j}]_X. \tag{9.5}$$

Similarly, for an odd index $2j+1$ we have $\widehat{Q}_{2j+1} = \widehat{P}^*_{a_{2j+1}}$ and

$$X_{2j+1} \mapsto T(\widehat{Q}_{2j+1} - e) \mapsto [a_{2j+1}]_Y + X - 1, \tag{9.6}$$

$$Y_{2j+1} \mapsto T(\widehat{Q}_{2j+1}/e) \mapsto Y^{a_{2j+1}}, \tag{9.7}$$

$$x_{2j+1} \mapsto T_L(\widehat{Q}_{2j+1}, e) \mapsto [a_{2j+1}]_Y, \tag{9.8}$$

$$y_{2j+1} \mapsto T_C(\widehat{Q}_{2j+1}, e) \mapsto 1. \tag{9.9}$$

These substitutions determine the parameters u_k and v_k:

$$u_k \mapsto \begin{cases} X^{a_{k-1}}, & \text{if } k \text{ is odd,} \\ Y^{a_{k-1}}, & \text{if } k \text{ is even,} \end{cases} \tag{9.10}$$

$$v_k \mapsto \begin{cases} [a_k]_Y, & \text{if } k \text{ is odd,} \\ [a_k]_X, & \text{if } k \text{ is even.} \end{cases} \tag{9.11}$$

In particular, we set

$$u_1 = y_1 \mapsto 1 = X^0, \qquad v_1 = x_1 \mapsto [a_1]_Y.$$

Substituting these rules into (4.8) yields the following formula.

Theorem 9.1. *Let G be the Tait graph of a standard rational link diagram encoded by the continued fraction*

$$[0, a_1, a_2, \ldots, a_n],$$

where $a_1, a_2, \ldots, a_n$ are positive. Then the Tutte polynomial $T(G; X, Y)$ is given by

$$T(G; X, Y) = \begin{bmatrix} 1 & 0 \end{bmatrix} A_n A_{n-1} \cdots A_1 \begin{bmatrix} 1 \\ X - 1 \end{bmatrix}.$$

Here $a_0 = 0$, and

$$A_k = \begin{cases} \begin{bmatrix} [a_k]_Y & X^{a_{k-1}} \\ 1 & 0 \end{bmatrix}, & \text{if } k \text{ is odd,} \\ \begin{bmatrix} [a_k]_X & Y^{a_{k-1}} \\ 1 & 0 \end{bmatrix}, & \text{if } k \text{ is even.} \end{cases}$$

Remark 9.2. Substituting $Y = X^{-1}$ in Theorem 9.1 yields

$$A_k \mapsto \begin{bmatrix} [a_k]_{X^{(-1)^k}} & X^{(-1)^{k-1} a_{k-1}} \\ 1 & 0 \end{bmatrix} \qquad \text{for all } k.$$

It should be noted that even for odd n, the formula stated in Theorem 9.1 looks very different from the one stated in [27, Theorem 3.4], which may be rewritten in our notation as follows.

Theorem 9.3 (Qazaqzeh–Yasein–Abu-Qamar)**.** *Let G be the Tait graph of a standard rational link diagram encoded by the continued fraction*

$$[0, a_1, a_2, \ldots, a_n],$$

where $a_1, a_2, \ldots, a_n$ are positive and $n = 2k+1$ is odd. Then the Tutte polynomial $T(G; X, Y)$ is given by

$$T(G; X, Y) = \sum_{\substack{0 \le l \le k \\ 1 \le i_1 < i_2 < \cdots < i_l \le k}} \prod_{j=1}^{l} [a_{2i_j}]_X \prod_{j=0}^{l} \big([c_j]_Y + X - 1\big),$$

where we set $i_0 = 0$, $i_{l+1} = k+1$, and

$$c_j = a_{2i_j+1} + a_{2i_j+3} + \cdots + a_{2i_{j+1}-1}, \qquad j = 0, 1, \ldots, l.$$

Our formula expresses $T(G; X, Y)$ directly in terms of the partial denominators a_j, without introducing the auxiliary quantities c_j, whose definition depends on the choice of a subset $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k\}$. Since the Tutte polynomial of a graph is uniquely determined, the two expressions must agree when n is odd, as demonstrated by the following simple example.

