FRAÏSSÉ LIMITS OF UNCOUNTABLE SIZE

by

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A dissertation submitted to the faculty of
The University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
Mathematics

Charlotte

2026

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ABSTRACT

JACOB D PAGE. Fraïssé Limits of Uncountable Size. (Under the direction of DR. WILLIAM BRIAN)

In 1954, Roland Fraïssé developed a theory involving countable objects that had what is known in this paper as the finite extension property. Countable structures with this property, known as Fraïssé limits, are unique up to isomorphism, as well as possess several other nice model theoretic properties. Fraïssé was able to show that the existence of these structures depended on properties of the space of its finite substructures, known as its age. This thesis aims to generalize this theory into the context of larger cardinalities. First, it will be confirmed that ages that produce Fraïssé limits can also produce similar structures at higher cardinals. Second, it will be shown that certain ages that cannot produce such countable structures are still able to produce uncountable structures with the same desired properties.

The first chapter of this dissertation is intended to provide the reader with the preliminary background knowledge of model theory necessary to discuss the concepts presented in this area of study. The second chapter is then meant to provide insight into the theory of Fraïssé limits and how they are constructed, so that the parallels between them and the later constructions are apparent.

The second and third chapters consider the $\aleph_1$ case. The second chapter focuses on a direct analog of the finite extension property, known as the countable extension property, and proceeds to show that under the continuum hypothesis $\mathfrak{c} = \aleph_1$, a class that produces a Fraïssé limit can also produce a structure of size $\aleph_1$ with the countable extension property. The third chapter shifts to a weaker instance of the countable extension property that not only implies the same or similar properties as Fraïssé limits, but also can be constructed from from an age that does not produce a Fraïssé limit, provided it satisfies certain similar properties.

The final chapter takes the results of chapters 2 and 3 and generalizes them further to apply to strongly inaccessible cardinals or regular cardinals under the generalized continuum hypothesis.

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CHAPTER 1: Introduction

This chapter is meant to provide the reader with enough preliminary knowledge in model theory so that the results presented in later chapters are clear. Much of this chapter is adapted from [1], with some minor adjustments.

1.1 Language and Structures

Consider the following space.

Definition. Let $\langle \mathbb{Q}, \leq \rangle$ denote the linear order of the rational number line.

In model theory, $\langle \mathbb{Q}, \leq \rangle$ is an example of a *structure*, where $\mathbb{Q}$ is the *domain* of $\langle \mathbb{Q}, \leq \rangle$ and $\{\leq\}$ is known as the *signature* of $\langle \mathbb{Q}, \leq \rangle$. In particular, the signature of a structure contains all of the necessary constant symbols, as well as any finitary relation or function symbols that are necessary to describe the structure. For example, to describe the structure of $\mathbb{Q}$ as a group under addition, symbols for '0', '+', and '$-$' would be included in the signature. Though it is not often listed when implied, it is also necessary for a structure to have an *interpretation*, which is a function that maps the symbols in the signature to actual elements, relations, and functions on the given domain. If A is a structure and s is a symbol in the signature of A, s^A formally denotes the interpretation of s in A.

Different structures can also rely on the same signature. For instance, $\langle \mathbb{N}, \leq \rangle$ and $\langle \mathbb{R}, \leq \rangle$ use the same signature as $\langle \mathbb{Q}, \leq \rangle$, but now describe the ordering of the natural or real number lines, respectively. If L is the signature of a structure A, then A is known as an *L-structure.* Additionally, when L is clear, a structure may be referred to by its domain.

1.2 Embeddings Between Structures

Consider the linear order of $\mathbb{Q}$ compared to the lonear order of $\omega + \omega$. Let φ be a function mapping $\omega + \omega$ to $\mathbb{Q}$ where for all $n < \omega$, $\varphi(n) = -\frac{1}{n+1}$ and $\varphi(\omega + n) = n$. In this case, the structure of $\omega + \omega$ is preserved by φ. It will be of particular interest when such functions occur between spaces.

Definition. For a signature L, let φ be a function with domain $A = \langle A, L\rangle$ and range $B = \langle B, L\rangle$. φ is an *embedding* if φ is injective and the interpretation of L in A aligns with the interpretation of L in the image of A under φ, $\varphi[A]$. More specifically,

- if c is a constant symbol in L, then $\varphi(c^A) = c^B$.
- if R is a finitary relation symbol in L and $\mathbf{a}, \mathbf{a}' \in A^{<\omega}$, then

  $(\mathbf{a}, \mathbf{a}') \in R^A \iff (\varphi(\mathbf{a}), \varphi(\mathbf{a}')) \in R^B$.
- if g is a finitary function symbol in L, then $\varphi \circ g^B = g^A \circ \varphi$.

If φ is also surjective, then φ is an *isomorphism*, and an isomorphism from a structure A to itself is an *automorphism*. If there exists an embedding from two structures A and B, then $A \preceq B$. Similarly $A \preceq_\varphi B$ if φ serves as an embedding from A to B.

It will also be useful to have an understanding of when two embeddings are similar. More precisely,

Definition. Let A, B, A', and B' all be L-structures and let $\varphi : A \to B$ and $\psi : A' \to B'$ be embeddings. φ is considered *isomorphic* to ψ if there exist isomorphisms $\eta_A : A \to A'$ and $\eta_B : B \to B'$ such that $\psi \circ \eta_A = \eta_B \circ \varphi$.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} \\ A' & \xrightarrow{\psi} & B' \end{array}$$

Figure 1.1: φ and ψ are isomorphic

Last on the topic of embeddings, it often be of interest to consider when two embeddings have the same domain.

Definition. Let A, B and C all be L-structures and let $\varphi : A \to B$ and $\psi : A \to C$ be embeddings. Then the tuple (A, φ, B, ψ, C) is known as an *amalgam* between L-structures.

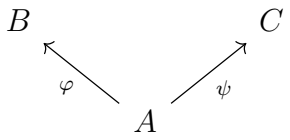


Figure 1.2: The Amalgam (A, φ, B, ψ, C)

1.3 Substructures and Ages

If φ is an embedding between L structures, A and B, then $\varphi[A]$ is an L-structure contained in B. More generally,

Definition. Given a signature L and a structure $B = \langle B, L\rangle$, the space $A = \langle A, L\rangle$ is a *substructure* of B if $A \subset B$ and A is *closed* under L, meaning

- if s is a constant in L, then $s^A = s^B$
- if f is a function symbol in L and $\mathbf{a} \in A^{<\omega}$, then $f(\mathbf{a}) \in A^{<\omega}$.

Additionally, if $A \subset B$, where A is not necessarily an L-structure, then the substructure of B *generated* by A (known as the *hull* of A) is the smallest substructure of B that contains A, denoted as $\langle A\rangle_B$.

Given an L-structure B and $A \subset B$, $\langle A\rangle_B$ can be explicitly defined as $\bigcup_{i<\omega} A_i$, where

- $A_0 = A \cup \{s^B : s \text{ is a constant in } L\}$, and
- $A_{i+1} = A_i \cup \bigcup\{f^B[A_i^n] : f \text{ is an } n\text{-ary function symbol in } L\}$

If $b \subset B$ is finite, then $\langle b \rangle_B$ is a *finitely generated* substructure of B. More generally, an L-structure b is finitely generated if there exists a finite $b' \subset b$ such that $b = \langle b' \rangle_b$.

Depending on the signature, it is possible for a finitely generated structure to be infinite. Consider the structure $\mathbb{Z} = \langle \mathbb{Z}, 0, +, - \rangle$ describing the integers as a group under addition. Then $\mathbb{Z} = \langle \{1\} \rangle_{\mathbb{Z}}$, meaning that $\mathbb{Z}$ is a finitely generated, infinite structure. In general, Given some subset A of an L-structure B, if $\kappa = \max\{\aleph_0, |L|\}$ and $\lambda = |A|$, then

$$\lambda \leq |\langle A \rangle_B| \leq \lambda + \kappa$$

For the sake of the results in this paper, it will always be assumed that the signature of a structure contains at most countably many symbols, meaning that finitely generated structures will be at most countable. It will also be assumed that L contains only finitely many constant symbols, so that a finitely generated structure can be build via the above process, with each set being finite.

It will be of particular interest to look at the space of all finitely generated substructures of a given L-structure.

Definition. Let L be a signature and D an L-structure. A class $\mathcal{K}$ of finitely generated L-structures contains the *age* of D if each finitely generated substructure of D is isomorphic to an element of $\mathcal{K}$. Additionally, if each element of $\mathcal{K}$ is isomorphic to a substructure of D, then $\mathcal{K}$ is simply called the age of D.

In other words, a class $\mathcal{K}$ is the age of D if the isomorphism *types* in $\mathcal{K}$ are exactly the isomorphism types of finitely generated substructures of B. A class of finitely generated structures is *essentially countable* if it contains only countably many different isomorphism types of structure. So to say that D has *countable age* is to say that D has countably many different isomorphism types for its finite substructures.

If $\mathcal{K}$ is the age of some L-structure, then $\mathcal{K}$ will have the following two properties:

Definition. For a signature L, a class $\mathcal{K}$ of finitely generated L-structures has the *Hereditary Property* (HP), if for any $B \in \mathcal{K}$, if A is a substructure of B, then there is some $C \in \mathcal{K}$ that is isomorphic to A.

Definition. For a signature L. a class $\mathcal{K}$ of finitely generated L-structures has the *Joint Embedding Property* (JEP), if for any $A, B \in \mathcal{K}$, there exists a $C \in \mathcal{K}$ such that A and B both embed into C.

Additionally, if $\mathcal{K}$ is an essentially countable class of finitely generated L-structures has the HP and the JEP, then $\mathcal{K}$ is the age of some countable structure. For example, let $\mathcal{L}$ be the class of finite linear orders. It is clear that $\mathcal{L}$ has the HP and JEP, and it is the age of $\langle \mathbb{Q}, \leq \rangle$. $\mathcal{L}$ is also the age of $\langle \mathbb{N}, \leq \rangle$, meaning that two structures can have the same age. If D is an L-structure with age $\mathcal{K}$, then D is called a $\mathcal{K}$-structure.

CHAPTER 2: The FEP

The framework for Fraïssé limits was first described in [2] and later in [3] and were devoloped from observations of $\langle \mathbb{Q}, \leq \rangle$. To summarize, $\mathbb{Q}$ has the following nice properties.

Fact. Every countable linear order embeds into $\mathbb{Q}$.

Fact. Up to isomorphism, $\mathbb{Q}$ is the unique countable, unbounded, dense linear order.

Fact. Every embedding of a finite linear order into $\mathbb{Q}$ can be extended to an automorphism of $\mathbb{Q}$.

It was through this study that Fraïssé was able to develop the notion of a Fraïssé limit.

2.1 Finite Extensions

The particular property of $\mathbb{Q}$ that allows for these nice properties is its ability to extend embeddings of finite substructures.

Definition. Let L be a signature and D an L-structure. D has the *Finite Extension Property* (FEP) if whenever $a, b \in \text{Age}(D)$ such that $a \subseteq b$ and $\varphi : a \to D$ is an embedding, then there exists an embedding $\psi : b \to D$ such that $\psi \restriction A = \varphi$.

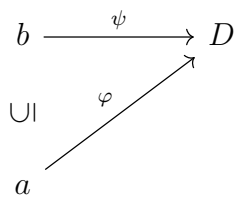


Figure 2.1: FEP

It is worth noting that the FEP can be rephrased to account for arbitrary embeddings, rather than requiring direct substructures.

Theorem 2.1.1. *Let L be a signature and D an L-structure. The following are equivalent:*

- *D has the FEP.*
- *Whenever $a, b \in \mathrm{Age}(D)$, and $\varphi : a \to D$ and $\rho : a \to b$ are embeddings, there exists an embedding $\psi : b \to D$ such that $\psi \circ \rho = \varphi$.*

Proof. The reverse implication is clear since the inclusion map is a case of an embedding.

Suppose that D is an L-structure with the FEP. Let $a, b \in \mathrm{Age}(D)$, and $\varphi : a \to D$ and $\rho : a \to b$ be embeddings. Let $a' = \rho[a]$. Then $\varphi \circ \rho^{-1}$ is an embedding from a' into D.

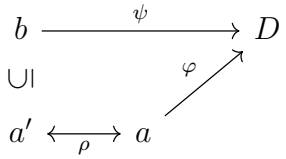


Figure 2.2: FEP generalized

By the FEP, there exists an embedding $\psi : B \to D$ which extends $\varphi \circ \rho^{-1}$, meaning $\psi \circ \rho = \varphi \circ \rho^{-1} \circ \rho = \varphi$. ⇞

Consider the linear ordering of the natural numbers, $\mathbb{N}$. Within the age of $\mathbb{N}$, there exist (isomorphic copies of) the finite orderings $\{1, 3\}$ and $\{1, 2, 3\}$. Let $f : \{1, 3\} \to \mathbb{N}$ where $f(1) = 1$ and $f(3) = 2$. In this case, f cannot be extended to an embedding $g : \{1, 2, 3\} \to \mathbb{N}$, since there is no $n \in \mathbb{N}$ such that $f(1) = 1 < n < 2 = f(3)$. On the other hand, $\mathbb{Q}$ does not have this issue.

Fact. $\mathbb{Q}$ has the FEP

This fact is clear since $\mathbb{Q}$ is both unbounded and dense.

2.2 Consequences of the FEP

Each of the nice properties of $\mathbb{Q}$ mentioned previously are a direct result of $\mathbb{Q}$ having the FEP and being countable. It can then be expected that any countable structure with the FEP, would share these nice properties.

Definition. Let L be a signature and let D be an L-structure with age $\mathcal{K}$. For a cardinal μ, D is *μ-universal (for $\mathcal{K}$)* if whenever A is a $\mathcal{K}$-structure such that $|A| \leq \mu$, then there exists an embedding from A into D. D is *universal (for $\mathcal{K}$)* if it is $|D|$-universal.

Theorem 2.2.1. *Let D be an L-structure for some signature L. If D has the FEP, then D is universal.*

Proof. To show that D is universal, Let $\mathcal{K}$ be the age of D and let A be a countable L-structure with age $\mathcal{K}$. Since A is countable, there exists an ascending countable sequence $(a_n : n < \omega)$ of finitely generated substructures of A whose union is A. Since $a_0 \in \mathcal{K}$, there is an embedding from a_0 into D. Let $\varphi_0 : a_0 \to D$ be an embedding. Then $a_0 \subset a_1 \in \mathcal{K}$ and, since D is countable and has the FEP, there exists an embedding $\varphi_1 : a_1 \to D$ which extends φ_0. This recursively creates a sequence of functions φ_n which embed a_n into D. Since for all n, $a_n \subset a_{a+1}$, we have that φ_n is extended by φ_{n+1}. Let $\varphi = \bigcup_n \varphi_n$. Then φ is an embedding from A into D. ⁂

So if D is a countable structure with the FEP and has age $\mathcal{K}$, then every countable $\mathcal{K}$-structure embeds into D. Additionally, if the other structure also has the FEP, then an isomorphism can be made between them.