Example 9.4. Consider the case $n = 3$ with $a_3 = 1$. By Theorem 9.1 the Tutte polynomial of the Tait graph is

$$\begin{aligned} T(G;X,Y) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & X^{a_2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} [a_2]_X & Y^{a_1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} [a_1]_Y & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ X-1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & X^{a_2} \end{bmatrix} \begin{bmatrix} [a_2]_X & Y^{a_1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} [a_1]_Y + X - 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} [a_2]_X + X^{a_2} & Y^{a_1} \end{bmatrix} \begin{bmatrix} [a_1]_Y + X - 1 \\ 1 \end{bmatrix} \\ &= \big([a_2]_X + X^{a_2}\big)\big([a_1]_Y + X - 1\big) + Y^{a_1} \\ &= [a_2+1]_X\big([a_1]_Y + X - 1\big) + Y^{a_1}. \end{aligned}$$

Translating the notation of [27] into ours, [27, Corollary 3.5] gives

$$\begin{aligned} T(G;X,Y) &= X[a_2]_X\big([a_1]_Y + X - 1\big) + [a_1+1]_Y + X - 1 \\ &= \big([a_2+1]_X - 1\big)\big([a_1]_Y + X - 1\big) + [a_1+1]_Y + X - 1 \\ &= [a_2+1]_X\big([a_1]_Y + X - 1\big) - \big([a_1]_Y + X - 1\big) + [a_1+1]_Y + X - 1 \\ &= [a_2+1]_X\big([a_1]_Y + X - 1\big) + Y^{a_1}, \end{aligned}$$

which coincides with the formula obtained above.

Using the substitution rules (9.10) and (9.11), Theorem 5.3 has the following consequence.

Corollary 9.5. *Let G be the Tait graph of a standard alternating rational link diagram encoded by the continued fraction $[0, a_1, a_2, \ldots, a_n]$, where $a_1, a_2, \ldots, a_n$ are positive. Define the Laurent polynomial $P(G;X,Y)$ by*

$$P(G;X,Y) = \begin{cases} X^{-(a_{n-1}+a_{n-3}+\cdots)}\, T(G;X,Y), & \text{if } n \text{ is odd,} \\ Y^{-(a_{n-1}+a_{n-3}+\cdots)}\, T(G;X,Y), & \text{if } n \text{ is even.} \end{cases}$$

Then

$$P(G;X,Y) = \begin{bmatrix} 1 & 0 \end{bmatrix} B_n B_{n-1} \cdots B_1 \begin{bmatrix} 1 \\ X-1 \end{bmatrix}.$$

Here the matrix B_k is given by

$$B_k = \begin{cases} \begin{bmatrix} \dfrac{[a_k]_Y\, Y^{a_{k-2}+a_{k-4}+\cdots}}{X^{a_{k-1}+a_{k-3}+\cdots}} & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } k \text{ is odd,} \\ \begin{bmatrix} \dfrac{[a_k]_X\, X^{a_{k-2}+a_{k-4}+\cdots}}{Y^{a_{k-1}+a_{k-3}+\cdots}} & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } k \text{ is even.} \end{cases}$$

Remark 9.6. Substituting $Y = X^{-1}$ in Corollary 9.5 yields

$$P(G;X,X^{-1}) = X^{(-1)^n(a_{n-1}+a_{n-3}+\cdots)}T(G;X,X^{-1}) \quad \text{and}$$

$$B_k \mapsto \begin{bmatrix} [a_k]_{X^{(-1)^k}} \cdot X^{(-1)^k(a_1+a_2+\cdots+a_{k-1})} & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} [a_1+\cdots+a_k]_{X^{(-1)^k}} - [a_1+\cdots+a_{k-1}]_{X^{(-1)^k}} & 1 \\ 1 & 0 \end{bmatrix}$$

Using the substitution rules (9.10) and (9.11), Theorem 5.5 has the following consequence.