Theorem 2.2.2. *Let L be a signature. Let C and D be countable L-structures with the FEP such that* $\mathrm{Age}(C) = \mathrm{Age}(D)$. *Then there exists an isomorphism $\psi : C \to D$.*

Additionally, if a is a finitely generated substructure of C and $\varphi : a \to D$ is an embedding, then φ can be extended to an isomorphism between C and D.

Proof. For a signature L, let C and D be countable L-structures with age $\mathcal{K}$ that both have the FEP. Let a be a finitely generated substructure of C and let $\varphi : a \to D$ be an embedding. The isomorphism ψ will be constructed recursively via a back and forth argument.

Using the fact that C and D are both countable, let $C = \bigcup_i c_i$ and $D = \bigcup_i d_i$ where $c_0 = a$ and that for each $i < \omega$, $c_i, d_i \in \mathcal{K}$, $c_i \subset c_{i+1}$, and $d_i \subset d_{i+1}$. Let $n > 0$ such that $\varphi[c_0] \subset d_n$. Since $d_n \in \mathcal{K} = \mathrm{Age}(C)$ and C has the FEP, there exists an embedding $\rho_n : d_n \to C$ which extends φ^{-1}. If $i < n$, let $\rho_i = \rho_n \restriction d_i$. Let $m > n$ such that $\rho_n[d_n] \subset c_m$. Similarly, since $c_m \in \mathcal{K} = \mathrm{Age}(D)$ and D has the FEP, there exists an embedding $\psi_m : c_m \to D$ which extends ρ_n^{-1}. If $i < m$, let $\psi_i = \psi_m \restriction c_i$.

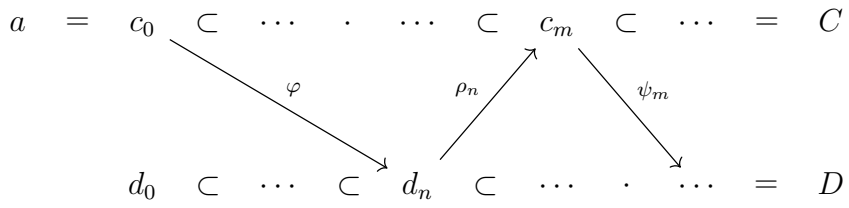


Figure 2.3: Creating the Isomorphism via a Back-and-Forth method

Following this back and forth argument recursively defines a sequence of functions $(\psi_n : n < \omega)$ and $(\rho_n : n < \omega)$ such that for all $n > 0$,

- $c_n = \mathrm{Dom}(\psi_n)$ and $d_n = \mathrm{Dom}(\rho_n)$
- ψ_n extends φ.
- There exists $k > m > n$ such that $\psi_n \subset \rho_m^{-1} \subset \psi_k$.

Let $\psi = \bigcup_n \psi_n$ and $\rho = \bigcup_n \rho_n$. Then ψ and ρ are both embeddings between C and D and $\psi^{-1} = \rho$. Hence, ψ is an isomorphism between C and D which extends φ.

⁂

Definition. Let L be a signature and let D be an L-structure with age $\mathcal{K}$. D is *homogeneous* if for any $a \in \mathcal{K}$ and embedding $\varphi : a \to D$, there exists an automorphism of D which extends φ.

Corollary 2.2.3. *Let D be a countable L-structure for some signature L. If D has the FEP, then D is homogeneous.*

Proof. The proof of 2.2.2 can be applied here directly by taking $C = D$. ⇝

2.3 Examples of the FEP

While Fraïssé started this study from observations of the rationals, many other spaces have been found to possess the FEP.

Definition. The *Countable Random Graph* is the countable undirected graph R, constructed (with probability 1) in the following way:

Let the vertex set for R be ω and for each $n, m < \omega$, whether there is an edge between n and m is decided randomly, such as via a coin flip.

Fact. R has the FEP.

The construction above is due to Erdős and Renyi in [4], but the graph was first described by Ackermann in [5], who had constructed it as a directed graph using the hereditarily finite sets as the vertex set. Rado provided a similar (but undirected) construction in [6] using the natural numbers as the vertex set to show that the graph is universal.

The age of R is the class of all finite graphs, $\mathcal{G}$. Note that by definition, a substructure of a graph is an induced subgraph. As it turns out, there are subclasses of $\mathcal{G}$ that also are the age of structures with the FEP.

Definition. For each $n < \omega$, let $\mathcal{G}_n \subset \mathcal{G}$ be the class of all finite graphs G such that the complete graph on n vertices, K_n, is not an induced subgraph of G.

Henson found in [7] that for each $n > 1$ there exists a countable graph H_n with the FEP whose age is $\mathcal{G}_n$. It was later shown in [8] that the only nontrivial countable graphs with the FEP are R, as well as H_n and its compliment for each $n \geq 3$.

Another example of a countable structure with the FEP would be the countable atomless boolean algebra.

Definition. The *Countable Atomless Boolean Algebra* (CABA) is the boolean algebra represented by the clopen subsets of the Cantor Space, C under the topology induced by its typical metric.

Fact. CABA has the FEP.

The age of CABA is the class of finite boolean algebras. Since CABA is a Fraïssé limit and C is dual to this limit, it is not surprising that C possesses some properties that are in a similar vein, namely the following:

Fact. For every compact metric space, K, there exists a continuous surjective function $\varphi : C \to K$.

This property is known as being *image universal*, and is in a sense dual to the notion of Fraïssé limits being universal for their age.

2.4 Fraïssé Limits

In 1954, Roland Fraïssé was interested in the connection between classes of finitely generated structures and countable structures with the FEP. In order to be the age of some structure, a class $\mathcal{K}$ must necessarily have the HP and JEP, but this is not enough to be the age of a structure with the FEP.

Definition. Let L be a signature. A class $\mathcal{K}$ of finitely generated L-structures has the *Amalgamation Property* (AP) if for any $a, b, c \in \mathcal{K}$ and embeddings $\varphi : a \to b$ and $\psi : a \to c$, there exists a $d \in \mathcal{K}$ and embeddings $\varphi' : b \to d$ and $\psi' : c \to d$ such that $\varphi' \circ \varphi = \psi' \circ \psi$.

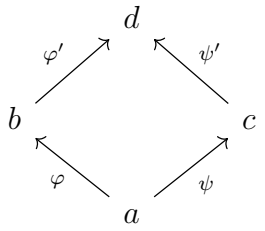


Figure 2.4: The Amalgamation Property

Theorem 2.4.1 (Fraïssé 1954. [3])**.** *Let L be a countable signature and let $\mathcal{K}$ be an essentially countable class of finitely generated L-structures with the HP, JEP, and AP. Then $\mathcal{K}$ is the age of a countable L-structure D with the FEP.*

Proof. For a countable signature L, let $\mathcal{K}$ be a class of finitely generated L-structures with the HP, JEP, and AP. It can be assumed without loss of generality that $\mathcal{K}$ is closed under taking isomorphic copies.

Let P be a countable set of pairs (a, b) of elements of $\mathcal{K}$ such that $a \subseteq b$ and for all embeddings, φ, between elements of $\mathcal{K}$, there exists an $(a, b) \in P$ such that the inclusion map from a to b is isomorphic to φ. The objective here will be to use P as a list of possible finite extensions, and use amalgamation to construct a chain of finitely generated structures whose union is a countable structure with the FEP.

Let $\pi : \omega \times \omega \to \omega$ be a bijection such that $\pi(i, j) \geq i$ for all $i, j < \omega$. Let $d_0 \in \mathcal{K}$. Inductively, whenever d_n is chosen, let $((\varphi_{nj}, a_{nj}, b_{nj}) : j < \omega)$ enumerate all embeddings $\varphi_{nj} : a_{nj} \to d_n$ such that $(a_{nj}, b_{nj}) \in P$. Then, if $\pi(i, j) = n$, then φ_{ij} embeds a_{ij} into d_n and the inclusion map ι embeds a_{ij} into b_{ij}. Since $\mathcal{K}$ has the AP, there exists an element of $\mathcal{K}$ which amalgamates φ_{ij} and ι. Since $\mathcal{K}$ is closed under taking isomorphic copies, let d_{n+1} amalgamate φ_{ij} and ι such that d_n is a substructure of d_{n+1}.

Then $(d_n : n < \omega)$ is an ascending sequence of elements in $\mathcal{K}$ such that each finite extension of elements in $\mathcal{K}$ is achieved in some d_n. Let $D = \bigcup_n d_n$. Then D is a

countable L-structure with the FEP.

↜

The object D in the above construction is known as the Fraïssé Limit for the class $\mathcal{K}$. A class $\mathcal{K}$ of finitely generated L-structures is called a Fraïssé Class if it has the HP, JEP, and AP. It is also worth noting that if D is a countable L-structure with the FEP, then the age of D is a Fraïssé Class.

Theorem 2.4.2. *Let L be a signature and D be an L-structure. If D has the FEP, then* Age(D) *has the AP.*

Proof. Let D be an L-structure with the FEP. Let $a, b, d \in \text{Age}(D)$ and let $\varphi : a \to b$ and $\psi : a \to c$ be embeddings. For Age(D) to have the AP, there needs to be a structure $d \in \text{Age}(D)$ as well as embeddings $\varphi' : b \to d$ and $\psi' : c \to d$ such that $\varphi' \circ \varphi = \psi' \circ \psi$. Since $a \in \text{Age}(D)$, there exists a finitely generated substructure of D that is isomorphic to a. Let a' be a finitely generated substructure of D and $\eta : a \to a'$ be an isomorphism. Then $\varphi \circ \eta^{-1}$ is an embedding from a' into b. Since D has the FEP, there exists a function $\varphi' : B \to D$ which extends the identity mapping on a'. Let $b' = f'[b]$. Similarly, there exists a function $\psi' : c \to D$ which extends the identity mapping on a'. Let $a' = \psi'[c]$. Then $d = \langle b' \cup c' \rangle_D \in \text{Age}(D)$ as well as $\varphi' : b \to d$ and $\psi' : c \to d$ are embeddings such that $\varphi' \circ \varphi = \psi' \circ \psi$. ↜

This means that if it is known that there does not exist a countable structure with the FEP for a given age $\mathcal{K}$, then $\mathcal{K}$ lacks the amalgamation property. One such example of this would be the class of finite planar graphs, which was shown by Pach in [9] to not have a countable universal object. Another example requires a brief discussion on what is known as a *projective Fraïssé limit.*

Irwin and Solecki developed the notion of a projective Fraïssé class in [10], where it was defined to be the topological space dual to a class of finite boolean algebras,

potentially with additional functions or relation defined on them. By focusing on continuous surjections in place of embeddings, they found that the duals of certain classes had similar properties to Fraïssé classes. There are several examples of these kinds of classes, but here are three that express the variety present in this dualization of Fraïssé's theory.

- The class of finite boolean algebras with no additional structure, where both the class and its dual are considered Fraïssé in their respective notions. The limits for these are CABA and the cantor set, mentioned earlier.

- The class of finite boolean algebras with an additional relation so that its dual is the space of finite single chains. While the class is not a Fraïssé class, it's dual class is a projective Fraïssé class and its limit is the pseudo arc, as described in [11].

- The class of finite boolean algebras with an additional relation so that its dual is the space of strongly connected directed graphs is not a Fraïssé class in either sense. More will be discussed on this space in chapter 4.

2.5 Genericity of the FEP

Up until this point, the fact that the age of a given structure is closed under isomorphism types has been used to construct a Fraïssé Limit as an ascending union of finitely generated structures. In order to describe another particular quality of Fraïssé Limits, it will be useful to describe them in the context of direct limits, instead.

Definition. Let L be a signature and κ a cardinal. Let $(D_\alpha : a < \kappa)$ be a list of L-structures and for each $\alpha \leq \beta < \kappa$ let $\varphi_{\alpha\beta} : D_\alpha \to D_\beta$ be an embedding such that $\varphi_{\alpha\alpha}$ is the identity map on D_α and $\varphi_{\alpha\beta} = \varphi_{\gamma\beta} \circ \varphi_{\alpha\gamma}$ whenever $\alpha \leq \gamma \leq \beta$. The set $\langle D_\alpha, \varphi_{\alpha\beta}\rangle = \{(D_\alpha, \varphi_{\alpha\beta}) : \alpha \leq \beta < \kappa\}$ is known as a *direct system* over κ.

For a direct system over κ, $\langle D_\alpha, \varphi_{\alpha\beta}\rangle$, let $\sim$ define the equivalence relation on the disjoint union $\bigsqcup D_\alpha$, where if $x \in D_\alpha$ and $y \in D_\beta$, then $x \sim y$ if there exists a $\gamma < \kappa$ such that $\varphi_{\alpha\gamma}(x) = \varphi_{\beta\gamma}(y)$. The *direct limit* of $\langle D_\alpha, \varphi_{\alpha\beta}\rangle$ is the structure $\varinjlim D_\alpha = \bigsqcup D_\alpha / \sim$.

Formally, the direct limit is often paired with the collection of *canonical functions* $\chi_\beta : D_\beta \to \varinjlim D_\alpha$, where χ_β maps each element of D_β to its congruence class in $\varinjlim D_\alpha$.

The idea of direct limits was introduced in [12] as a way of building a structure from a sequence of embeddings such that the structures used in the embeddings are essentially subsets of the limit.

To briefly revisit the proof of Fraïssé's Theorem, rather than leveraging isomorphism types to make sure that d_n is a substructure of d_{n+1}, it is possible to instead take any structures satisfying the amalgamation for the d_n's and to then let D be the direct limit of the direct system $\langle d_n, \varphi_{nk}\rangle$, where $\varphi_{nk} = \varphi_k \circ \ldots \circ \varphi_n$ for any $n < k < \omega$.

In practice it is typically easier to use direct limits when going through Fraïssé's construction on a given class. It is also worth mentioning that in certain cases, such as the class of finite boolean algebras, taking the *inverse limit*, The last property of Fraïssé limits worth mentioning here involves looking at all possible direct limits simultaneously.

Definition. For a signature L, let $\mathcal{K}$ be a countable set of finitely generated L-structures. Define $S_\mathcal{K}$ to be the set of all sequences $((\varphi_n, d_n) : n < \omega)$ of pairs (φ_n, d_n) such that φ_0 is the empty embedding and for all $n > 0$,

- $d_n \in \mathcal{K}$
- $\varphi_n : d_{n-1} \to d_n$.

Define the topology on $S_{\mathcal{K}}$ from initial segments, meaning a basic open set of $S_{\mathcal{K}}$, $[\![((\varphi_0, d_0), \ldots, (\varphi_n, d_n))]\!]$, would be the set of all sequences who agree on the first n many embeddings for some n.

$S_{\mathcal{K}}$ is a complete metric space with the topology defined above. This means that the Baire Category Theorem [13] applies, so taking the intersection of countably many dense open sets remains dense. Additionally, a given sequence in $S_{\mathcal{K}}$ induces a direct system. From this, it is possible to create a dense set in this way such that each element in the set "limits to" the Fraïssé limit for $\mathcal{K}$.

Theorem 2.5.1. *Let L be a countable signature and let $\mathcal{K}$ be a countable set of finitely generated L-structures. If $\mathcal{K}$ is a Fraïssé class with limit F, then there exists a dense G_δ set $X \subset S_{\mathcal{K}}$ such that if $((d_n, \varphi_n) : n < \omega) \in X$, then $\varinjlim d_n$ is isomorphic to F.*

Proof. Let P be the set of all pairs of embeddings (ψ, ρ) between structures in $\mathcal{K}$ such that $\mathrm{Dom}(\psi) = \mathrm{Dom}(\rho)$. Since $\mathcal{K}$ is countable, P is also countable.

For a given $(\psi, \rho) \in P$, let $a = \mathrm{Dom}(\psi) = \mathrm{Dom}(\rho)$ and $b = \mathrm{Ran}(\psi)$. Define $U_{\psi\rho}$ to be the union of all basic open sets $[\![(\varphi_0, d_0), \ldots, (\varphi_n, d_n), (\varphi_{n+1}, d_{n+1})]\!]$ in $S_{\mathcal{K}}$ such that either

- $\mathrm{Ran}(\rho) \neq d_n$ and $a \preceq d_{n+1}$, or
- $\mathrm{Ran}(\rho) = d_n$ and there exists an embedding $\psi' : b \to d_{n+1}$ such that $\varphi_{n+1} \circ \rho = \psi' \circ \psi$

Let $(\psi, \rho) \in P$ where $a = \mathrm{Dom}(\psi) = \mathrm{Dom}(\rho)$ and $b = \mathrm{Ran}(\psi)$. Clearly, $U_{\psi\rho}$ forms an open subset of $S_{\mathcal{K}}$. To show that $U_{\psi\rho}$ is dense, let $[\![(\varphi_0, d_0), \ldots, (\varphi_n, d_n)]\!]$ be a basic open subset of $S_{\mathcal{K}}$. Suppose that $\mathrm{Ran}(\rho) \neq d_n$. Since $\mathcal{K}$ has the JEP, there exists an $e \in \mathcal{K}$ such that $d_n \preceq e$ and $a \preceq e$. By letting $d_{n+1} = e$ and φ_{n+1} be an embedding from d_n to d_{n+1}, the set $[\![(\varphi_0, d_0), \ldots, (\varphi_n, d_n), (\varphi_{n+1}, d_{n+1})]\!]$ is a basic open contained in $U_{\psi\rho} \cap [\![(\varphi_0, d_0), \ldots, (\varphi_n, d_n)]\!]$. Now suppose $\mathrm{Ran}(\rho) = d_n$, since $\mathcal{K}$ has the AP, there

exists an $e \in \mathcal{K}$ and embeddings $\varphi' : d_n \to e$ and $\psi' : b \to e$ such that $\varphi' \circ \rho = \psi' \circ \psi$. Let $d_{n+1} = e$ and $\varphi_{n+1} = \varphi'$. Then $[\![(\varphi_0, d_0), \ldots, (\varphi_n, d_n), (\varphi_{n+1}, d_{n+1})]\!]$ is a basic open contained in $U_{\psi\rho} \cap [\![(\varphi_0, d_0), \ldots, (\varphi_n, d_n)]\!]$

Let

$$X = \bigcap \{U_{\psi,\rho} : (\psi, \rho) \in P\}$$

Then X is a countable intersection of dense open subsets of $S_{\mathcal{K}}$ and thusly forms a dense G_δ set in $S_{\mathcal{K}}$.

Let $((\varphi_n, d_n) : n < \omega) \in X$ and let $D = \varinjlim d_n$. Note that D has age $\mathcal{K}$, since for each $a \in \mathcal{K}$, there is exists an $n < \omega$ such that $a \preceq d_n \preceq D$. Then, for D to be isomorphic to F, D must satisfy the FEP.

Let $a, b \in \mathcal{K}$ and let $\psi : a \to b$ and $\rho : a \to D$ be embeddings. Since a is finitely generated and D is a direct limit, there exists some n such that every element of $\rho[a]$ has a representative in d_n. Recall that $\chi_n : d_n \to D$ is the embedding mapping each element in d_n to its congruence class. Then the function $\rho' = \chi_n^{-1} \circ \rho$ is an embedding from a into d_n. Since $D \in X \subset U_{\psi\rho'}$ there exists some $\psi' : b \to d_{n+1}$ such that $\varphi_{n+1} \circ \rho' = \psi' \circ \psi$.

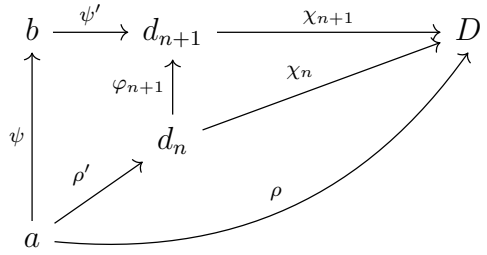


Figure 2.5: FEP in D

Then $\chi_{n+1} \circ \psi'$ is an embedding from b into D and, since D is a direct limit,

$$\chi_{n+1} \circ \psi' \circ \psi = \chi_n \circ \chi_n^{-1} \circ \rho = \rho$$

To say that a property holds for a *generic* point in a topological space means that the set of all points in the space with that property contains a dense G_δ set. The above theorem then shows that a *generic* sequence of embeddings of finite substructures in a Fraïssé class creates a structure isomorphic to the associated Fraïssé limit. Additionally, a class $\mathcal{K}$ of L-structures has a *generic isomorphism type* if there exists a G_δ set in $S_\mathcal{K}$ such that any two sequences in the set have isomorphic direct limits.

Given an age $\mathcal{K}$, it is natural that $\mathcal{K}$ must have the AP in order for the FEP to be a generic property in $S_\mathcal{K}$. However, the AP is not necessary for $\mathcal{K}$ to have a generic isomorphism type. Consider the following weakening of AP.

Definition. A class $\mathcal{K}$ of finitely generated L-structures has the *weak-Amalgamation Property* if for each $a \in \mathcal{K}$, there exists an $a' \in \mathcal{K}$ and an embedding $\eta : a \to a'$ such that whenever $b, c \in \mathcal{K}$ and $\varphi : a' \to b$ and $\psi : a' \to c$ are embeddings, there exists a $d \in \mathcal{K}$ and embeddings $\varphi' : b \to d$ and $\psi' : c \to d$ such that $\varphi' \circ \varphi \circ \eta = \psi' \circ \psi \circ \eta$.

Alex Kruckman showed in [14] that for an age $\mathcal{K}$, $\mathcal{K}$ has a generic isomorphism type exactly when it has the weak Amalgamation Property. Consider for example the class $\mathcal{A}_{\leq 2}$ of finite acyclic graphs where the degree of any vertex is at most two.

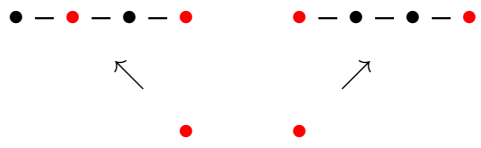


Figure 2.6: Incomplete Amalgam in $\mathcal{A}_{\leq 2}$

The above amalgam demonstrates that $\mathcal{A}_{\leq 2}$ lacks the amalgamation property. However, any graph $A \in \mathcal{A}_{\leq 2}$ of size n can be embedded into a chain of length $2n$, and any amalgam in $\mathcal{A}_{\leq 2}$ whose root is a chain can be completed in $\mathcal{A}_{\leq 2}$, so this class has the w-AP. The generic isomorphism type for this graph is the countable

bi-infinite chain, and the graph of countably copies of this chain forms a universal $\mathcal{A}_{\leq 2}$-structure.

Another example would be the class of lattices whose Wallman dual [15] corresponds to the class of graphs that are single finite chains. Irwin and Solecki showed in [10] that this class of graphs provides a dual nature to Fraïssé limits.

CHAPTER 3: The CEP

The primary focus of this research was to identify structures with properties that are similar to Fraïssé limits, but whose size is $\aleph_1$ under the continuum hypothesis. This chapter will focus in particular on $\aleph_1$ sized structures whose age is a Fraïssé class. Certain results in this chapter assume the continuum hypothesis, $\mathfrak{c} = \aleph_1$, which will be acknowledged when that is the case.

3.1 Countable Extensions

In the case of uncountable structures, the FEP is not a strong enough property to guarantee similar qualities of a Fraïssé limit. Consider the linear order of the real number line. $\mathbb{R}$ has the same age as the rationals and, for the same reasons, the real number line has the FEP. However, if $\varphi : (0,1) \cap \mathbb{Q} \to \mathbb{Q}$ is an isomorphism and, therefore, an embedding in $\mathbb{R}$, then there is no automorphism of $\mathbb{R}$ which extends φ, meaning that $\mathbb{R}$ is not a homogeneous structure. Additionally, the linear ordering made by taking ω_1 many copies of the rationals also has the FEP, but cannot be embedded into the reals, so $\mathbb{R}$ is neither uniquely defined by the FEP nor universal.

In order for size-$\aleph_1$ structure to retain these properties, it needs to not only be able to allow for finite extensions, but countable ones as well.

Definition. Let L be a signature and D an L-structure. D has the *Countable Extension Property* (CEP) if whenever A and B are countable substructures of D such that $A \subseteq B$ and $\varphi : A \to D$ is an embedding, then there exists an embedding $\psi : B \to D$ such that $\psi \restriction A = \varphi$.

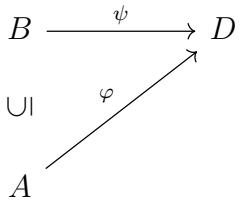


Figure 3.1: CEP

3.2 Consequences of the CEP under CH

Recall that if a countable structure D has the FEP, then it is universal, homogeneous, and unique up to isomorphism. In the case of $\aleph_1$ size structures, the CEP is strong enough to regain all of these properties.

Theorem 3.2.1. *Let D be an L-structure for some signature L. If D has the CEP, then D is $\aleph_1$-universal. Hence, D is universal if $|D| = \aleph_1$*

Proof. Note that if D has the CEP, then $|D| > \aleph_0$, since otherwise an automorphism of D could not be extended to a larger structure. Let $\mathcal{K}$ be the age of D and let A be an L-structure of size $\aleph_1$ with age $\mathcal{K}$. Since A has size $\aleph_1$, A can be written as a union of $\aleph_1$ many countable substructures of A. Let $(A_\alpha : \alpha < \omega_1)$ be a sequence of countable sustructures of A such that for all $\alpha < \omega_1$, $A_\alpha \subset A_{\alpha+1}$ and $A_\gamma = \bigcup\{A_\alpha : \alpha < \gamma\}$ in the case that γ is a limit ordinal.

Since D has the CEP, D also has the FEP and is therefore $\aleph_0$ universal. So for each $\alpha < \omega_1$, A_α can be embedded into D. Let $\varphi_0 : A_0 \to D$ be an embedding. Then given $\varphi_\alpha : A_\alpha \to D$, since D has the CEP, there exists an embedding $\varphi_{\alpha+1} : A_{\alpha+1} \to D$ which extends φ_α. In the case where γ is a limit ordinal, let $\varphi_\gamma = \bigcup\{f_\alpha : \alpha < \gamma\}$. Continuing this method recursively creates a sequence of functions φ_α which embed A_α into D, for all $\alpha < \omega_1$. Let $\varphi = \bigcup\{\varphi_\alpha : \alpha < \omega_1\}$. Then φ is an embedding from A into D. ⇝

Similar to Fraïssé limits, this process of extending embeddings via countable substructures can create an isomorphism between two size-$\aleph_1$ structures with the CEP.

Theorem 3.2.2. *Let C and D be L-structures of size $\aleph_1$ for some signature L. If both C and D have the CEP and $\mathrm{Age}(C) = \mathrm{Age}(D)$, then C is isomorphic to D. Moreover, if A is a countable substructure of C and $\varphi : A \to D$ is an embedding, then φ can be extended to an isomorphism between C and D.*

Proof. For a signature L, let C and D be $\aleph_1$ size L-structures with age $\mathcal{K}$ that both have the CEP. Let A be a countable substructure of C and let $\varphi : A \to D$ be an embedding. The isomorphism ψ will be constructed recursively via a back and forth argument.

Using the fact that $|C| = |D| = \aleph_1$, let $C = \bigcup_{\alpha<\omega_1} C_\alpha$ and $D = \bigcup_{\alpha<\omega_1} D_\alpha$ where $C_0 = A$ and that for each $\alpha < \omega$, C_α and D_α are countable $\mathcal{K}$-structures, $C_\alpha \subset C_{\alpha+1}$, $D_\alpha \subset D_{\alpha+1}$, and if $\gamma < \omega_1$ is a limit ordinal, $C_\gamma = \bigcup_{\alpha<\gamma} C_\alpha$ and $D_\gamma = \bigcup_{\alpha<\gamma} D_\alpha$. Let $\beta > 0$ such that $\varphi[C_0] \subset D_\beta$. Since D_β is a countable $\mathcal{K}$-structure and C has the CEP, there exists an embedding $\rho_\beta : D_\beta \to C$ which extends φ^{-1}. If $\alpha < \beta$, let $\rho_\alpha = \rho_\beta \restriction D_\alpha$. Let $\delta > \beta$ such that $\rho_\beta[D_\beta] \subset C_\delta$. Similarly, since C_δ is a countable $\mathcal{K}$-structure and D has the CEP, there exists an embedding $\psi_\delta : C_\delta \to D$ which extends ρ_β^{-1}. If $\alpha < \delta$, let $\psi_\alpha = \psi_\delta \restriction C_\alpha$.

Following this back and forth argument recursively and taking unions at limit steps defines a sequence of functions $(\psi_\alpha : \alpha < \omega_1)$ and $(\rho_\alpha : \alpha < \omega_1)$ such that for all such α,

- $C_\alpha = \mathrm{Dom}(\psi_\alpha)$ and $D_\alpha = \mathrm{Dom}(\rho_\alpha)$
- ψ_α extends φ.
- There exists a $\delta > \beta > \alpha$ such that $\psi_\alpha \subset \rho_\beta^{-1} \subset \psi_\delta$.