Corollary 9.7. *Let G be the Tait graph of a standard alternating rational link diagram encoded by the continued fraction $[0, a_1, a_2, \ldots, a_n]$, where $a_1, a_2, \ldots, a_n$ are positive. Then the Laurent polynomial*

$$R(G; X, Y) = \frac{T(G; X, Y)}{\prod_{1\le j\le \lfloor n/2\rfloor} X^{a_{2j}} \prod_{1\le j\le \lfloor (n+1)/2\rfloor} Y^{a_{2j-1}}}$$

is given by

$$R(G; X, Y) = \begin{bmatrix} 1 & 0 \end{bmatrix} C_n C_{n-1} \cdots C_1 \begin{bmatrix} 1 \\ X-1 \end{bmatrix}.$$

Here

$$C_k = \begin{cases} \begin{bmatrix} \frac{[a_k]_X}{X^{a_k}} & \frac{1}{X^{a_k}} \\ 1 & 0 \end{bmatrix}, & \text{if } k \text{ is even;} \\ \begin{bmatrix} \frac{[a_k]_Y}{Y^{a_k}} & \frac{1}{Y^{a_k}} \\ 1 & 0 \end{bmatrix}, & \text{if } k \text{ is odd.} \end{cases}$$

Remark 9.8. Substituting $Y = X^{-1}$ in Corollary 9.7 yields

$$R(G; X, X^{-1}) = T(G; X, X^{-1}) \cdot \prod_{1\le k\le n} X^{(-1)^{k-1}a_k} \quad \text{and}$$

$$C_k \mapsto \begin{bmatrix} [a_k]_{X^{(-1)^k}} X^{(-1)^{k-1}a_k} & X^{(-1)^{k-1}a_k} \\ 1 & 0 \end{bmatrix}.$$

10. Computing the writhe of a rational link diagram

Inspired by a finite automaton introduced in [8] and by the illustrations in [11], we introduce two finite automata that can be used to compute the writhe of any oriented rational link diagram in standard form, together with a third automaton that computes the number of connected components. These automata allow us to simplify the proofs of the results of Qazaqzeh, Yasein, and Abu-Qamar on the writhe of certain rational link diagrams [27, Propositions 4.2 and 4.3]. Moreover, our approach also applies to rational link diagrams with an even number of twist boxes, whereas in [27] rational link diagrams are required to be in *canonical form*, meaning that they must have an odd number of twist boxes. This is the point where our approach is genuinely more general. Extending the method beyond alternating diagrams is in fact not essential, because of Lemma 10.1 below. Given an oriented link diagram $\mathcal{D}$, we call the unoriented diagram obtained by simply "forgetting" the orientation of its strands the *underlying unoriented link diagram.*

Lemma 10.1. *Let $\mathcal{D}$ be an oriented rational link diagram whose underlying unoriented diagram is in standard form and encoded by the continued fraction $[0, a_1, a_2, \ldots, a_n]$. Let $\mathcal{D}'$ be the alternating oriented link diagram obtained from $\mathcal{D}$ by changing all overcrossings to undercrossings in the twist boxes associated with negative partial denominators a_i. The underlying unoriented diagram of $\mathcal{D}'$ is then encoded by $[0, |a_1|, |a_2|, \ldots, |a_n|]$.*

If the writhe of $\mathcal{D}'$ *is*

$$\sum_{i=1}^{n} s_i\,|a_i|, \qquad s_i \in \{1,-1\},$$

then the writhe of $\mathcal{D}$ *is*

$$\sum_{i=1}^{n} s_i\,a_i.$$

Proof. The crossings in the twist box represented by a_i contribute $s_i a_i$ to the writhe for some $s_i \in \{1,-1\}$. The same also holds for $\mathcal{D}'$. Thus it suffices to show that the same coefficients s_i occur in both expressions.

Consider the operation of changing all overcrossings to undercrossings in a twist box corresponding to some $a_i < 0$. This move changes the link represented by the diagram, but it does not change the number of components or the orientations of the strands. Consequently the sign of every crossing in that twist box changes to its opposite. At the same time we replace $a_i < 0$ with $|a_i| > 0$. These two changes cancel each other in the writhe computation. If the original twist box contributes $s_i a_i$ to the writhe of $\mathcal{D}$, then the modified twist box contributes $s_i|a_i|$ to the writhe of $\mathcal{D}'$. □

Consider an unoriented rational link diagram in standard form encoded by the continued fraction $[0, a_1, a_2, \ldots, a_n]$. We number its strands 1, 2, 3, and 4 from top to bottom. Next we turn our unoriented link diagram into an oriented link diagram. Without loss of generality we may assume that the lowest strand is oriented from right to left, as shown in Figure 1. This choice also determines the orientation at the left end of the second lowest strand. To construct our automaton we will use the sample links shown in Figure 6 as a guide.

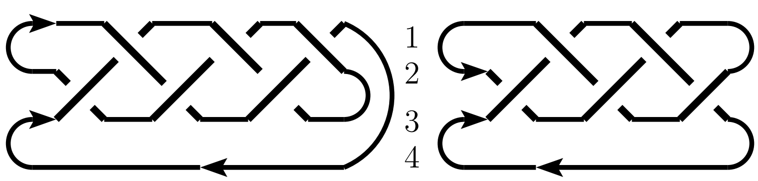

FIGURE 6. Two sample links

At the left end of each link, strand 1 is connected with strand 2 and strand 3 is connected with strand 4. Taking the orientation of the strands into account, we may write these connections as ordered pairs: if, following the orientation, we go from strand i to strand j, we write the ordered pair (i, j). Initially we may assume that the orientation of the bottom two strands is $(4, 3)$ and we have two possible initial states: $(2, 1)(4, 3)^*$ (as on the left-hand side of Figure 6) and $(1, 2)(4, 3)^*$ (as on the right-hand side of Figure 6).

We use a star to indicate that currently we have an even number of twist boxes (at the beginning this number is zero). If we want to complete the drawing of our link at this point, we must use the closing shown in the second row of Figure 1. We omit the star for the states reached after parsing an odd number of twist boxes. If we stop in any of those states, we must use the closing shown in the first row of Figure 1. We refer to the starred states as *even states* and to the unstarred states as *odd states*, and to the presence or absence of the star as the *parity* of a state.

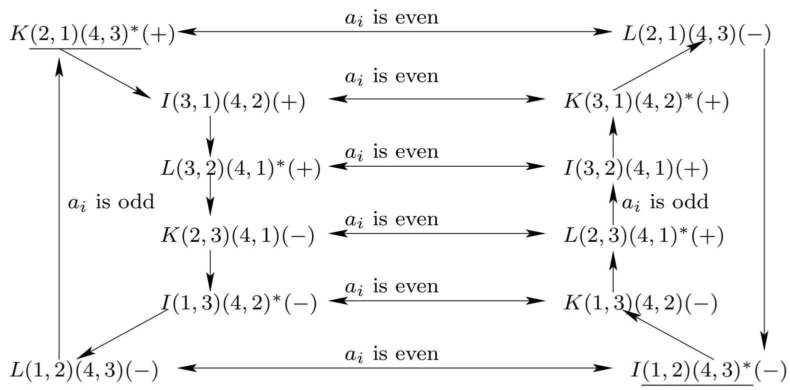

FIGURE 7. An automaton that helps compute the writhe and the number of connected components of a rational link diagram

These two possible initial states are underlined in Figure 7. A common feature of the links shown in Figure 6 is that there is exactly one crossing (an odd number) in each twist box. Parsing these sample links we may easily describe the states reached from the initial states by using the transition rule corresponding to the case when a_i is odd. Parsing the sample link on the left-hand side yields the oriented cycle

$$(2,1)(4,3)^* \to (3,1)(4,2) \to (3,2)(4,1)^* \to (2,3)(4,1)$$
$$\to (1,3)(4,2)^* \to (1,2)(4,3) \to (2,1)(4,3)^*.$$

We arrive again at the original state, and to close the link we must connect strand 1 with strand 4 and strand 2 with strand 3. The result is a knot. Note that we only need to know the final state to determine the number of connected components: if we stop at an even state we always pair the strands $\{1,4\}$ and $\{2,3\}$, whereas if we stop at an odd state we must pair the strands $\{1,2\}$ and $\{3,4\}$.

It is easy to check that the state $(2,3)(4,1)$ also represents a knot, the states $(3,2)(4,1)^*$ and $(1,2)(4,3)$ represent two-component links, whereas $(3,1)(4,2)$ and $(1,3)(4,2)^*$ are not valid stopping states: closing the strands in these cases yields a conflict of orientations. The letters K, L, and I in Figure 7 indicate whether stopping at the corresponding state yields a knot, a (two-component) link, or an invalid result.