Let $\psi = \bigcup_{\alpha<\omega_1} \psi_\alpha$ and $\rho = \bigcup_{\alpha<\omega_1} \rho_\alpha$. Then ψ and ρ are both embeddings between C and D and $\psi^{-1} = \rho$. Hence, ψ is an isomorphism between C and D which extends φ.

Corollary 3.2.3. *Let D be an L-structure of size-$\aleph_1$ for some signature L. If D has the CEP, then D is Homogeneous.*

Proof. Letting $C = D$, the proof of theorem 3.2.2 shows that D is homogeneous. ⁂

While none of the above proofs require the continuum hypothesis (CH), it will be necessary to assert there exists structures of size-$\aleph_1$ with these properties.

3.3 Structures with the CEP

Since structures with the CEP behave in essence the exact same as a Fraïssé limit, it is not surprising that such a structure can be built from one.

Lemma 3.3.1. *Let $\mathcal{K}$ be a Fraïssé class with F its limit. If A, B and C are countable $\mathcal{K}$-structures and $\varphi : A \to B$ and $\psi : A \to C$ are embeddings, then there exist embeddings $\varphi' : B \to F$ and $\psi' : C \to F$ such that $\varphi' \circ \varphi = \psi' \circ \psi$.*

Proof. Let $\mathcal{K}$ be a Fraïssé class with limit F and let A, B and C be countable $\mathcal{K}$-structures. Let $\varphi : A \to B$ and $\psi : A \to C$ be embeddings. Since A, B and C are countable, let $A = \bigcup a_n$, $B = \bigcup b_n$ and $C = \bigcup c_n$ such that for all $n < \omega$

- $a_n, b_n, c_n \in \mathcal{K}$
- $a_n = b_n \cap A = c_n \cap A$

Let $\varphi_n = \varphi \restriction a_n$ and $\psi_n = \psi \restriction a_n$. Note that if $m > n$ that $a_n \preceq_{\varphi_n} b_m$ and $a_n \preceq_{\psi_n} c_m$.

Since $a_0 \in \mathcal{K}$, $a_0 \preceq F$. Let $\rho_0 : a_0 \to F$ be an embedding. Since F has the FEP, there exists an embedding $\varphi_1' : b_1 \to F$ that extends ρ_0. Define $\rho_1 : a_1 \to F$ to be the embedding given by $\varphi_1' \circ \varphi_1$. Again since F has the FEP, there exists an embedding $\psi_2' : c_2 \to F$ that extends ρ_1.

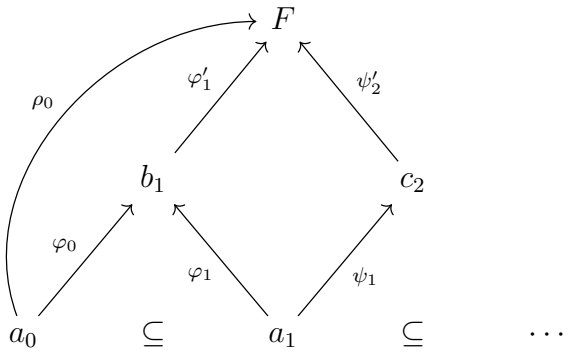


Figure 3.2: Iteration of the FEP

Continuing in this process recursively produces for each $n < \omega$ the embeddings $\varphi'_{2n+1} : b_{2n+1} \to F$ and $\psi'_{2n+2} : c_{2n+2} \to F$ such that if $x \in a_n$, then $\varphi'_{2n+1}(\varphi(x)) = \psi'_{2n+2}(\psi(x))$. Let $\varphi' = \bigcup_n \varphi_{2_n+1}$ and $\psi' = \bigcup_n \psi_{2_n+2}$. Then $\varphi' : B \to F$ and $\psi' : C \to F$ are embeddings such that $\varphi' \circ \varphi = \psi' \circ \psi$. ℵ

Theorem 3.3.2. *(CH) Let $\mathcal{K}$ be a Fraïssé class. Then there exists a size-$\aleph_1$ $\mathcal{K}$-structure with the CEP.*

Proof. This proof follows a similar structure to the construction of a Fraïssé limit shown in Chapter 2, but with a few key differences.

Let F be the Fraïssé limit for $\mathcal{K}$. Assuming CH, there exist up to isomorphism, $\aleph_1$ many countable $\mathcal{K}$-structures. Let P be a size-$\aleph_1$ set of pairs (A, B) of countable $\mathcal{K}$-structures such that $A \subseteq B$ and for all embeddings, φ, between countable $\mathcal{K}$-structures, there exists an $(A, B) \in P$ such that the inclusion map from A to B is isomorphic to φ. The objective here will be to use P as a list of possible countable extensions, and use 3.3.1 construct a chain of embeddings between countable structures whose direct limit will have the CEP.

Let $\pi : \omega_1 \times \omega_1 \to \omega_1$ be a bijection such that $\pi(\alpha, \beta) \geq \alpha$ for all $\alpha, \beta < \omega_1$. Let $F_0 = F$. Inductively, whenever F_γ is chosen, let $((\varphi_{\gamma\beta}, A_{\gamma\beta}, B_{\gamma\beta}) : \beta < \omega_1)$ enumerate all embeddings $\varphi_{\gamma\beta} : A_{\gamma\beta} \to F_\gamma$ such that $(A_{\gamma\beta}, B_{\gamma\beta}) \in P$. Then, if $\pi(\alpha, \beta) = \gamma$, then $\varphi_{\alpha\beta}$ embeds $A_{\alpha\beta}$ into F_γ and the inclusion map ι embeds $A_{\alpha\beta}$ into $B_{\alpha\beta}$. By

3.3.1, there exists an embeddings $\rho_\gamma : F_\gamma \to F$ and $\psi : B_{\alpha\beta} \to F$ such that ψ extends $\rho_\gamma \circ \varphi_{\alpha\beta}$. If γ is a limit, for each $\alpha \leq \beta < \gamma$ define the embedding $\rho_{\alpha\beta} : F_\alpha \to F_\beta$ such that

- If $\alpha = \beta$ $\rho_{\alpha\beta}$ is the identity map on F_α
- If $\alpha < \beta$ and there is no limit ordinal between α and β, $\rho_{\alpha\beta} = \rho_\beta \circ \rho_{\beta-1} \circ \ldots \circ \rho_{\alpha+1} \circ \rho_\alpha$
- If there is a limit ordinal μ such that $\alpha < \mu \leq \beta$, then $\rho_{\alpha\beta} = \rho_{\mu\beta} \circ \chi^\mu_\alpha$, where χ^μ_α is the canonical embedding from F_α into F_μ, as described in Chapter 2.

Let $F_\gamma = \varinjlim_{\alpha<\gamma} F_\alpha$. Note that $\gamma < \omega_1$ means that F_γ is countable, since $\bigsqcup_{\alpha<\gamma} F_\alpha$ is the disjoint union of countably many countable sets.

Let $D = \varinjlim_{\alpha<\omega_1} F_\alpha$, then $|D| = \aleph_1$ and, by construction, D has the CEP. ℵ

So it is possible to create a size-$\aleph_1$ $\mathcal{K}$-structure with the CEP, provided that $\mathcal{K}$ is a Fraïssé class. If it is also assumed that all of the structures in $\mathcal{K}$ are finite, then the desired structure can be constructed using what is known as an ultrapower of the Fraïssé limit.

Definition. Let λ be an infinite cardinal and let $A_\alpha : \alpha < \lambda$ be a family of L-structure for a signature L. Let u be an ultrafilter for λ. The *ultraproduct* of the A_α's over u is defined as the space $\prod A_\alpha/u$, where if $f, g \in \prod A_\alpha$, then $f \cong g \mod u$ if $\{n : f(n) = g(n)\} \in u$. If Each A_α is the same structure, then it is instead called an *ultrapower* and written as A^λ/u.

Ultrapowers have been used multiple times (such as in [1], [16], [17], and [18]) as a means of creating new, often much larger structures that retain many of the nice properties of the original structure. In particular, Łoś showed in [19] that for any finite $B \subset A^\lambda$ and a first order statement p with parameters in $\bar{B}$, p is true in D

if and only if the set of α such that p (with parameters in B_α) is true in A_α is an element of u. This theorem will be used to show the following:

Theorem 3.3.3. *Let $\mathcal{K}$ be a class of finite L-structures. If $\mathcal{K}$ is a Fraïssé class, Then F^ω/u has the CEP, where F is the Fraïssé limit for $\mathcal{K}$ and u is a nonprincipal ultrafilter on ω.*

Proof. Let F be the Fraïssé limit for $\mathcal{K}$. Let u be a non-principal ultrafilter on ω and let $D = F^\omega/u$ be the ultrapower of F. Let A and B be countable substructures of D such that $A \subset B$. Since elements of A and B are congruence classes of functions, let $(f_n : n < \omega)$ and $(g_n : n < \omega)$ be subsets of F^ω such that $A = \{[f_n] : n < \omega\}$, $B = \{[g_n] : n < \omega\}$, and for each $n < \omega$ there is some $m < \omega$ such that $f_n = g_m$.

Let $\varphi : A \to D$ be an embedding. Let $(f'_n : n < \omega) \subset F^\omega$ such that for all n, $\varphi([f_n]) = [f'_n]$ and let $A' = \{[f'_n] : n < \omega\} = \varphi[A]$. For each $n < \omega$, let

- $a_n = \langle f_i(n) : i \leq n\rangle_F$
- $a'_n = \langle f'_i(n) : i \leq n\rangle_F$ and
- $b_n = \langle a_n \cup \{g_i(n) : i \leq n\}\rangle_F$.

Since D is an ultrapower, Łoś's Theorem shows that since $A \cong A'$ and are finite, the set $\{m \in \omega : a_m \cong a'_m\}$ is an element of the ultrafilter u. For each aforementioned m, let $\varphi_m : a_m \to F$ be the embedding where for each $i \leq m$, $\varphi_m(f_i(m)) = f'_i(m)$. Since F has the FEP, there exists an embedding $\psi_m : b_m \to F$ which extends φ_m.

Let $(g'_n : n < \omega)$ be a sequence of functions in F^ω such that if $g_m = f_n$, then $g'_m = f'_n$, and the set $\{m \in \omega : b'_m = \psi_m[b_m]\}$ is an element of u, where $b'_m = \langle g'_i(n) : i \leq n\rangle_F$. Let $\psi : B \to D$ where for each $n < \omega$, $\psi([g_n]) = [g'_n]$. Again due to Łoś's Theorem, $\{m \in \omega : b_n \cong b'_n\}$ is an element of u, so ψ is an embedding from B to D, and $\psi \restriction A = \varphi$. Hence, $D = F^\omega/u$ has the CEP. ⋈

It is worth mentioning that CH is not necessary for the ultrapower of a Fraïssé limit to have the CEP, but it is necessary for F^ω/u to be size-$\aleph_1$, and it is only when the given structure is of this size that the CEP implies the other previously mentioned properties. For example, Roitman showed in [16] that by adding $\aleph_2$ many Cohen reals, for instance, it is possible to find nonprincipal ultrafilters on omega u and v such that F^ω/u and F^ω/v are not isomorphic.

Lastly, it is worth acknowledging that the amalgamation property is again a necessity for a class $\mathcal{K}$ to be the age of a structure with the CEP.

Theorem 3.3.4. *If D is an L-structure with the CEP, then* Age(D) *is a Fraïssé class.*

Proof. Let D be an L-structure with the CEP. Then D clearly has the FEP and by theorem 2.4.2, Age(D) is a Fraïssé class. ⁂

Meaning that in order for a class $\mathcal{K}$ to be the age of a structure with the CEP, it must already be able to construct a structure with with the FEP.

3.4 Genericity of the CEP

Under CH, If D is a size-$\aleph_1$ L-structure with the CEP for some signature L, then D will possess the same properties as a Fraïssé limit, but one cardinal higher than before. However, the generic nature of the Fraïssé limit as described in chapter 1 does not translate as directly. This is because unlike with the FEP, there are uncountably many instances of the CEP. This would mean needing to take an uncountable intersection of dense open sets in the corresponding sequence space, to which Baire Category Theorem would no longer apply.

However, if the topology were made finer, then the concept of genericity in this sense could be generalized so that it aligns with larger structures.

Definition. Let κ be a regular infinite cardinal and let κ^κ be the space of all functions $f : \kappa \to \kappa$. Consider the topology on κ^κ defined by a basis of initial segments, meaning that given a function $k : \beta \to \kappa$ for some $\beta < \kappa$, the set $[\![k]\!] = \{f \in \kappa^\kappa : k \subset f\}$ is a basic open set.

For a regular cardinal κ, κ^κ with the above topology acts as a generalization of the Baire Space, ω^ω, where the intersection of $\lambda < \kappa$ many dense open sets is a dense open set. It was loosely shown in [20] and [21] that the Baire Category Theorem has a generalized form in these spaces.

Theorem 3.4.1. *If $\{U_\gamma : \gamma < \kappa\}$ is a collection of dense open subsets of κ^κ, then $\bigcap\{U_\gamma : \gamma < \kappa\}$ is dense in κ^κ.*

Proof. Let $[\![k_0]\!]$ be a nonempty basic open subset of κ^κ, where $k_0 : \beta_0 \to \kappa$ for some $\beta_0 < \kappa$. Since U_0 is a dense open subset of κ^κ, there exists a $\beta_1 < \kappa$ and a function $k_1 : \beta_1 \to \kappa$ such that $[\![k_1]\!]$ is a nonempty open subset of $[\![k_0]\!] \cap U_0$.

This process can be continued recursively at successor steps, where if $[\![k_\gamma]\!]$ is a nonempty open subset of $[\![k]\!] \cap \bigcap\{U_\alpha : \alpha < \gamma\}$ for some $\gamma < \kappa$, then there exists a $\beta_{\gamma+1} < \kappa$ and a function $k_{\gamma+1} : \beta_{\gamma+1} \to \kappa$ such that $[\![k_{\gamma+1}]\!]$ is a nonempty open subset of $[\![k_0]\!] \cap \bigcap\{U_\alpha : \alpha \leq \gamma\}$.

If γ is a limit ordinal below κ, then since κ is regular, $\beta_\gamma = \lim_{\alpha\to\gamma} \beta_\alpha$ is also below κ. Let $k_\gamma = \bigcup\{k_\alpha : \alpha < \gamma\}$. then $k_\gamma : \beta_\gamma \to \kappa$ and $[\![k_\gamma]\!]$ is a nonempty open subset of $[\![k_0]\!] \cap \bigcap\{U_\alpha : \alpha < \gamma\}$ and thus there exists a $\beta_{\gamma+1} < \kappa$ as well as a function $k_{\gamma+1} : \beta_{\gamma+1} \to \kappa$ such that $[\![k_{\gamma+1}]\!]$ is a nonempty open subset of $[\![k_0]\!] \cap \bigcap\{U_\alpha : \alpha \leq \gamma\}$.