Similarly to the above explanation, parsing the sample link on the right-hand side yields the oriented path

$$(1,2)(4,3)^* \to (1,3)(4,2) \to (2,3)(4,1)^* \to (3,2)(4,1)$$
$$\to (3,1)(4,2)^* \to (2,1)(4,3),$$

and our sample link has two components. The completion of Figure 7 is easy using the following two observations:

(1) Reading a twist box containing an even number of crossings takes the state $(i,j)(k,l)$ into the state $(i,j)(k,l)^*$ and vice versa.
(2) Applying a central symmetry to Figure 7 corresponds to reversing the ordered pair not containing 4: $(1,2)(4,3)^*$ goes into $(2,1)(4,3)^*$, and $(2,3)(4,1)$ goes into $(3,2)(4,1)$. This involution matches an L-state to another L-state and an I-state to a K-state.

The main results of this section state that the labelings in Figure 7 are correct.

Theorem 10.2. *Consider an unoriented rational link diagram encoded by* $[0, a_1, a_2, \ldots, a_n]$*. The left ends of this diagram may be oriented as indicated in a starting state of the automaton shown in Figure 7 if and only if reading the vector* $(a_1, a_2, \ldots, a_n)$ *results in a final state that is not marked* I*. The unoriented link diagram represents a knot if and only if the corresponding final state in Figure 7 is marked* K *or* I*.*

Proof. We prove this statement by presenting a "quotient" of the automaton shown in Fig. 7. This quotient, shown in Fig. 8, is obtained by identifying the state $(a, b)(4, c)^*$ with the state $(b, a)(4, c)^*$ and the state $(a, b)(4, c)$ with the state $(b, a)(4, c)$.

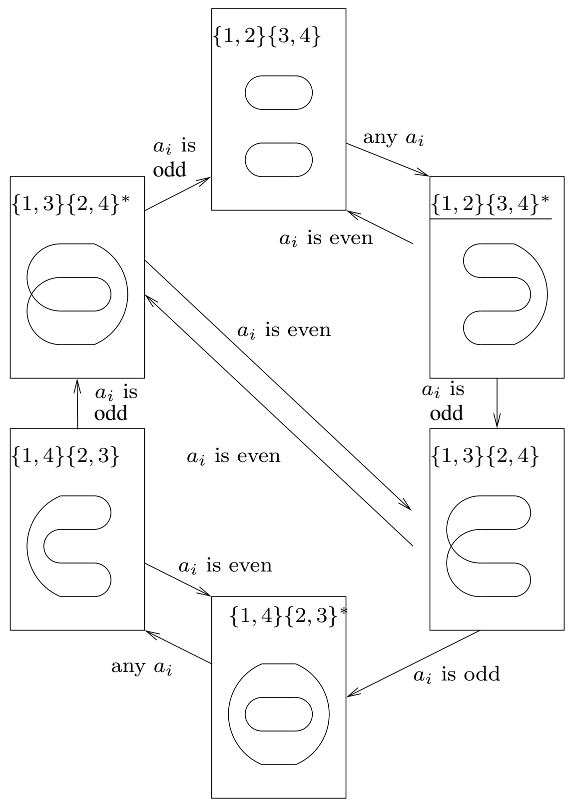

FIGURE 8. A simplified automaton that helps compute the number of connected components of a rational link diagram

This identification corresponds to forgetting the orientation of the strands, hence we label the states with unordered pairs of numbers. Regardless of which state we are in that corresponds to the same pairing of strands on the left, the number of connected components depends only on the pairing of the strands on the right, which in turn depends only on the parity of the number of twist boxes parsed so far. It depends only on the parity of a_i which pairing of the four strands we obtain next. The visual left and right pairings of the strands at each state are shown in Fig. 8. It is immediate to see which states represent (unoriented) knots and which represent two-component links. Each state of Fig. 8 corresponds to a pair of states in Fig. 7: for two-component links both states represent valid orientations, whereas for knots one state corresponds to a valid oriented knot diagram and the other to an invalid final state. □

Theorem 10.3. *To compute the writhe of the corresponding oriented link diagram, multiply each a_i by the sign associated with the state reached immediately before reading a_i.*

Proof. By Lemma 10.1, it suffices to prove the statement for the special case when all a_i are positive. In this case the diagram represents an alternating link. By inspecting the sample oriented links shown in Figure 6, it is easy to see that the signs of the crossings in the twist boxes are exactly as indicated in the states of our automaton given in Figure 7.