Let $k = \bigcup\{k_\alpha : \alpha < \kappa\}$. Then k is a function in κ^κ such that $k \in [\![k_0]\!]$ and $k \in \bigcap\{U_\alpha : \alpha < \kappa\}$. Since the choice of $[\![k_0]\!]$ was arbitrary, this shows that the intersection is dense in κ^κ. ⁂

Now that it has been shown that, under this finer topology, density can be retained after taking larger intersections, the idea of genericity scaled up to a larger cardinal can be made clear.

Definition. Let κ be a regular infinite cardinal and Let X be a topological space homeomorphic to κ^κ with the topology described above. To say that a *κ-generic* $x \in X$ has property P means that there exists a size κ intersection of dense open subsets of X such that each element of the intersection has property P.

Definition. For a signature L, let $\mathcal{K}$ be a countable set of finitely generated L-structures. For a cardinal λ, let $\mathcal{K}(\lambda)$ be a set of $\mathcal{K}$-structures D such that $|D| < \lambda$ and for each $\mathcal{K}$-structure C with $|C| < \lambda$, there is exactly one $D \in \mathcal{K}(\lambda)$ such that $D \cong C$.

Define $S_\mathcal{K}(\lambda)$ to be the set of all sequences $((\varphi_\alpha, D_\alpha) : \alpha < \lambda)$ of pairs $(\varphi_\alpha, D_\alpha)$ such that φ_0 is the empty embedding and for all subsequent $\alpha, \beta, \gamma < \lambda$,

- $D_\alpha \in \mathcal{K}(\lambda)$
- $\varphi_\alpha : D_{\alpha-1} \to D_\alpha$
- if γ is a limit, then $D_\gamma = \varinjlim_{\alpha<\gamma} D_\alpha$ and φ_γ is the identity map on D_γ.

Define the topology on $S_\mathcal{K}(\lambda)$ from initial segments, meaning a basic open set of $S_\mathcal{K}(\lambda)$, $[\![((\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha))]\!]$, would be the set of all sequences who agree on the first α many embeddings for some α.

Since $\mathcal{K}$ is countable, under CH there are $\aleph_1$ many distinct countable $\mathcal{K}$-structures, as well as $\aleph_1$ many possible embeddings between them. This means that $S_\mathcal{K}(\omega_1)$ is homeomorphic to the space of all functions from ω_1 to ω_1 under this new topology. It can now be shown that *almost surely*, a sequence in $S_\mathcal{K}(\omega_1)$ will produce a structure with the CEP, so long as $\mathcal{K}$ is a Fraïssé class.

Theorem 3.4.2. *(CH) Let L be a countable signature and let $\mathcal{K}$ be a countable set of finitely generated L-structures. If $\mathcal{K}$ is a Fraïssé class, then the direct limit of an ω_1-generic sequence in $S_{\mathcal{K}}(\omega_1)$ will have the CEP.*

Proof. Let P be the set of all pairs of embeddings (ψ, ρ) between structures in $\mathcal{K}(\omega_1)$ such that $\mathrm{Dom}(\psi) = \mathrm{Dom}(\rho)$. Note that P has size $\aleph_1$ under CH.

For a given $(\psi, \rho) \in P$, let $A = \mathrm{Dom}(\psi) = \mathrm{Dom}(\rho)$ and $B = \mathrm{Ran}(\psi)$. Define $U_{\psi\rho}$ to be the union of all basic open sets $[\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha), (\varphi_{\alpha+1}, D_{\alpha+1})]\!]$ in $S_{\mathcal{K}}$ such that either

- $\mathrm{Ran}(\rho) \neq D_\alpha$, or
- $\mathrm{Ran}(\rho) = D_\alpha$ and there exists an embedding $\psi' : B \to D_{\alpha+1}$ such that $\varphi_{\alpha+1} \circ \rho = \psi' \circ \psi$

Let $(\psi, \rho) \in P$ where $A = \mathrm{Dom}(\psi) = \mathrm{Dom}(\rho)$ and $B = \mathrm{Ran}(\psi)$. Clearly, $U_{\psi\rho}$ forms an open subset of $S_{\mathcal{K}}(\omega_1)$. To show that $U_{\psi\rho}$ is dense, let $[\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha)]\!]$ be a basic open subset of $S_{\mathcal{K}}(\omega_1)$. Suppose that $\mathrm{Ran}(\rho) \neq D_\alpha$. By letting $D_{\alpha+1} = D_\alpha$ and $\varphi_{\alpha+1}$ be identity map on D_α, the set $[\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha), (\varphi_{\alpha+1}, D_{\alpha+1})]\!]$ is a basic open contained in $U_{\psi\rho} \cap [\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha)]\!]$.

Now suppose $\mathrm{Ran}(\rho) = D_\alpha$. Since A, B and D_α are each countable $\mathcal{K}$-structures, they can be written as an ascending union of finitely generated substructures $(a_n : n < \omega)$, $(b_n : n < \omega)$, and $(d_n : n < \omega)$ such that for each $n < \omega$, $\rho[a_n] \subset d_n$ and $\psi[a_n] \subset b_n$. Since $\mathcal{K}$ is a Fraïssé class, there exists an ascending sequence of $\mathcal{K}$-structures $(e_n : n < \omega)$ whose direct limit is a countable $\mathcal{K}$-structure E (without loss of generality) in $\mathcal{K}(\omega_1)$ and for each $n < \omega$, there exist embeddings $\rho'_n : d_n \to e_n$ and $\psi'_n : b_n \to e_n$ satisfying $\rho'_n \circ \rho = \psi'_n \circ \psi$. Let $f_{\beta+1} = \bigcup f'_n$ and $g = \bigcup g'_n$. Let $D_{\alpha+1} = E$ and define $\varphi_{\alpha+1}$ such that for all n, $\varphi_{\alpha+1} \restriction d_n = \rho'_n$. Then $[\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha), (\varphi_{\alpha+1}, D_{\alpha+1})]\!]$ is a basic open contained in $U_{\psi\rho} \cap [\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha)]\!]$.

Let

$$X = \bigcap \{U_{\psi,\rho} : (\psi, \rho) \in P\}$$

Then X is an intersection of $\aleph_1$ many dense open subsets of $S_{\mathcal{K}}(\omega_1)$ and thusly forms a dense G_δ.

Let $((\varphi_\alpha, D_\alpha) : \alpha < \omega_1) \in X$ and let $D = \varinjlim D_\alpha$. To show that D has the CEP, Let A, B be countable substructures of D and let $\psi : A \to B$ and $\rho : A \to D$ be embeddings. Since A is countable and D is a direct limit with size-$\aleph_1$, there exists some $\alpha < \omega_1$ such that every element of $\rho[A]$ has a representative in D_α. Recall that $\chi_\alpha : D_\alpha \to D$ is the embedding mapping each element in D_α to its congruence class. Then the function $\rho' = \chi_\alpha^{-1} \circ \rho$ is an embedding from A into D_α. Since $D \in X \subset U_{\psi\rho'}$ there exists some $\psi' : B \to D_{\alpha+1}$ such that $\varphi_{\alpha+1} \circ \rho' = \psi' \circ \psi$.

Then $\chi_{\alpha+1} \circ \psi'$ is an embedding from B into D and, since D is a direct limit,

$$\chi_{\alpha+1} \circ \psi' \circ \psi = \chi_\alpha \circ \chi_\alpha^{-1} \circ \rho = \rho$$

$\aleph$

CHAPTER 4: The w-CEP

As seen in the previous chapter, if a class $\mathcal{K}$ is a Fraïssé class, then $\mathcal{K}$ admits a structure of size-$\aleph_1$ with the CEP under CH. In this chapter, focus will shift to certain classes with the HP and the JEP, but lack amalgamation. It is known that without Amalgamation, a class cannot have a Fraïssé limit. That being said, it will be shown that there is a weakening of the extension properties discussed thus far that still achieves interesting results at $\aleph_1$, and will have age equal to a class with a similarly weaker form of amalgamation.

4.1 Ages Without Amalgamation

In order to begin this discussion, it would help to first provide some examples of non-Fraïssé classes. The first example worth mentioning has ties to the following algebraic dynamical system.

Definition. *The Shift Map* is the name given to the space $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$, where σ is the function on $\mathcal{P}(\omega)/\mathrm{Fin}$ such that for any $A \subset \omega$, $\sigma([A]) = [A+1]$.

The dual of $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$ corresponds to an automorphism of the space of non-principal ultrafilters on ω, known as ω^*. It had been a longstanding open problem as to whether replacing φ with its inverse produced a structure isomorphic to $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$. Discussions on this problem can be found in [22], [23], [24] and [25]. It was then shown in [26] that the two spaces are indeed isomorphic under the assumption of CH. Much of this work stemmed from making arguments regarding the age of $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$.

Definition. The age of $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$ is the class $\mathcal{W}$ of boolean algebras paired with a relation R such that the dual class $\mathcal{W}^*$ is the class of strongly connected finite

directed graphs.

Because of the dual nature of these classes, it is often easier to acquire results in one class due to arguments in the other. In the case of $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$, the converse of the following fact can be shown that its age is not a Fraïssé class.

Fact. If $\mathcal{K}$ is a Fraïssé class, then its projective dual is a projective Fraïssé class.

Theorem 4.1.1. *$\mathcal{W}^*$ is not a projective Fraïssé class, meaning that $\mathcal{W}$ is thusly not a Fraïssé class.*

Without going into too much detail, it is enough to provide an example of a surjective amalgam in the class of finite strongly completed directed graphs that has no completion. In [26], the following amalgam, where each vertex is implied to have an edge with itself, is shown to not have a completion in the space of strongly connected directed graphs.

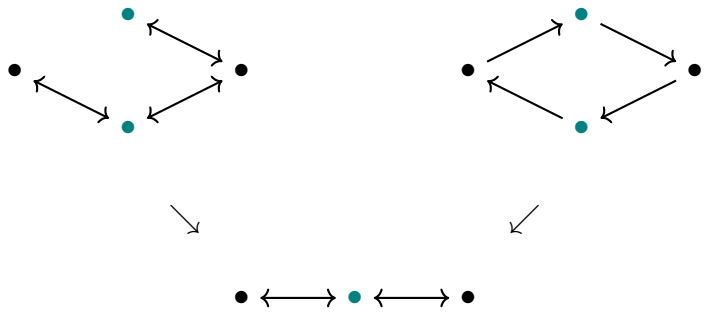


Figure 4.1: An Amalgam in $\mathcal{W}^*$

So the class of boolean algebras that is the age of $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$ lacks amalgamation. The following class serves as another example of a space without amalgamation.

Definition. Let $\mathcal{P}$ denote the class of finite planar graphs.

It was already mentioned earlier that a result in [9] shows that $\mathcal{P}$ does not admit a countable universal object, which is enough to confirm that it is not a Fraïssé class. This can be taken a step further to show that $\mathcal{P}$ is not even a weak Fraïssé class, meaning that it also lacks a generic countable isomorphism type.

Theorem 4.1.2. *$\mathcal{P}$ has the HP and JEP, but does not have the weak AP. Hence, $\mathcal{P}$ is not a Fraïssé class, nor does it have a generic isomorphism type.*

Proof. If $G \in \mathcal{P}$, then any induced subgraph of G will also be planar and hence in $\mathcal{P}$. Also, if G and H are planar, so is their disjoint union. Because of this, $\mathcal{P}$ has the HP and JEP.

To show that $\mathcal{P}$ lacks weak amalgamation, it is sufficient to show that there exists a graph such that for any extension of the graph A, there is an amalgam of planar graphs (A, φ, B, ψ, C) that cannot be completed by a planar graph.

Consider the graph K_3. Let A be a planar extension of K_3. Without loss of generality it can be assumed that A is connected. It is a well known result from Euler in [27] that a planar extension of a graph cannot reduce the number of faces, so the number of faces in A is at least two.

Let C be a boundary cycle in A, meaning a cycle of vertices that provide a boundary for a face in A. Given two vertices $v \in C$ and $u \in A - C$, say that u is independently connected to v with respect to C if there is a path p from u to v that does not contain any other vertices in C. From this definition, there are two possibilities to be concerned with in regards to C. Either

- there exist vertices $v_1, v_2, v_3 \in C$ and $u \in A - C$ such that u is independently connected to v_1, v_2 and v_3, or
- for any $v_1, v_2, v_3 \in C$, there is no $u \in A - C$ that is independently connected to all of them

with respect to C. In either case, the following amalgam can be made on A with respect to C (and u in the event that case one holds).

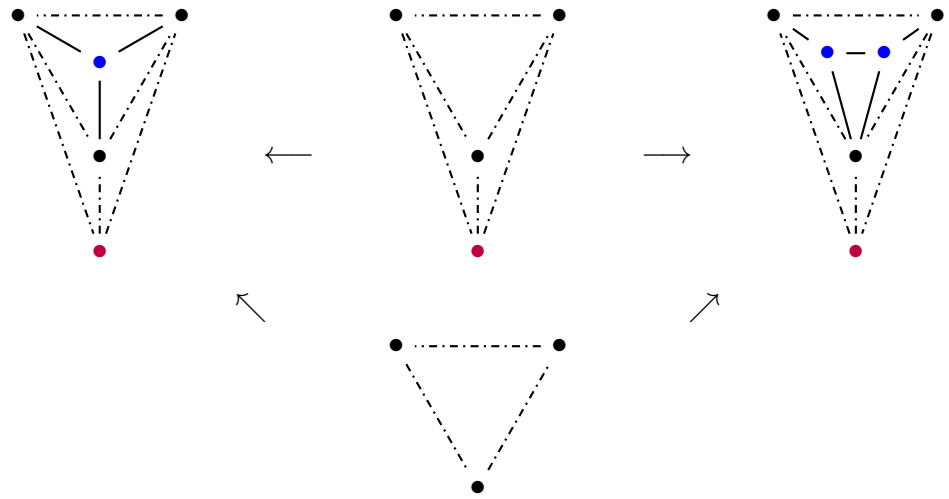


Figure 4.2: An amalgam of A given either case one (upper) or case two (lower)

Consider a graph D that completes the above amalgam. Note that since embeddings have to preserve the existence or nonexistence of an edge between vertices, the added vertices in the amalgam above must each be mapped to a different vertex in D. Because of this, it is possible to show that D must admit the graph $K_{3,3}$ as a minor. To see this, start by contracting all paths in D to the vertices and edges derived from the amalgam, then perform the following actions:

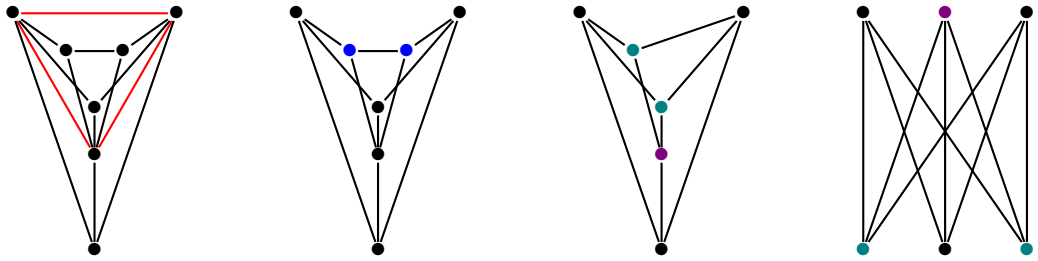


Figure 4.3: Caption

It is known thanks to the work of Kuratowski and Wagner in [28] and [29] that a graph cannot be planar if it admits $K_{3,3}$ as a minor. Because of this, $D \notin \mathcal{P}$. Since A was an arbitrary extension of K_3 in $\mathcal{P}$, this shows that $\mathcal{P}$ lacks the weak amalgamation property, so it neither has a Fraïssé limit nor a countable generic structure.

In the previous chapter, it was acknowledged in [1] and [16] that amalgamation is necessary for a class to be a Fraïssé class and thusly have a Fraïssé limit. Since $\mathcal{P}$ is not a Fraïssé class, there does not exist a countable structure with age $\mathcal{P}$ that has any of the nice properties seen thus far. Similarly, there does not exist a size-$\aleph_1$ structure with age $\mathcal{P}$ that has the CEP. However, while the CEP is sufficient, there is a weaker property that can be found in certain $\aleph_1$ size structures with ages that are not Fraïssé classes, yet still retain many of the other properties of Fraïssé limits.

4.2 Weak Countable Extensions

Until now, the ability for a structure to have a nice extension property depended on whether or not the space of its finitely generated substructures possessed amalgamation. Moving forward, it will be of interest to look at structures where extensions are denied only in instances where amalgamation would fail.

Definition. Let L be a signature and let D be an L-structure with age $\mathcal{K}$. Suppose that A and B are substructures of D such that $A \subset B$ and $\varphi : A \to D$ is an embedding. A structure $c \in \text{Age}(D)$ *obstructs* the pair (φ, B) if there exists an $a \in \text{Age}(A)$ and $b \in \text{Age}(B)$ such that $a \subset b$, $\varphi[a] \subset c$ and the amalgam created from the inclusion of a in b and $\varphi : a \to c$ cannot be completed in $\mathcal{K}$.

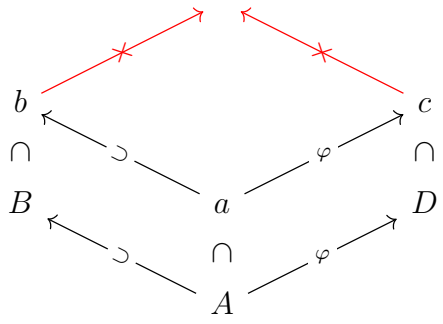


Figure 4.4: An obstruction of (φ, B)

Definition. Let L be a signature and D an L-structure. D has the *weak* Countable Extension Property (w-CEP) if whenever A and B are countable substructures of D such that $A \subset B$ and $\varphi : A \to D$ is an embedding, either

- there exists an embedding $\psi : B \to D$ which extends φ.
- there exists a $c \in \mathrm{Age}(D)$ that obstructs (φ, B).

In essence, An $\aleph_1$ sized structure with the w-CEP will not be able to achieve a particular countable extension only if there is a finitary piece of that extension that cannot be achieved.

4.3 Consequences of the w-CEP

In this case, the methods for showing that structures with the CEP have nice properties can be used as a framework for structures with the w-CEP by modifying which countable structures to extend.

In order to show that structures with the w-CEP possess the similar properties to the structures mentioned thus far, it will be useful to show that it possesses some additional properties not yet discussed.

Definition. Let A and B be L-structures. A and B are *elementarily equivalent*, denoted $A \equiv B$, if any first order statement constructed from L is true in A if and only if it is true in B. If $A \subset B$, then A is an elementary substructure, $A \preccurlyeq B$ if any first order statement constructed from L with parameters in A is true in A if and only if it is true in B. An embedding $\varphi : A \to B$ is an elementary embedding if $\varphi[A] \preccurlyeq B$.

Definition. Let A and B be L-structures. A and B are *elementarily equivalent*, denoted $A \equiv B$, if any first order statement constructed from L is true in A if and only if it is true in B. If $A \subset B$, then A is an elementary substructure, $A \preccurlyeq B$ if any

first order statement constructed from L with parameters in A is true in A if and only if it is true in B. An embedding $\varphi : A \to B$ is an elementary embedding if $\varphi[A] \preccurlyeq B$.

Theorem 4.3.1. *Let L be a signature and D be an L-structure. If D has the w-CEP, then D is $\aleph_0$-universal for its age.*

Proof. Let D be an L-structure with the w-CEP and let $\mathrm{Age}(D) = \mathcal{K}$. Let B be a countable $\mathcal{K}$-structure. Let η be the empty embedding into D. Since $\mathrm{Age}(B) = \mathcal{K} = \mathrm{Age}(D)$, for each $c \in \mathrm{Age}(B)$, there exists a substructure $d_c \subset D$ that is isomorphic to c. Let $\psi_c : c \to d_c$ be an isomorphism. Then ψ_c is an embedding from c into D which extends η. This means that no $c \in \mathrm{Age}(D)$ obstructs the empty embedding and, since D has the w-CEP, there must exist an embedding $\psi : B \to D$. Since B was arbitrary, this shows that D is $\aleph_0$ universal. ⋘

It is worth mentioning here that since the w-CEP implies $\aleph_0$-universality, the w-CEP can be generalized in a similar fashion to 2.1.1, where A and B are allowed to be arbitrary countable $\mathcal{K}$-structures. Since every countable structure embeds into a structure with the w-CEP, focusing on elementary structures will allow the next property to be achieved.

Theorem 4.3.2. *Let L be a signature and D be an uncountable L-structure. If D has the w-CEP, then D is $\aleph_1$-universal for its age.*

Proof. Let D be an L-structure with the w-CEP and let $\mathrm{Age}(D) = \mathcal{K}$. Let E be a size-$\aleph_1$ $\mathcal{K}$-structure. Let $E = \bigcup\{E_\alpha : \alpha < \omega_1\}$, where $(E_\alpha : \alpha < \omega_1)$ is an ascending sequence of countable elementary substructures of E such that, if γ is a limit ordinal below ω_1, then $E_\gamma = \bigcup\{E_\alpha : \alpha < \gamma\}$. Since D is $\aleph_0$ universal, there exists an embedding $\varphi_0 : E_0 \to D$.

Suppose $\varphi_\alpha : E_\alpha \to D$ is an embedding. Suppose for contradiction that there does not exist an embedding $\varphi' : E_{\alpha+1} \to D$ that extends φ_α. Since D has the w-CEP, this means that there exists a $c \in \mathcal{K}$ that obstructs $(\varphi_\alpha, E_{\alpha+1})$. Let $a \in \mathrm{Age}(E_\alpha)$ and $b \in \mathrm{Age}(E_{\alpha+1})$ such that $a \subset b$, φ_α embeds a into c, and the amalgam created from the inclusion map from a into b and φ_α cannot be completed. Because E_α is an elementary substructure of E, there exists a $b' \subset E_\alpha$ such that $b' \cong b$ and $a \subset b'$. Then $\langle \varphi(b') \cup c \rangle_D \in \mathcal{K}$ completes the previously mentioned amalgam, which is a contradiction. Hence, there must exist an embedding from $E_{\alpha+1} \to D$ which extends φ_α. Let φ_1 be the desired embedding from $E_{\alpha+1}$ into D.

If γ is a limit ordinal and $(\varphi_\alpha : \alpha < \gamma)$ is an ascending sequence of embeddings from E_α into D, then $\varphi_\gamma = \bigcup\{e_\alpha : \alpha < \gamma\}$ is an embedding from $E_\gamma \to D$. Let $\varphi = \bigcup\{\varphi_\alpha : \alpha < \omega_1\}$. Then φ is an embedding from E into D, showing that D is universal. ℵ

By taking advantage of elementarity, it was possible to show that any structure of size $\aleph_1$ with the w-CEP will be universal. By continuing to focus on elementarity, it is also possible to show that two structures with the w-CEP have uncountable isomorphic substructures.

Lemma 4.3.3. *Let L be a signature. Suppose that C and D are uncountable L-structures with the same age and both have the w-CEP. Then C and D have elementary substructures that are isomorphic. In fact, if $\varphi : A \to D$ is an embedding for some countable elementary substructure A of C, then there exist $C' \preccurlyeq C$, $D' \preccurlyeq D$ and an isomorphism $\psi : C' \to D'$ which extends φ.*

Proof. For a signature L, let C and D be uncountable L-structures with age $\mathcal{K}$ that both have the CEP. Let A be a countable elementary substructure of C and let

$\varphi : A \to D$ be an embedding. The isomorphism ψ will be constructed recursively via a back and forth argument.

Using the fact that C and D are uncountable, let $(x_\alpha : \alpha < \omega_1)$ and $(y_\alpha : \alpha < \omega_1)$ be sequences of elements in C and D respectively. Let C_0 be a countable substructure of C such that $x_0 \in C$ and $A \preccurlyeq C_0 \preccurlyeq C$. Then, since D has the w-CEP, by 4.3.2 there exists an embedding $\psi_0 : C_0 \to D$ that extends φ. Let D_0 be a countable substructure of D such that $y_0 \in D$ and $\psi_0[C_0] \preccurlyeq D_0 \preccurlyeq D$. Similarly, since C has the w-CEP, by 4.3.2 there exists an embedding $\rho_0 : D_0 \to C$ that extends ψ_0^{-1}.

Following this back and forth argument recursively and taking unions at limit steps defines a sequence of functions $(\psi_\alpha : \alpha < \omega_1)$ and $(\rho_\alpha : \alpha < \omega_1)$ as well as a sequence of countable structures $(C_\alpha : \alpha < \omega)$ and $(D_\alpha : \alpha < \omega)$ such that for all such α,

- $C_\alpha = \mathrm{Dom}(\psi_\alpha)$ and $D_\alpha = \mathrm{Dom}(\rho_\alpha)$
- $x_\alpha \in C_\alpha$ and $y_\alpha \in D_\alpha$.
- $C_\alpha \preccurlyeq C$ and $D_\alpha \preccurlyeq D$
- ψ_α extends φ.
- $\psi_\alpha \subset \rho_\beta^{-1} \subset \psi_\delta$ whenever $\alpha < \beta < \delta$.

Let $C' = \bigcup_{\alpha<\omega_1} C_\alpha$ and $D' = \bigcup_{\alpha<\omega_1} D_\alpha$, as well as let $\psi = \bigcup_{\alpha<\omega_1} \psi_\alpha$ and $\rho = \bigcup_{\alpha<\omega_1} \rho_\alpha$. Then C' and D' are both elementary substructures of C and D, and ψ and ρ are both embeddings between C and D and $\psi^{-1} = \rho$. Hence, ψ is an isomorphism between C' and D' which extends φ. ⇜

Lemma 4.3.4. *Structures with the w-CEP are elementarily equivalent*

Proof. Let C and D be $\mathcal{K}$-structures with the w-CEP. Then by 4.3.3, There exists elementary substructures $C' \preccurlyeq C$ and $D' \preccurlyeq D$ that are isomorphic. Since isomorphic structures are elementarily equivalent,

$$C \equiv C' \equiv D' \equiv D$$

meaning that $C \equiv D$. ⇜

So structures with the w-CEP are elementarily equivalent as well as $\aleph_1$-universal. If the structure has cardinality $\aleph_1$, then it will additionally satisfy a different form of homogeneity that focuses on elementary structures (which will be referred to as *elementary homogeneity*), as well as be isomorphic to similar structures of the same size.

Theorem 4.3.5. *Let L be a signature and C and D be size-$\aleph_1$ L-structures with age $\mathcal{K}$ and have the w-CEP. Then C and D are isomorphic and any elementary embedding of a countable substructure of D into D can be extended to an automorphism.*

Proof. If $(x_\alpha : \alpha < \omega_1)$ and $(y_\alpha : \alpha < \omega_1)$ are enumerations of C and D, then the proof of 4.3.2 will produce an isomorphism between them. Additionally, by letting $C = D$, the proof shows that elementary embeddings can be extended to automorphisms for structures with the w-CEP. ⇜

The last new property that will be shown as a consequence of the w-CEP involves a structures ability to satisfy countably many first order properties simultaneously.

Definition. Let D be an L-structure and let $A \subset D$. Let p be a collection of existential first order L-formulas with parameters in A. p is a *type* (over A) in D if for any finite subset of p, there exists an $x \in D$ that realizes all of them. A type is considered *complete* if it is maximal.

Let κ be a cardinal. D is *κ-saturated* if every complete type over a subset of D with cardinality less than κ is realized by some $x \in D$. If additionally $|D| = \kappa$, then D is *saturated*.

Theorem 4.3.6. *Let D be an L-structure with size $\aleph_1$. If D has the w-CEP, then D saturated.*

The proof for 4.3.6 immediately follows from 4.3.2 and 4.3.5 thanks to the following fact given in [17].

Fact. Given an L-structure D, D is saturated if and only if it is both universal and elementarily homogeneous.

4.4 psuedo-Fraïssé Structures

Recall that the age of any structure will have the hereditary and joint embedding properties. If a class $\mathcal{K}$ has these properties, but is not a Fraïssé class, it must then lack the amalgamation property. In order to guarantee that $\mathcal{K}$ can admit a size-$\aleph_1$ structure with the w-CEP, there needs to be a way to guarantee that if there is an ascending chain of amalgamations that can be satisfied in $\mathcal{K}$, then there is a way to construct countable structures from $\mathcal{K}$ that satisfy them all simultaneously.

Definition. Let $\mathcal{K}$ be a countable set of finitely generated L-structures, for some signature L. $\mathcal{K}$ is a *pseudo-Fraïssé Class* if $\mathcal{K}$ has the HP, JEP, and the following additional property:

$\star$ for any $a, b, c \in \mathcal{K}$ and embeddings $\varphi : a \to b$ and $\psi : a \to c$, if there exists a $d \in \mathcal{K}$ that completes the amalgam (a, φ, b, ψ, c), then there exists a finite collection $(d_i : i \leq n)$ of structures in $\mathcal{K}$ such that if $e \in \mathcal{K}$ completes (a, φ, b, ψ, c), then d_i embeds into e for some $i \leq n$.