Another way to verify the correctness of the signs is to consider the simplified automaton shown in Figure 9.

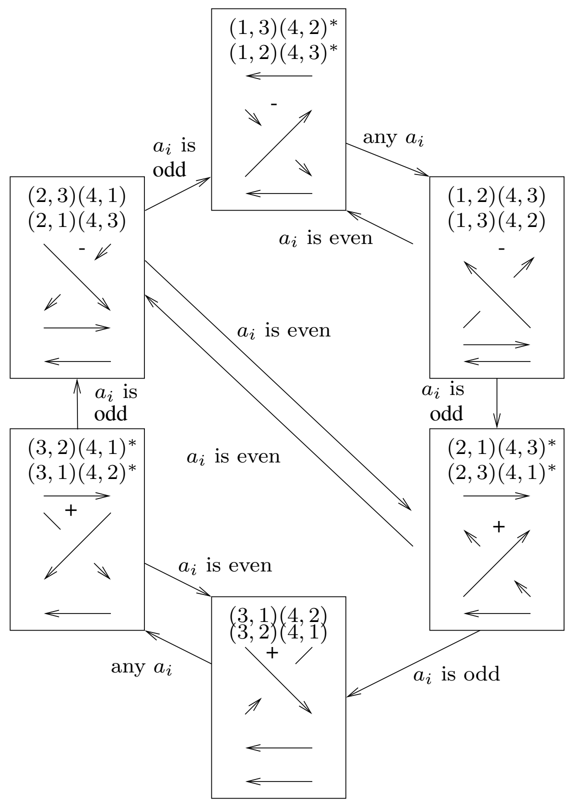

Figure 9. A simplified automaton that helps compute the writhe of a rational link diagram

This automaton is obtained by merging pairs of states of the same parity that represent the same orientation of the strands immediately before the next twist box. For example, the states $(1,3)(4,2)^*$ and $(1,2)(4,3)^*$ are the two even states in which strands 1 and 4 are oriented from right to left and strands 2 and 3 are oriented from left to right. Because these are even states, the crossings in the next twist box involve strands 2 and 3. When all a_i are positive, the crossings appear exactly as shown in the diagram associated with the state. The description of the states $(1,3)(4,2)$ and $(1,2)(4,3)$ is completely analogous, the only difference being that the crossings in the next twist box involve strands 1 and 2.

In general the state $(u_1,v_1)(u_2,v_2)$ is paired with the state $(u_1,v_2)(u_2,v_1)$, and the state $(u_1,v_1)(u_2,v_2)^*$ is paired with the state $(u_1,v_2)(u_2,v_1)^*$. If we start from either state of such a pair, after reading the next a_i we arrive at one of two states that are again paired. Thus the automaton shown in Figure 9 is also a homomorphic image of the automaton shown in Figure 7. □

We conclude this section with a relabeled variant of the automaton shown in Figure 7. The pairs of permutations labeling the states in Figure 10 extend the notation introduced in [27]. Our labeling of the strands is the same as theirs. In [27] each canonical oriented link diagram D is represented by a permutation

$$\sigma_D = (23)^{a_1}(12)^{a_2}\cdots(12)^{a_{n-1}}(23)^{a_n}$$

of the set $\{1,2,3\}$, where n is an odd integer and $[0, a_1, \ldots, a_n]$ is the continued fraction representing the underlying unoriented rational link. In [27] the partial denominators a_i are all positive integers, and multiplication of the involutions is performed from left to right.

We extend this permutation labeling to all states of our automaton shown in Figure 7. Allowing negative integers a_i does not make a significant difference, since the involutions (12) and (23) are their own inverses. Multiplication by such an involution occurs when we read an odd a_i, otherwise we copy the same permutation and toggle the presence of a star. If i is odd and a_i is odd, then we follow an arrow leading from an even state to an odd state and multiply by (23). If i is even and a_i is odd, then we follow an arrow from an odd state to an even state and multiply by (12). Note that the parity of i may be determined simply by checking which end of the arrow is marked by a star.