Lemma 4.4.1. *Let L be a countable signature and let $\mathcal{K}$ be a pseudo-Fraïssé class of L-structures. Suppose that A, B, and C are countable $\mathcal{K}$-structures and $\varphi : A \to B$ and $\psi : A \to C$ are embeddings. If for all $a \in \mathrm{Age}(A)$, $b \in \mathrm{Age}(B)$ and $c \in \mathrm{Age}(C)$, the amalgam (a, φ, b, ψ, c) can be completed in $\mathcal{K}$, then there exists a countable $\mathcal{K}$-structure D and embeddings $\varphi' : B \to D$ and $\psi' : C \to D$ such that $\varphi' \circ \varphi = \psi' \circ \psi$*

Proof. Let A, B, and C be countable $\mathcal{K}$-structures and $\varphi : A \to B$, $\psi : A \to C$ be embeddings such that for all $a \in \mathrm{Age}(A)$, $b \in \mathrm{Age}(B)$ and $c \in \mathrm{Age}(C)$, the amalgam (a, φ, b, ψ, c) can be completed in $\mathcal{K}$. Since A is countable, let $A = \bigcup(a_n : n < \omega)$, where each $a_n \in \mathrm{Age}(A)$. Similarly, let $B = \bigcup(b_n : n < \omega)$ and $C = \bigcup(c_n : n < \omega)$ with the additional feature that for all $n < \omega$, $\varphi(a_n) \subset b_n$ and $\psi(A_n) \subset c_n$. Since for each $n < \omega$, the amalgam $(a_n, \varphi, b_n, \psi, c_n)$ can be completed and $\mathcal{K}$ is a pseudo-Fraïssé class, there exists a finite list of k_n many $\preceq$-minimal structures $d_n^0, \ldots, d_n^{k_n} \in \mathcal{K}$ such that for each $j \leq k_n$, d_n^j completes the amalgam $(a_n, \varphi, b_n, \psi, c_n)$.

Note that if $m > n$, then each structure in the list $d_m^0, \ldots d_m^{k_m}$ also completes $(a_n, \varphi, b_n, \psi, c_n)$. For $m > n$, $i < k_n$ and $j < k_m$, an embedding $\rho : d_n^i \to d_m^j$ is *amalgamation preserving* if there exist embeddings $\varphi'_n : b_n \to d_n^i$, $\varphi'_m : b_m \to d_m^j$, $\psi'_n : c_n \to d_n^i$ and $\psi'_m : c_m \to d_m^j$ such that $\varphi'_m \circ \varphi = \rho \circ \varphi'_n \circ \varphi$ and $\psi'_m \circ \psi = \rho \circ \psi'_n \circ \psi$

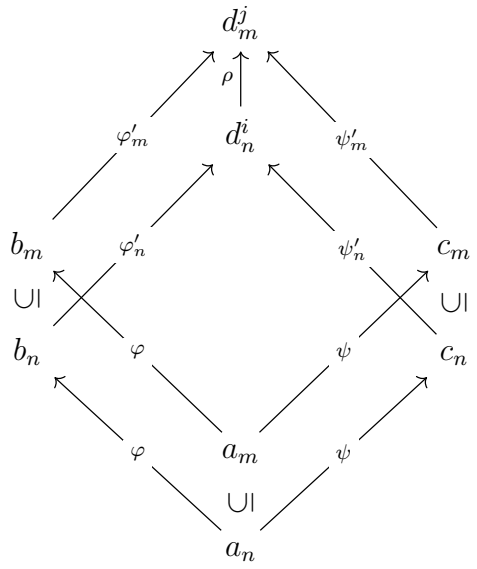


Figure 4.5: Preserving Amalgamation

Let $\mathcal{D}$ be the space consisting of the empty set and all such $d_n^0, \ldots, d_n^{k_n}$ for $n < \omega$. Define a relation $\hookrightarrow$ on the $\mathcal{D}$ where $\varnothing \hookrightarrow d_0^j$ for each $j < k_0$ and $d_n^i \hookrightarrow d_m^j$ if and only if $m = n + 1$ and there exists an amalgamation preserving embedding from d_n^i to d_m^j. Note that since each of these structures are $\preceq$-minimal, for a given d_m^j, if φ'_m and ψ'_m are embeddings from b_m and c_m into d_m^j that witness the completion of the amalgamation of $(a_m, \varphi, b_m, \psi, c_m)$, then for each $n < m$, there exists a $i \leq k_n$ such that $d_n^i \preceq \langle \varphi'_m(b_n) \cup \psi'_m(c_n) \rangle_{d_m^j}$. This means that for each d_m^j, if $n < m$ then there exists an amalgamation preserving embedding $\rho : d_n^i \to d_m^j$ for some $i \leq k$. Then $(\mathcal{D}, \hookrightarrow)$ forms a connected, countably infinite, locally finite graph. By König's Lemma [30], there exists a countably infinite chain of structures $(d_n : n < \omega)$ and amalgamation preserving embeddings $(\rho_n : n < \omega)$ through $\mathcal{D}$. Let D be the direct limit of this chain. Define the embeddings $\varphi' : B \to D$ and $\psi' : C \to D$, where if $x \in b_n$ and $y \in c_n$ for some $n < \omega$, then $\varphi'(x) = \chi_n(\varphi'_n(x))$ and $\psi'(x) = \chi_n(\psi'_n(x))$, where φ'_n and ψ'_n were witnesses to d_n's completion of $(a_n, \varphi, b_n, \psi, c_n)$ and χ_n is the canonical embedding from d_n into D.

$\aleph$

Theorem 4.4.2. *(CH) Let L be a countable signature and let $\mathcal{K}$ be a countable set of finitely generated L-structures. If $\mathcal{K}$ is a psuedo-Fraïssé class, then there exists a $\mathcal{K}$-structure of size-$\aleph_1$ with the w-CEP.*

Proof. Let $\mathcal{K}$ be a psuedo-Fraïssé class. Since $\mathcal{K}$ is essentially countable, there exist $\aleph_1$ many distinct countable $\mathcal{K}$-structures.

Let P be an $\aleph_1$ size set of pairs (A, B) of countable $\mathcal{K}$-structures such that $A \subseteq B$ and for all countable $\mathcal{K}$-structures, A', B', if there exists an embedding $\varphi : A' \to B'$, then there is an $(A, B) \in P$ such that A' is isomorphic to A, B' is isomorphic to B, and the inlusion map of A into B is isomorphic to φ. Similar to in Chapter 3, P will be used as a list of (potential) possible extensions and, leveraging the fact that $\mathcal{K}$ is a psuedo-Fraïssé class, construct a chain of embeddings between countable $\mathcal{K}$-structures whose direct limit is an $\aleph_1$ size structure with the w-CEP.

Let $\pi : \omega_1 \times \omega_1 \to \omega_1$ be a bijection such that $\pi(\alpha, \beta) \geq \alpha$ for all $\alpha, \beta < \omega_1$. Let D_0 be a countable L-structure with age $\mathcal{K}$. Inductively, whenever D_γ is chosen, let $((\varphi_{\gamma\beta}, A_{\gamma\beta}, B_{\gamma\beta}) : \beta < \omega_1)$ enumerate all embeddings $\varphi_{\gamma\beta} : A_{\gamma\beta} \to D_\gamma$ such that $(A_{\gamma\beta}, B_{\gamma\beta}) \in P$. Then, if $\pi(\alpha, \beta) = \gamma$, $\varphi_{\alpha\beta}$ embeds $A_{\alpha\beta}$ into D_γ and the inclusion map ι embeds $A_{\alpha\beta}$ into $B_{\alpha\beta}$.

Suppose that there exists a structure $c \in \mathrm{Age}(D_\gamma)$ that obstructs $(\varphi_{\alpha\beta}, B_{\alpha\beta})$. In this case, let $D_{\gamma+1} =_\gamma$ and let ρ_γ be the identity map on D_γ. Now suppose no such c exists. Then, by 4.4.1, there exists a countable $\mathcal{K}$-structure E and embeddings $\rho' : D_\gamma \to E$ and $\psi_B : B \to E$ such that

$$\psi_B \restriction A = \rho_\gamma \circ \varphi_{\alpha\beta}$$

In this case, let $D_{\gamma+1} = E$ and let $\rho_\gamma = \rho'$.

At limit stages, let $D_\gamma = \varinjlim\{D_\alpha : \alpha < \gamma\}$. Let $D = \varinjlim\{D_\gamma : \gamma < \omega_1\}$. Since D_0

is embeddable in D, D has Age(K). Lastly, if A and B are countable substructures of D such that $A \subset B$ and $\varphi : A \to D$ is an embedding, then there exists an α and β such that $A \cong A_{\alpha\beta}$, $B \cong B_{\alpha\beta}$ and φ is isomorphic to $\varphi_{\alpha\beta}$. This means that for $\gamma = \pi(\alpha, \beta)$, either there exists a $c \in \text{Age}(D_\gamma)$ that obstructs ($\varphi_{\alpha\beta}$, or there exists an embedding $\psi : B \to D_{\gamma+1}$ such that if $\chi_{\gamma+1}$ is the canonical embedding of $D_{\gamma+1} \to D$, then $\chi_\gamma \circ \psi$ extends φ. This shows that D has the w-CEP. ⇜

4.5 Genericity of the w-CEP

Theorem 4.5.1. *(CH) Let L be a countable signature and let $\mathcal{K}$ be a countable set of finitely generated L-structures. If $\mathcal{K}$ is a pseudo-Fraïssé class, then the direct limit of an ω_1-generic sequence in $S_\mathcal{K}(\omega_1)$ will have the w-CEP.*

Proof. Let P be the set of all pairs of embeddings (ψ, ρ) between structures in $\mathcal{K}(\omega_1)$ such that $\text{Dom}(\psi) = \text{Dom}(\rho)$. Note that P has size $\aleph_1$ under CH.

For a given $(\psi, \rho) \in P$, let $A = \text{Dom}(\psi) = \text{Dom}(\rho)$ and $B = \text{Ran}(\psi)$. Define $U_{\psi\rho}$ to be the union of all basic open sets $[\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha), (\varphi_{\alpha+1}, D_{\alpha+1})]\!]$ in $S_\mathcal{K}$ such that either

- $\text{Ran}(\rho) \neq D_\alpha$,
- $\text{Ran}(\rho) = D_\alpha$ and there exists a $c \in \text{Age}(D_\alpha)$ that obstructs (ρ, B), or
- $\text{Ran}(\rho) = D_\alpha$ and there exists an embedding $\psi' : B \to D_{\alpha+1}$ such that $\varphi_{\alpha+1} \circ \rho = \psi' \circ \psi$

Let $(\psi, \rho) \in P$ where $A = \text{Dom}(\psi) = \text{Dom}(\rho)$ and $B = \text{Ran}(\psi)$. Clearly, $U_{\psi\rho}$ forms an open subset of $S_\mathcal{K}(\omega_1)$. To show that $U_{\psi\rho}$ is dense, let $[\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha)]\!]$ be a basic open subset of $S_\mathcal{K}(\omega_1)$. Suppose that either $\text{Ran}(\rho) \neq D_\alpha$ or $\text{Ran}(\rho) = D_\alpha$ and there exists a $c \in \text{Age}(D_\alpha)$ that obstructs (ρ, B). In either case, by letting $D_{\alpha+1} = D_\alpha$

and $\varphi_{\alpha+1}$ be identity map on D_α, the set $[\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha), (\varphi_{\alpha+1}, D_{\alpha+1})]\!]$ is a basic open contained in $U_{\psi\rho} \cap [\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha)]\!]$.

Now suppose $\mathrm{Ran}(\rho) = D_\alpha$ and no such obstruction exists. Then, by lemma 4.4.1, there exists a countable $\mathcal{K}$-structure E and embeddings $\varphi' : D_\alpha \to E$ and $\psi' : B \to E$ such that $\varphi' \circ \rho = \psi' \circ \psi$. Let $(\varphi_{\alpha+1}, D_{\alpha+1}) = (\varphi', E)$. Then $[\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha), (\varphi_{\alpha+1}, D_{\alpha+1})]\!]$ is a basic open contained in $U_{\psi\rho} \cap [\![(\varphi_0, D_0), \ldots, (\varphi_\alpha, D_\alpha)]\!]$.

Let

$$X = \bigcap \{U_{\psi,\rho} : (\psi, \rho) \in P\}$$

Then X is an intersection of $\aleph_1$ many dense open subsets of $S_{\mathcal{K}}(\omega_1)$ and thusly forms a dense G_δ.

Let $((\varphi_\alpha, D_\alpha) : \alpha < \omega_1) \in X$ and let $D = \varinjlim D_\alpha$. To show that D has the w-CEP, Let A, B be countable substructures of D and let $\psi : A \to B$ and $\rho : A \to D$ be embeddings. Further suppose there is no $c \in \mathcal{K}$ that obstructs (ρ, B). Since A is countable and D is a direct limit with size-$\aleph_1$, there exists some $\alpha < \omega_1$ such that every element of $\rho[A]$ has a representative in D_α. Recall that $\chi_\alpha : D_\alpha \to D$ is the embedding mapping each element in D_α to its congruence class. Then the function $\rho' = \chi_\alpha^{-1} \circ \rho$ is an embedding from A into D_α. Since $D \in X \subset U_{\psi\rho'}$ there exists some $\psi' : B \to D_{\alpha+1}$ such that $\varphi_{\alpha+1} \circ \rho' = \psi' \circ \psi$.

Then $\chi_{\alpha+1} \circ \psi'$ is an embedding from B into D and, since D is a direct limit,

$$\chi_{\alpha+1} \circ \psi' \circ \psi = \chi_\alpha \circ \chi_\alpha^{-1} \circ \rho = \rho$$

$\aleph$

So under CH, these continuum size structures are uniquely characterized by the w-CEP. In addition, they represent both a generic and universal object in the class of continuum size structures with the same age. This makes them serve the same role as the extensions of the Fraïssé limits discussed in Chapter 3, even though the age of

these structures is not a Fraïssé class.

Also similar to in Chapter 3, it is important to mention the necessity of CH for these objects to be uniquely characterized by the w-CEP. As an example, consider again the space $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$. Thanks to the work of van Douwen and Shelah in [31] and [32], it was shown that assuming the proper forcing axiom (which implies $\mathfrak{c} = \aleph_2$), that $\langle \mathcal{P}(\omega)/\mathrm{Fin}, \sigma \rangle$ and its inverse are not isomorphic. Through the more recent results in [33], [34], and [35] it was shown that the open coloring axiom (which implies $\mathfrak{c} \geq \aleph_2$) was sufficient in proving this claim. This results assert that having $\mathfrak{c} = \aleph_1$ is essential for these structures and other structures of the same size with the w-CEP to be unique up to isomorphism.

CHAPTER 5: Generalizations of Extension Properties

A natural question that arises from these results would be whether or not these concepts translate to higher cardinals. In order for similar properties to hold for a given cardinal λ, the following assumption will be made.

$$\lambda^{<\lambda} = \lambda$$

This hypothesis is frequently made in model theoretic works such as [36] and [21]. In this case, the primary benefit to this assumption is that it makes it possible to construct a sequence of length λ that includes, up to isomorphism, every $\mathcal{K}$-structure with size less than λ. Note that under the GCH, which states that $\lambda^+ = 2^\lambda$ for each cardinal λ, that every regular cardinal satisfies this hypothesis. In the more general case, this hypothesis is satisfied when λ is a strong limit cardinal, as described in [37].

5.1 The weak κ-Extension Property

Recall from chapter 3 that a structure D has the countable extension property if whenever A is a countable structure embedded into D and B a countable structure containing A, there is a way to extend the embedding to B. Then the weak countable extension property described in chapter 4 says such an extension is possible provided that there is no finitary obstruction. Therefore the results presented here will focus on generalizing the weak version, since a structure with the stronger property will in turn also satisfy the weaker.

Definition. Let L be a signature and D an L-structure. For an infinite cardinal λ, D has the *weak* λ-Extension Property (weak λ-EP) if whenever A and B are substructures of D such that $|B| < \lambda$ and $A \subset B$, and $\varphi : A \to D$ is an embedding, either

- there exists an embedding $\psi : B \to D$ which extends φ.
- there exists a $c \in \text{Age}(D)$ that obstructs (φ, B).

Since much of the work of chapter 4 translates directly into the higher cardinal case, several of the nice properties of structures with the weak λ-EP can rely on the exact techniques used for structures with the weak CEP.

Theorem 5.1.1. *Let λ be an infinite cardinal and let D be an L-structure with size λ. If D has the weak λ-EP, then with respect to its age D is, up to isomorphism, the unique universal, generic, elementarily homogeneous, saturated structure with this property.*

Proof. Apply the proofs of 4.3.2, 4.3.3, 4.3.5, and 4.5.1, but with a recursive argument of length λ. ⁂

5.2 Building Structures with the weak λ-EP

While it is quick at this point to assert that structures with the weak λ-EP have the desired properties discusses in this paper, constructing them requires a more technical argument. In order to build these structures, it will be beneficial to identify properties in the space of $\mathcal{K}$-structures with size less than λ.

Theorem 5.2.1. *Let $\mathcal{K}$ be either a Fraïssé or pseudo-Fraïssé class for a signature L. Let μ be an infinite cardinal and let A, B and C be $\mathcal{K}$-structures of size μ. Suppose that $\varphi : A \to B$ and $\psi : A \to C$ are embeddings such that every finite sub-amalgam of (A, φ, B, ψ, C) can be completed in $\mathcal{K}$, then there exists a $\mathcal{K}$-structure D with size at most 2^μ and embeddings $\varphi' : B \to D$ and $\psi' : C \to D$ such that $\varphi' \circ \varphi = \psi' \circ \psi$.*

Before going into the proof of this theorem, it will be necessary to introduce a new type of ultrafilter that respects the change in cardinality. An ultrafilter u on a cardinal μ is *regular* if it contains a collection of sets $\{A_\alpha : \alpha < \mu\}$ such that each $\beta < \mu$ is only contained in finitely many A_α's. Regular ultrafilters are used frequently in model theoretic arguments because they have the ability to create saturated ultrapowers of models, provided that the the theory relatively simple. For this proof, the following fact will be useful:

Fact. If u is a regular ultrafilter over μ and A is a subset of μ such that $|A| < \mu$, then $A \notin u$.

Proof. This proof will be done via induction on μ. Let A, B and C be $\mathcal{K}$-structures with size μ and let $\varphi : A \to B$ and $\psi : A \to C$ be embeddings. Suppose that every finite sub-amalgam of (A, φ, B, ψ, C) can be completed in $\mathcal{K}$ and further suppose that if $\aleph_0 \leq \kappa < \mu$, then any size κ sub-amalgam can be completed with a $\mathcal{K}$-structure of size less than or equal to 2^κ. Note that if $\mu = \aleph_1$ and the first supposition holds, the second holds by 3.3.1 if $\mathcal{K}$ is a Fraïssé class and by 4.4.1 if it is a psuedo-Fraïssé class.

Let A, B and C be written as an increasing union of structures $(A_\alpha, \alpha < \mu)$, $(B_\alpha, \alpha < \mu)$ and $(C_\alpha, \alpha < \mu)$ such that for all $\alpha < \mu$, $\varphi[A_\alpha] \subset B_\alpha$, $\psi[A_\alpha] \subset C_\alpha$, and $|A_\alpha| = |B_\alpha| = |C_\alpha| < \mu$. For each sub-amalgam $(A_\alpha, \varphi, B_\alpha, \psi, C_\alpha)$, let D_α be a $\mathcal{K}$-structure with size less than or equal to 2^κ that completes it, where $\kappa = |A|$. Let φ'_α and ψ'_α be embeddings from B_α and C_α into D_α that witness D_α as a completion to the amalgam.

Let u be a regular ultrafilter over μ and let $D = \prod D_\alpha / u$. Note that $|D| \leq (2^{<\mu})^\mu = 2^\mu$. For each $x \in B$ and $y \in C$, let $f_x, g_y \in \prod D_\alpha$ such that $f_x(\alpha) = \varphi'_\alpha(x)$ and $g_y(\alpha) = \psi'_\alpha(y)$ whenever x or y is an element of B_α or C_α respectively. Let $\varphi' : B \to D$ and $\psi' : C \to D$ where $\varphi'(x) = [f_x]$ and $\psi'(y) = [g_y]$ for all $x \in B$ and $y \in C$. To confirm that φ' (and similarly ψ') is an embedding, it is sufficient to show that φ' preserves function symbols, since nonprincipal ultrafilters already preserve

constant and relation symbols in the ultraproduct. Let h be a finitary function symbol in L and $\bar{x}$ be a tuple in B. Since $\bar{x}$ is finite, there exists an $\alpha < \mu$ such that for all $\beta \geq \alpha$, $\bar{x} \subset B_\beta$. Then

$$\{\beta < \mu : h^{D_\beta}(\varphi_\beta[\bar{x}] = \varphi_\beta[h^{B_\beta}(\bar{x})]\} \subset \{\beta < \mu : \beta \geq \alpha\} \in u$$

since u is regular. Then by Łoś's Theorem, $h^D(\varphi'[\bar{x}]) = \varphi'[h^B(\bar{x})]$, showing that φ' preserves any function symbols in L. Since a similar argument works for ψ', this shows that φ' and ψ' are embeddings from B and C into D. Lastly, let $z \in A$. Then there exists an $\alpha < \mu$ such that $a \in A_\beta$ for all $\beta \leq \alpha$. This means that

$$\{\beta < \mu : \varphi_\beta(\varphi(z)) = \psi_\beta(\psi(z))\} \subset \{\beta < \mu : \beta \geq \alpha\} \in u$$

So, again by Łoś's theorem, $\varphi'(\varphi(z)) = \psi'(\psi(z))$ for all $z \in A$. Therefore, D is a $\mathcal{K}$-structure with size at most 2^μ that completes the amalgam (A, φ, B, ψ, C). ⋈

So, in a sense, the space of $\mathcal{K}$-structures up to a certain size are similar to that of a Fraïssé class. In particular, if λ is a strongly inaccessible cardinal (meaning $\mu < \lambda$ implies $2^\mu < \lambda$), then 5.2.1 implies that the class of all $\mathcal{K}$-structures with size less than λ has a slightly weakened version of the amalgamation property. This property can be used in a Fraïssé like construction to create a structure with the weak λ-EP.

Theorem 5.2.2. *Let λ be an uncountable, strongly inaccessible cardinal and let $\mathcal{K}$ be either a Fraïssé class or a pseudo-Fraïssé class. Then there exists a $\mathcal{K}$-structure with size λ that has the weak λ-EP.*

Proof. Let λ be an infinite cardinal. Assuming $\lambda^{<\lambda} = \lambda$, there exist up to isomorphism, λ many $\mathcal{K}$-structures with size less than λ. Let P be a size-λ set of pairs (A, B) of $\mathcal{K}$-structures with size less than λ such that $A \subseteq B$ and for all embeddings, φ, between similarly sized $\mathcal{K}$-structures, there exists an $(A, B) \in P$ such that the inclusion map from A to B is isomorphic to φ.

Let $\pi : \lambda \times \lambda \to \lambda$ be a bijection such that $\pi(\alpha, \beta) \geq \alpha$ for all $\alpha, \beta < \lambda$. Let D_0 be any $\mathcal{K}$-structure with size less than λ. Inductively, whenever D_γ is chosen, let $((\varphi_{\gamma\beta}, A_{\gamma\beta}, B_{\gamma\beta}) : \beta < \lambda)$ enumerate all embeddings $\varphi_{\gamma\beta} : A_{\gamma\beta} \to D_\gamma$ such that $(A_{\gamma\beta}, B_{\gamma\beta}) \in P$. Then, if $\pi(\alpha, \beta) = \gamma$, then $\varphi_{\alpha\beta}$ embeds $A_{\alpha\beta}$ into F_γ and the inclusion map ι embeds $A_{\alpha\beta}$ into $B_{\alpha\beta}$.

Suppose that there exists a structure $c \in \mathrm{Age}(D_\gamma)$ that obstructs $(\varphi_{\alpha\beta}, B_{\alpha\beta})$. In this case, let $D_{\gamma+1} =_\gamma$ and let ρ_γ be the identity map on D_γ. Now suppose no such c exists. Then, by 5.2.1, there exists a $\mathcal{K}$-structure E with size less than λ and embeddings $\rho' : D_\gamma \to E$ and $\psi_B : B \to E$ such that

$$\psi_B \restriction A = \rho_\gamma \circ \varphi_{\alpha\beta}$$

In this case, let $D_{\gamma+1} = E$ and let $\rho_\gamma = \rho'$.

At limit stages, let $D_\gamma = \varinjlim\{D_\alpha : \alpha < \gamma\}$. Let $D = \varinjlim\{D_\gamma : \gamma < \omega_1\}$. Note that D has size λ and since D_0 is embeddable in D, D has $\mathrm{Age}(K)$. Lastly, if A and B are substructures of D with size less than λ such that $A \subset B$ and $\varphi : A \to D$ is an embedding, then there exists an α and β such that $A \cong A_{\alpha\beta}$, $B \cong B_{\alpha\beta}$ and φ is isomorphic to $\varphi_{\alpha\beta}$. This means that for $\gamma = \pi(\alpha, \beta)$, either there exists a $c \in \mathrm{Age}(D_\gamma)$ that obstructs ($\varphi_{\alpha\beta}$, or there exists an embedding $\psi : B \to D_{\gamma+1}$ such that if $\chi_{\gamma+1}$ is the canonical embedding of $D_{\gamma+1} \to D$, then $\chi_\gamma \circ \psi$ extends φ. This shows that D has the weak λ-EP. ⇝

So assuming λ is a strong limit cardinal, it is possible for a Fraïssé or pseudo-Fraïssé class $\mathcal{K}$ to be the age of a structure of size-λ with the weak λ-EP. If it is also assumed that $\mathcal{K}$ is a Fraïssé class whose structures are all finite, these structures can be obtained by looking at saturated elementary extensions.

Theorem 5.2.3. *Let F be a $\mathcal{K}$-structure that has the FEP, where $\mathcal{K}$ is a Fraïssé class of finite structures. If E is an elementary extension of F that is λ-saturated, then E has the weak λ-EP.*

Proof. Let A and B be substructures of E where $A \subset B$ and $|B| < \lambda$. Let $\varphi : A \to E$ be an embedding. Since $\mathcal{K}$ is a Fraïssé class there are no finitary obstructions of (φ, B). Also note that since F has the FEP, E does as well. Let $|B| = \mu$ and let $(x_\alpha : \alpha < \mu)$ enumerate the elements of $B - A$ and let $A_\alpha = A \cup \bigcup\{x_\beta : \beta < \alpha\}$. The embedding $\psi : B \to E$ will be constructed recursively in the following way.

Let $\psi(z) = \varphi(z)$ for all $z \in A$. Then, whenever x_α is the least element of B yet to be added to the domain of ψ, Let $p_\alpha = tp(x_\alpha/A_\alpha)$ be the complete type over A_α such that x_α realizes p_α. Let $\psi^*(p_\alpha)$ be the collection of formulas that are the formulas of p_α transported by ψ, meaning that if $\sigma(z, \bar{x})$ is a formula in p_α if and only if $\sigma(z, \psi[\bar{x}])$ is a formula in $\psi^*(p_\alpha)$. Consider a finite subset of p_α, p'. Since $\mathcal{K}$ is a class of finite structures, the parameters in p' form a finite structure. Then, since E has the FEP, there must exist a $y \in E$ such that y realizes $\psi^*(p')$. Since this is true for all such p', $\psi^*(p_\alpha)$ forms a type over $\psi(A_\alpha)$. Since E is λ-saturated, there exists a y_α that realizes $\psi^*(p_\alpha)$. Let $\psi(x_\alpha) = y_\alpha$. continuing this process for all $\alpha < \mu$ produces an embedding $\psi : B \to E$ that extends φ, which shows that E had the λ-EP. ⋈

This means that if an elementary extension E of a Fraïssé limit is $|E|$-saturated, then it will possess all of the nice properties of a Fraïssé limit at a higher cardinality. To construct one requires an ultrapower that uses a specially crafted ultrafilter.

It is discussed in [38], [39], and [40] that depending on the choice of ultrafilter, u, it is possible for the corresponding ultrapower A^λ/u of a model A to be λ^+ saturated. These ultrafilters are known as λ^+-good ultrafilters, and combining this result and the necessary cardinal hypotheses produces the following result.

Theorem 5.2.4. *Let $\mathcal{K}$ be a Fraïssé class of finite structures and let λ be an infinite*

cardinal such that $2^\lambda = \lambda^+$. *Then there exists a* $\mathcal{K}$*-structure with size-*λ^+ *with the weak* λ^+*-EP.*

Proof. Let F be the Fraïssé limit for $\mathcal{K}$ and let u be a λ^+-good ultrafilter on λ. Let $D = F^\lambda/u$. Then D has size-λ^+ and is a saturated elementary extension of F. So, by 5.2.3, D has the weak λ-EP. ⇜

5.3 Closing Remarks

Over the seventy years since Roland Fraïssé developed the idea behind Fraïssé limits, model theorists have been able to use his framework to classify special spaces of structures with a single universal object. Through this research it is hopefully now clear to the reader that it is also possible to classify larger analogs of these same spaces using the same framework. In addition, by increasing the cardinality to the uncountable case, it is now possible to interpret classes previously excluded from this theory under a similar lens.

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