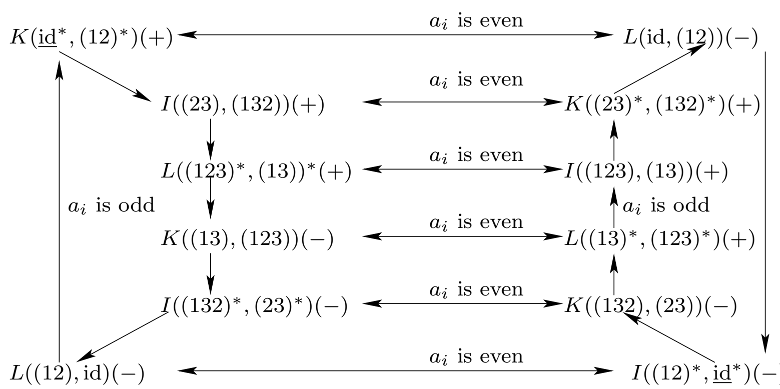

FIGURE 10. The automaton shown in Figure 7 with permutations labeling its states

Each state is labeled by a pair of permutations because there are two valid starting states. The first permutation corresponds to starting at the state labeled with the underlined identity permutation id* in the upper left corner, while the second permutation corresponds to starting at the state labeled with id* in the lower right corner. We preserve the symbols K, I, and L indicating whether the current state is a knot state, a link state, or an invalid stopping state.

Qazaqzeh, Yasein, and Abu-Qamar impose the additional restriction that the diagram must be *canonical*, meaning that it contains an odd number of twist boxes. In terms of our diagram this means that they allow stopping only in an odd state. To obtain a canonical knot we must either start in the upper left corner and stop at the state whose label has first coordinate (13) or (132), or start in the lower right corner and stop at the state whose label has second coordinate (23) or (123). Note that these permutations are pairwise distinct, so the starting state can be reconstructed from the final permutation together with the assumption that the diagram represents a canonical knot.

These possibilities are discussed in [27, Proposition 4.3]. Three cases are distinguished there:

(1) If the label of the final state is (23) or (123), then we must have started in the lower right corner. This state carries a negative sign, and so do both states reachable from it in a single step. The writhe formula therefore begins with $-a_1 - a_2$, to which we add the contributions of the remaining partial denominators.
(2) If the label of the final state is (13) or (132), then we must have started in the upper left corner. This state has a positive sign, but the sign of the next state depends on the parity of a_1. This gives rise to two cases in [27, Proposition 4.3]: the writhe formula begins with $a_1 - a_2$ if a_1 is even and $a_1 + a_2$ if a_1 is odd.

A similar analysis of arriving at the even link states is left to the reader. A proof of [27, Proposition 4.4] may be recovered in this way. Here we only highlight the following observation made in [27]. In all cases a writhe formula of the form

$$w(D) = \sum_{i=1}^{n} s_i a_i$$

can be obtained with each $s_i \in \{1, -1\}$. The determination of s_1 and s_2 varies from case to case, but for the remaining coefficients the following rule holds in all cases:

$$s_i = \begin{cases} -s_{i-2} & \text{if } a_{i-1} \text{ is odd, } i-1 \text{ is even, and } s_{i-1} = -1, \\ -s_{i-2} & \text{if } a_{i-1} \text{ is odd, } i-1 \text{ is odd, and } s_{i-1} = 1, \\ s_{i-2} & \text{otherwise.} \end{cases} \tag{10.1}$$

Equation (10.1) can be verified by inspecting Figure 9. Consider the state reached immediately before reading a_{i-2}. The sign associated with this state is s_{i-2}, while s_i is the sign of the state reached after reading a_{i-2} and a_{i-1}, that is, after moving along two consecutive arrows. For each state there is exactly one pair of arrows that leads to a state of opposite sign in two steps. These possibilities are listed in Table 1.

State before a_{i-2}	a_{i-2}	a_{i-1}	$i-1$	s_{i-1}
$(1,3)(4,2)^*/(1,2)(4,3)^*$	any	odd	even	-1
$(1,2)(4,3)/(1,3)(4,2)$	odd	odd	odd	1
$(2,1)(4,3)^*/(2,3)(4,1)^*$	even	odd	even	-1
$(3,1)(4,2)/(3,2)(4,1)$	any	odd	odd	1
$(3,2)(4,1)^*/(3,1)(4,2)^*$	odd	odd	even	-1
$(2,3)(4,1)/(2,1)(4,3)$	any	odd	odd	-1

TABLE 1. A complete list of cases where $s_i = -s_{i-2}$

The choices are made in the first three columns; the last two columns follow from these. In particular, $i-1$ is odd if and only if the state reached after the first step is an even state, and the last column gives the sign of the state reached after one step along the unique sign-changing directed path of length two. Since the table represents a complete enumeration of possibilities, equation (10.1) follows.

Using our automaton also allows the computation of the writhe formula for rational link diagrams containing an even number of twist boxes. Rather than producing separate formulas case by case, we emphasize that only the determination of s_1 and s_2 requires special consideration; equation (10.1) always determines the remaining coefficients in the writhe formula.

Example 10.4. Consider the unoriented rational link diagram encoded by the continued fraction

$$[0, 3, 7, -5, 3, -4, 6, 5, 4, 6, -8, 9, 3, 2, 4].$$

For the orientation assignment corresponding to the initial state $(2, 1)(4, 3)^*$, the final state determined by the automaton in Figure 7 is $I(1, 2)(4, 3)^*$. Hence this orientation assignment is invalid. The other orientation assignment corresponds to the initial state $(1, 2)(4, 3)^*$ and leads to the final state $K(2, 1)(4, 3)^*$. In this case we obtain $s_1 = s_2 = -1$, $s_3 = s_4 = s_5 = s_7 = 1$, $s_8 = -1$, $s_9 = 1$, $s_{10} = -1$, $s_{11} = s_{12} = s_{13} = s_{14} = 1$. It follows that

$$w(D) = -3 - 7 - 5 + 3 - 4 + 6 + 5 - 4 - 6 + 8 + 9 + 3 + 2 + 4 = 11.$$

Note that this computation does not require drawing the link diagram, and the procedure is readily programmable.

11. Concluding remarks

The Jones polynomial $V(t)$ of a link may be obtained from its HOMFLY polynomial by setting $l = it^{-1}$ and $m = i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})$ according to the notation of [24], or equivalently by setting

$$a = -t^{-1} \quad \text{and} \quad z = t^{-\frac{1}{2}} - t^{\frac{1}{2}}, \tag{11.1}$$

according to the notation in [8]. A formula for the HOMFLY polynomial of an oriented rational link is given by Lickorish and Millett [24]. In their formula the rational link must be represented by a diagram encoded by $[0, a_1, a_2, \ldots, a_n]$ in which all partial denominators a_i are even. A translation of the Lickorish–Millett result to alternating links may be found in [8, Theorem 7.5]. The Jones polynomial formulas obtained by substituting into these HOMFLY polynomial formulas appear somewhat cumbersome, although computationally they are about as "efficient" as the Tait graph approach. Rather than reproducing the explicit computations, we highlight the following observation.

The formulas in [8] contain substitutions into the *Fibonacci polynomials* $F_n(x)$, defined by the initial conditions

$$F_0(x) = 0 \quad \text{and} \quad F_1(x) = 1, \tag{11.2}$$

and the recurrence

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \quad \text{for } n \geq 1. \tag{11.3}$$

These polynomials satisfy the following remarkable identity

$$F_n\left(x - \frac{1}{x}\right) = x^{n-1}\,[n]_{-x^{-2}}. \tag{11.4}$$

A straightforward induction proof is left to the reader. As a consequence of (11.4), the substitutions into the Fibonacci polynomials in [8, Theorem 7.5] take the form of q-analogues of integers, similarly to the formulas obtained using the colored Tait graph approach.

By Theorem 7.1, the Jones polynomial encodes at least the numerator (or the denominator, if we use a variant of our encoding) of the rational number representing the rational link. It is well known that a rational link cannot be recovered from its Jones polynomial; infinitely many counterexamples were constructed by Kanenobu [15], and a more systematic approach was developed by Lawrence and

Rosenstein [21]. The structure of our formulas, which involve q-analogues of the partial denominators, suggests the possible existence of another link invariant of comparable complexity to the Jones polynomial such that the two invariants together uniquely determine a rational link.

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